The Hamiltonian number of graphs with prescribed connectivity

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Abstract

A Hamiltonian walk in a connected graph G is a closed walk of minimum length which contains every vertex of G. The Hamiltonian number $h(G)$ of a connected graph G is the length of a Hamiltonian walk in G. Let $\mathcal{G}(n)$ be the set of all connected graphs of order n, $\mathcal{G}(n, \kappa = k)$ be the set of all graphs in $\mathcal{G}(n)$ having connectivity $\kappa = k$, and $h(n, k) = \{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$. We prove in this paper that for any pair of integers n and k with $1 \leq k \leq n - 1$, there exist positive integers $a := \min(h; n, k) = \min\{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$ and $b := \max(h; n, k) = \max\{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$ such that $h(n, k) = \{x \in \mathbb{Z} : a \leq x \leq b\}$. The values of $\min(h; n, k)$ and $\max(h; n, k)$ are obtained in all situations.

Key Words: Hamiltonian walk, Hamiltonian number, cubic graph.

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1 Introduction

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Chartrand and Lesniak [4]. A *walk* W in a graph G is a sequence $x_0, x_1, x_2, \ldots, x_t$ of vertices of G in which $x_{i-1}x_i \in E(G)$ for all $i = 1, 2, \ldots, t$. If $x_0 = x_t$, then W is called a closed walk. A walk in G which contains all vertices of G is called a *spanning walk* of G and a closed walk in G which contains all vertices is

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called a *closed spanning walk* of G. For a walk W of G the *length* of W, denoted by $|W|$, is the number of edges used in W.

Given a connected graph G, it is possible to start at an arbitrary vertex u of G, walk in some sequence along the edges of G and return to the starting vertex u having passed through every vertex in G at least once. In general such a walk might pass through some vertices, and traverse some edges, more than once. We call such a walk a *closed spanning walk* in G. A *Hamiltonian walk* in G is a closed spanning walk of minimum length. The length of a Hamiltonian walk in G will be denoted by $h(G)$. Thus if G is a connected graph of order n, then $h(G) = n$ if and only if G is Hamiltonian. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian.

It is well known that there is no satisfactory characterization of Hamiltonian graphs. Goodman and Hedetniemi [8] introduced the concept of Hamiltonian walk and obtained some significant results on this graph parameter. Hamiltonian walks were also studied further by Asano, Nishizeki, and Watanabe [1, 2], Bermond [3], Vacek [9], Chartrand, Thomas, Saenpholphat, and Zhang [5]. In particular, the following results are known (see [5, 8]).

Theorem A *For every connected graph G of order $n \geq 2$,*

$$n \leq h(G) \leq 2n - 2.$$

Moreover,

1. *$h(G) = 2n - 2$ if and only if G is a tree, and*
2. *for every pair n, p of integers with $3 \leq n \leq p \leq 2n - 2$, there exists a connected graph G of order n having $h(G) = p$.*

Theorem B *Let G be a connected graph and $B_1, B_2, \ldots, B_k$ be the blocks of G. Then $h(G) = \sum_{i=1}^{k} h(B_i)$.*

Theorem C *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph on $n_1 + n_2 + \ldots + n_k = n$ vertices, where $n_1 \leq n_2 \leq \ldots \leq n_k$. Then*

1. *G is Hamiltonian if and only if $n_1 + n_2 + \ldots + n_{k-1} \geq n_k$.*
2. *If $n_1 + n_2 + \ldots + n_{k-1} < n_k$, then $h(G) = 2n_k$.*

A *vertex-cut* in a graph G is a set U of vertices of G such that $G - U$ is disconnected. The *vertex-connectivity* or simply the *connectivity*, denoted by $\kappa(G)$, of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n - 1$ if $G = K_n$ for some positive integer n.

Consequently, $\kappa(G) \le \delta(G)$. A graph G is said to be *k-connected*, $k \ge 1$, if $\kappa(G) \ge k$.

One of the interesting properties of 2-connected graphs is that every two vertices of such graphs lie on a common cycle. There is a generalization of this fact to k-connected graphs by Dirac [7] as we state in the following theorem.

Theorem D *Let G be a k-connected graph, $k \ge 2$. Then every k vertices of G lie on a common cycle of G.*

Our next result involves the independent sets of vertices and the connectivity of a graph. This result is due to Chvátal and Erdős [6].

