# The Graphs $C_{13}^{(t)}$ are Graceful for $t \equiv 0, 3 \pmod 4$ *

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**Abstract**

Let $C_n$ denote the cycle with $n$ vertices, and $C_n^{(t)}$ denote the graphs consisting of $t$ copies of $C_n$ with a vertex in common. Koh et al. conjectured that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0, 3 \pmod 4$. The conjecture has been shown true for $n = 3, 5, 6, 7, 9, 11, 4k$. In this paper, the conjecture is shown to be true for $n = 13$.
**Keywords:** *graceful graph, vertex labeling, edge labeling*

## 1 Introduction

Let $C_n$ denote the cycle with $n$ vertices, and $C_n^{(t)}$ denote the graphs consisting of $t$ copies of $C_n$ with a vertex in common. Koh et al. [4] conjectured that the graphs $C_n^{(t)}$ are graceful if and only if $nt \equiv 0, 3 \pmod 4$, and proved that the graphs $C_{4k}^{(t)}$ and $C_6^{(2t)}$ are graceful for $t \geqslant 1$. Qian [7] proved that the graphs $C_{2k}^{(2)}$ are graceful. Bermond et al. [1, 2] proved that the graphs $C_3^{(t)}$ (i.e, the friendship graph or Dutch t-windmill) are graceful if and only if $t \equiv 0$ or $1 \pmod 4$. The first author [6, 8, 9, 10] of this paper proved that

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the graphs $C_5^{(t)}$ and $C_9^{(t)}$ are graceful for $t \equiv 0,3 \pmod 4$, and $C_7^{(t)}$ and $C_{11}^{(t)}$ are graceful for $t \equiv 0,1 \pmod 4$. So the conjecture has been shown true for $n = 3,5,6,7,9,11,4k$. In this paper, the conjecture is shown to be true for $n = 13$. For the literature on graceful graphs we refer to [3] and the relevant references given in it.

## 2 The graphs $C_{13}^{(t)}$

Now, we consider the graphs $C_{13}^{(t)}$. Let $v_0^i, v_1^i, v_2^i, v_3^i, v_4^i, v_5^i, v_6^i, v_7^i, v_8^i, v_9^i, v_{10}^i$, $v_{11}^i, v_{12}^i$ be the vertices of the $i$-th cycle, $v_0^i = v$ for all $i$. Then we have

**Theorem 2.1.** The graphs $C_{13}^{(t)}$ are graceful for $t \equiv 0,3 \pmod 4$.
**Proof.** Case 1. $t \equiv 0 \pmod 4$, say $t = 4k$, i.e. $C_{13}^{(4k)}$

We define a vertex labeling $f$ as follows.

$$
\begin{aligned}
f(v) &= 0, \\
f(v_1^i) &= 52k+1-i, && 1 \le i \le 4k, \\
f(v_2^i) &= 8k+1-2i, && 1 \le i \le 4k, \\
f(v_3^i) &= 48k+1-i, && 1 \le i \le 4k, \\
f(v_4^i) &= \begin{cases} 12k+1+i, & 1 \le i \le 2k, \\ 8k+i, & 2k+1 \le i \le 4k, \end{cases} \\
f(v_5^i) &= 44k+1-i, && 1 \le i \le 4k, \\
f(v_6^i) &= 8k+2-2i, && 1 \le i \le 4k, \\
f(v_7^i) &= \begin{cases} 28k+1-i, & 1 \le i \le 2k, \\ 32k+1-i, & 2k+1 \le i \le 4k, \end{cases} \\
f(v_8^i) &= \begin{cases} 8k+i, & 1 \le i \le 2k, \\ 34k-1+i, & 2k+1 \le i \le 4k, \end{cases} \\
f(v_9^i) &= \begin{cases} 22k-i, & 1 \le i \le 2k, \\ 38k-i, & 2k+1 \le i \le 4k, \end{cases} \\
f(v_{10}^i) &= \begin{cases} 32k-1+i, & 1 \le i \le 2k, \\ 14k+i, & 2k+1 \le i \le 4k, \end{cases} \\
f(v_{11}^i) &= \begin{cases} 40k-i, & 1 \le i \le 2k, \\ 18k-i, & 2k+1 \le i \le 4k-3 \wedge i = 1 \bmod 2, \\ 22k-i, & 2k+2 \le i \le 4k-2 \wedge i = 0 \bmod 2, \\ 12k+1, & i = 4k-1, \\ 14k+2, & i = 4k, \end{cases} \\
f(v_{12}^i) &= \begin{cases} 26k+1-2i, & 1 \le i \le 2k, \\ 30k-2-2i, & 2k+1 \le i \le 4k-1, \\ 40k, & i = 4k. \end{cases}
\end{aligned}
$$

Now we prove that $f$ is a graceful labeling of $C_{13}^{(4k)}$ as follows.
Denote by

$$S_j = \{f(v_j^i) \mid 1 \le i \le 4k\}, \quad 0 \le j \le 12.$$

Then

$$\begin{aligned}
S_0 =& \{0\},\\
S_1 =& \{52k, 52k-1, \ldots, 48k+1\},\\
S_2 =& \{8k-1, 8k-3, \ldots, 1\},\\
S_3 =& \{48k, 48k-1, \ldots, 44k+1\},\\
S_4 =& S_{4.1} \cup S_{4.2}\\
=& \{12k+2, 12k+3, \ldots, 14k+1\} \cup \{10k+1, 10k+2, \ldots, 12k\},\\
S_5 =& \{44k, 44k-1, \ldots, 40k+1\},\\
S_6 =& \{8k, 8k-2, \ldots, 2\},\\
S_7 =& S_{7.1} \cup S_{7.2}\\
=& \{28k, 28k-1, \ldots, 26k+1\} \cup \{30k, 30k-1, \ldots, 28k+1\},\\
S_8 =& S_{8.1} \cup S_{8.2}\\
=& \{8k+1, 8k+2, \ldots, 10k\} \cup \{36k, 36k+1, \ldots, 38k-1\},\\
S_9 =& S_{9.1} \cup S_{9.2}\\
=& \{22k-1, 22k-2, \ldots, 20k\} \cup \{36k-1, 36k-2, \ldots, 34k\},\\
S_{10} =& S_{10.1} \cup S_{10.2}\\
=& \{32k, 32k+1, \ldots, 34k-1\} \cup \{16k+1, 16k+2, \ldots, 18k\},\\
S_{11} =& S_{11.1} \cup S_{11.2} \cup S_{11.3} \cup S_{11.4} \cup S_{11.5}\\
=& \{40k-1, 40k-2, \ldots, 38k\} \cup \{16k-1, 16k-3, \ldots, 14k+3\}\\
& \cup \{20k-2, 20k-4, \ldots, 18k+2\} \cup \{12k+1\} \cup \{14k+2\},\\
S_{12} =& S_{12.1} \cup S_{12.2} \cup S_{12.3}\\
=& \{26k-1, 26k-3, \ldots, 22k+1\} \cup \{26k-4, 26k-6, \ldots, 22k\} \cup \{40k\}
\end{aligned}$$

Hence

$$\begin{aligned}
& S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_{10} \cup S_{11} \cup S_{12}\\
=\ & S_0 \cup S_2 \cup S_6 \cup S_{8.1} \cup S_{4.2} \cup S_{11.4} \cup S_{4.1} \cup S_{11.5} \cup S_{11.2} \cup