Theorem E *Let G be a graph with at least three vertices. If $\kappa(G) \ge \beta(G)$, then G is Hamiltonian.*

2 Main results

Let $\mathcal{G}(n)$ be the set of all connected graphs of order n. Then $\mathcal{G}(n)$ can be partitioned according to the connectivity. For integers n and k with $1 \le k \le n-1$, we put $\mathcal{G}(n, \kappa = k) = \{G \in \mathcal{G}(n) : \kappa(G) = k\}$ and $h(n,k) = \{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$. Furthermore, we denote by $\min(h; n, k) := \min\{h(G) : G \in \mathcal{G}(n,k)\}$ and $\max(h; n, k) := \max\{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$. We prove in this section that for any pair of integers n, k with $1 \le k \le n-1$, there exist positive integers $a := \min(h; n, k)$ and $b := \max(h; n, k)$ such that $h(n,k) = \{x \in \mathbb{Z} : a \le x \le b\}$. Moreover, the values of $\min(h; n, k)$ and $\max(h; n, k)$ are obtained in all situations.

We first consider when $k = 1$ and $n \ge 3$. Since a Hamiltonian graph of order $n \ge 3$ is 2-connected, it follows that $\min(h; n, 1) \ge n + 1$. Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_n v_i : i = 1, 2, \ldots, n-1\}$. Thus G is a star of order n with center at v_n, $G \in \mathcal{G}(n, \kappa = 1)$ and, by Theorem A, $h(G) = 2n - 2$. Put $G_0 = G, G_1 = G_0 + v_1 v_2, G_2 = G_1 + v_2 v_3, \ldots, G_{n-3} = G_{n-4} + v_{n-3} v_{n-2}$. Thus $G_i \in \mathcal{G}(n, \kappa = 1)$, for each $i = 1, 2, \ldots, n-3$, and, by Theorem B, $h(G_i) = 2n - 2 - i$. Therefore, $h(n, 1) = \{x \in \mathbb{Z} : n + 1 \le x \le 2n - 2\}$. Thus we have proved the following theorem.

Theorem 2.1 *Let n be a positive integer with $n \ge 3$. Then $h(n, 1) = \{x \in \mathbb{Z} : n + 1 \le x \le 2n - 2\}$.*

For given integers n and k with $2 \le k \le n - 1$, a graph G obtained from K_{n-1} by joining a new vertex v to k vertices of K_{n-1} satisfies $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = n$. Thus $\min(h; n, k) = n$.

Lemma 2.2 *Let $G = (V, E)$ be a connected graph of order n and $E_1 = \{e_1, e_2, \ldots, e_t\} \subseteq E(G)$. If $\langle E_1 \rangle$ contains no cycle, then there exists a spanning tree T of G such that $E_1 \subseteq E(T)$.*

Proof. We will proceed by induction on t. Suppose that $t = 1$. Let T_1 be a spanning tree of G and $e_1 \notin E(T_1)$. Then $T_1 + e_1$ contains a cycle. Thus there exists $f \in E(T_1)$ such that $T_1 + e - f$ is a spanning tree of G containing e_1. Therefore the result holds for $t = 1$. We now suppose that $t \geq 2$ and the result holds for the graph $E_1 - \{e_t\}$. That is, there exists a spanning tree T_1 of G such that $E_1 - \{e_t\} \subseteq E(T_1)$. Thus $T_1 + e_t$ contains a unique cycle C. Since $\langle E_1 \rangle$ is a subgraph of $T_1 + e_t$ and $\langle E_1 \rangle$ does not contains a cycle, C contains an edge f in which $f \notin E_1$. Therefore $T = T_1 + e_t - f$ is a spanning tree of G such that $E_1 \subseteq E(T)$ as required.

■

As an application we obtain an upper bound of the Hamiltonian number for a connected graph containing a cycle.

Lemma 2.3 *Let G be a connected graph of order n. If G contains a cycle of order k, then $h(G) \leq 2n - k$.*

Proof. We first note that if G is a connected graph and $e \in E(G)$ such that $G - e$ is connected, then $h(G) \leq h(G - e)$. Let C be a cycle in G of order k and $e \in E(C)$. By Lemma 2.2, let T be a spanning tree of G containing $E(C - e)$. Thus $T + e$ consists of $n - k + 1$ blocks $B_1, B_2, \ldots B_{n-k+1}$ such that B_1 is a cycle of order k and the rest are blocks of order two. Thus, by Theorem B, $h(G) \leq h(T + e) = k + 2(n - k) = 2n - k$.

■

The following lemma provides a lower bound for the Hamiltonian number of a graph in term of the independence number of the graph.

Lemma 2.4 *Let G be a connected graph of order n. Then $h(G) \geq 2\beta(G)$. In particular, if G is Hamiltonian, then $\beta(G) \leq \frac{n}{2}$.*

Proof. Let $W : u_0, u_1, \ldots, u_t = u_0$ be a Hamiltonian walk of G. Let $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$ be a maximum independent set of G such that $0 \leq i_1 < i_2 < \ldots < i_r \leq t$. Thus $r = \beta(G)$. Since S is an independent set of vertices, it follows that for $j = 1, 2, \ldots, r - 1$, $i_{j+1} - i_j \geq 2$. Thus $t \geq 2r$. This completes the proof.

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Lemma 2.5 *Let n and k be positive integers. Then $h(n, k) = \{n\}$ if and only if $n \leq 2k$.*

Proof. Suppose that $n \leq 2k$. Let G be a k-connected graph of order n and I be a maximum independent set of vertices of G. Since G is not $(k+1)$-connected, $k \leq \delta(G)$. Thus for each $v \in I$, v has at least k neighbors in $V(G) - I$. It follows that G has at least $|I| + k$ vertices and hence $|I|+k \leq n$. Since $n \leq 2k$, $\beta(G) = |I| \leq n-k \leq k$. Thus, by Theorem E, G is Hamiltonian. Conversely, suppose that $n > 2k$. Let G be a graph with $V(G) = I \cup K$, where $I = \{v_1, v_2, \ldots, v_{n-k}\}$ and $K = \{w_1, w_2, \ldots, w_k\}$, and $E(G) = \{w_i w_j : 1 \leq i < j \leq k\} \cup \{v_i w_j : i = 1, 2, \ldots, n-k, j = 1, 2, \ldots, k\}$. It is clear that $G \in \mathcal{G}(n, \kappa = k)$. Since I is an independent set of vertices of G of cardinality $n-k$ and Lemma 2.4, $h(G) \geq 2(n-k) = n + (n-2k) > n$. Therefore $h(n,k) \neq \{n\}$.

■

The result of Lemma 2.5 gives a characterization of $h(n,k) = \{n\}$ as $k \geq n/2$. So we may assume from now on that $k < n/2$.

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that the subgraphs $\langle I \rangle$ and $\langle K \rangle$ of G induced by I and K are empty and complete graphs, respectively. Note that if $G = (V, E)$ is a split graph, then the corresponding partition $V = I \cup K$ may not be unique. It is unique if we choose the corresponding partition $V = I \cup K$ with minimum cardinality $|K|$. Thus for a split graph $G = (V, E)$, we understand that the corresponding partition $V = I \cup K$ is chosen in such a way that K has minimum cardinality. We will denote such a graph by $S(I \cup K, E)$. Further, a split graph $G = S(I \cup K, E)$ is called a *complete split graph* if for every vertex $v \in I$, v is adjacent to every vertex in K. Thus if G is a complete split graph of order n, then there exists a unique pair of integers k and $n-k$ such that $|K| = k$ and $|I| = n-k$. In this particular case, we write $G = CS(n-k, k)$. Thus $K_n = CS(1, n-1)$, for all $n \geq 2$. It is easy to see that $\kappa(CS(n-k,k)) = k$.

A split graph $G = S(I \cup K, E)$ with $|I| = |K|$ has a Hamiltonian cycle if and only if the bipartite graph $G' = G - E(\langle K \rangle)$ has a Hamiltonian cycle. It is not difficult to show that a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ contains a Hamiltonian cycle if and only if the graph $\langle I \cup N_G(I) \rangle$ contains a Hamiltonian cycle. Further, G contains no Hamiltonian cycle if $|I| > |K|$.

The complete split graph $G = CS(n-k, k)$, $k \geq 2$, satisfies the conditions that $\kappa(G) = k$ and $\beta(G) = n-k$. The following result can be considered as a direct consequence of Lemma 2.5 and Theorem C.

Corollary 2.6 *Let $G = CS(n-k, k)$ be a complete split graph of order n and $k \geq 1$. Then G has a Hamiltonian cycle if and only if $n \leq 2k$. Moreover, if $n > 2k$, then $h(G) = 2(n-k)$.*

We are now ready to prove the following main results.

Theorem 2.7 *Let n and k be integers such that $k \geq 2$ and $n > 2k$. Then $\min(h; n, k) = n$ and $\max(h; n, k) = 2(n-k)$. Moreover, for any positive integer i such that $0 \leq i \leq n-2k$, there exists $G_i \in \mathcal{G}(n, \kappa = k)$ with $h(G_i) = 2(n-k) - i$.*

Proof. We have already mentioned earlier that $\min(h; n, k) = n$ for all pairs of integers n, k such that $k \geq 2$ and $n \geq 2k$. It is clear that $CS(n-k, k) \in \mathcal{G}(n, \kappa = k)$. Since $h(CS(n-k, k)) = 2(n-k)$, $\max(h; n, k) \geq 2(n-k)$. On the other hand, let $G \in \mathcal{G}(n, \kappa = k)$. If $\beta(G) \leq k$, then, by Theorem E, $h(G) = n < n + (n-2k) = 2(n-k)$. Now suppose that $\beta(G) > k$. Let $X = \{v_1, v_2, \ldots, v_k\}$ be a set of k independent vertices of G. By Theorem D, there exists a cycle C in G containing $v_1, v_2, \ldots, v_k$. Since X is an independent set, C has order at least $2k$. By Lemma 2.3, $h(G) \leq 2n - 2k = 2(n-k)$. Thus $\max(h; n, k) = 2(n-k)$.

Let $G = CS(n-k, k)$ such that $V(G) = I \cup K$, $I = \{v_1, v_2, \ldots, v_{n-k}\}$ and $K = \{w_1, w_2, \ldots, w_k\}$. Put $G = G_0, G_1 = G_0 + v_k v_{k+1}, G_2 = G_1 + v_{k+1}v_{k+2}, \ldots, G_{n-2k} = G_{n-2k-1} + v_{n-k-1}v_{n-k}$. Thus $\beta(G_i) = n-k-\lceil i/2 \rceil$, for all $i = 0, 1, 2, \ldots, n-2k$. Also, G_i contains a cycle of order $2k+i$, for all $i = 1, 2, \ldots, n-2k$. Thus, by Lemmas 2.3 and 2.4, we have that for all $i = 0, 1, 2, \ldots, n-2k$, $2(n-k-\lceil i/2 \rceil) \leq h(G_i) \leq 2n-2k-i$. Further, G_{n-2k} contains a cycle of order $2k+n-2k = n$. Thus $h(G_{n-2k}) = n$. Since $2(n-k-\lceil i/2 \rceil) = 2(n-k)-i$ if i is even and $2(n-k-\lceil i/2 \rceil) = 2(n-k)-i-1$ if i is odd, it follows that $h(G_i) = 2(n-k)-i$, for all even integers i with $0 \leq i < n-2k$. We now consider for odd integer i. Let $W : u_0, u_1, \ldots, u_{t-1}, u_t = u_0$ be a Hamiltonian walk of G_i. Then there exist $u_{i_1}, u_{i_2}, \ldots, u_{i_{n-k}}$ such that $0 \leq i_1 < i_2 < \ldots < i_{n-k} \leq t$ and $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{n-k}}\} = \{v_1, v_2, \ldots, v_{n-k}\}$. Since $\{v_1, v_2, \ldots, v_{k-1}, v_{k+i+1}, \ldots, v_{n-k}\}$ is an independent set of $n-k-i-1$ vertices and $v_k, v_{k+1}, \ldots, v_{k+i}$ is a path of order $i+1$ of G_i, it follows that $|W| \geq 2(n-k-i-1)+i+1+1 = 2(n-k)-i$. Therefore $h(G_i) = 2(n-k)-i$ as required. Thus we have $h(n,k) = \{x \in \mathbb{Z}^+ : n \leq x \leq 2(n-k)\}$.

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We have seen that for integers $n \geq 3$ and $k \geq 1$ such that $n > 2k$, the graph $CS(n-k, k)$ satisfies the following properties:

1. $CS(n-k, k)$ is not Hamiltonian and $h(CS(n-k, k)) = 2(n-k)$,
2. $CS(n-k, k) \in \mathcal{G}(n, \kappa = k)$,
3. if $G \in \mathcal{G}(n, \kappa = k)$, then $h(G) \leq h(CS(n-k, k)) = 2(n-k)$,
4. $CS(n-k, k)$ is a graph of size $\binom{k}{2} + k(n-k)$.

If $k = 1$, then a characterization of graph G of order n with $h(G) = 2(n-1)$ can be obtained by result of Theorem A. Let $n \geq 3$ and $k \geq 2$ be integers with $n > 2k$. If $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$, then we have the following facts.

1. Since $h(G) = 2(n-k) = n + (n-2k) > n$, it follows that G is not Hamiltonian. Thus, by Theorem E, $\beta(G) \geq k+1$.

2. If $\{v_1, v_2, \ldots, v_k\}$ is an independent set of k vertices of G, then, by Theorem D, G contains a cycle of order at least $2k$. Since $h(G) = 2(n-k)$ and by Lemma 2.3, it follows that G contains a cycle of order at most $2k$. Thus G contains a cycle of order $2k$.

The following theorem is a characterization of k-connected graph of order n having Hamiltonian number $2(n-k)$.

Theorem 2.8 *Let $n \geq 3$ and $k \geq 2$ be integers with $n > 2k$. If $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$, then $m(G) \leq m(CS(n-k,k))$. Further, if $G \in \mathcal{G}(n, \kappa = k)$, then $h(G) = 2(n-k)$ and $m(G) = m(CS(n-k,k))$ if and only if $G \cong CS(n-k,k)$.*

Proof. Let $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$. By above observation there exists a cycle C of G of order $2k$ containing $\{v_1, v_2, \ldots, v_k\}$ and G does not contain a cycle of order more than $2k$. Without loss of generality, we may assume that $C : v_1, w_1, v_2, w_2, \ldots, v_k, w_k, v_1$. Let $X = V(G) - V(C)$. Then $|X| = n - 2k$. Since $h(G) = 2(n-k)$, $\langle X \rangle$ contains no cycle. Let K be a component of $\langle X \rangle$. Then $|N_G(V(K)) \cap V(C)| \leq k$ since otherwise G must contain a cycle of order at least $2k+1$. Suppose that K has order at least 2. If there exist two vertices of K have a common neighbor in C, then $h(G) < 2(n-k)$. Thus the average degree of all vertices of K is less than k. This is a contradiction. Thus $\langle X \rangle$ is an empty graph and for each $v \in X$ and $d(v) = k$. Let $v_{k+1} \in X$ such that $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ forms an independent set of G. Thus v_{k+1} is adjacent to $w_1, w_2, \ldots, w_k$. Further, for each $v \in X$, v is adjacent to either $v_1, v_2, \ldots, v_k$ or $w_1, w_2, \ldots, w_k$, otherwise, G must contain a cycle of order at least $2k+1$. Suppose that there exists $v \in X$ such that v is adjacent to $v_1, v_2, \ldots, v_k$. Then $\langle X \cup \{v, v_{k+1}\} \rangle$ contains a cycle of order $2k+2$. Thus for each $v \in X$, $N_G(v) = \{w_1, w_2, \ldots, w_k\}$. Therefore, $\{v_1, v_2, \ldots, v_k\} \cup X$ is an independent set of G of cardinality $n-k$ which implies that G is a subgraph of $CS(n-k,k)$. Thus, $m(G) = m(CS(n-k,k))$ if and only if $G \cong CS(n-k,k)$.

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By Theorem 2.8, we have that the complete split graph $CS(n-k,k)$ is the only k-connected graph of order n with Hamiltonian number $2(n-k)$ and of maximum size. We close this paper by asking the following problems.

Problem 1 Let n, k and i be integers with $k \geq 1$, $n > 2k$ and $1 \leq i \leq n-2k$. Find the maximum size of a connected graph G of order n with $\kappa(G) = k$ and $h(G) = 2(n-k) - i$.

Problem 2 Let n and ℓ be integers with $2 \leq \ell \leq n$. Find the maximum size of a connected graph G of order n with $h(G) = 2n - \ell$.

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References

[1] T. Asano, T. Nishizeki, and T. Watanabe, An upper bound on the length of a Hamiltonian walk of a maximal planar graph, *J. Graph Theory* **4** (1980) 315-336.

[2] T. Asano, T. Nishizeki, and T. Watanabe, An approximation algorithm for the Hamiltonian walk problems on maximal planar graphs, *Discrete Appl. Math.* **5** (1983) 211-222.

[3] J. C. Bermond, On Hamiltonian walks, *Congr. Numer.* **15** (1976) 41-51.

[4] G. Chartrand and L. Lesniak, "GRAPHS & DIGRAPHS", 4^{th} Edition, Chapman & Hall/CRC, A CRC Press Company, 2004.

[5] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang, A new look at Hamiltonian walks, *Bull. Inst. Combin. Appl.* **42** (2004) 37-52.

[6] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, *Discrete Math.* **2** (1972), 111-113.

[7] G. A. Dirac, In abstrakten Graphen vorhande vollstóndige 4-Graphen und ihre unterteilungen, *Math. Nachr.* **22** (1960) 61-85.

[8] S. E. Goodman and S. T. Hedetniemi, On Hamiltonian walks in graphs, *Congr. Numer.* (1973) 335-342.

[9] P. Vacek, On open Hamiltonian walks in graphs, *Arch Math. (Brno)* **27A** (1991) 105-111.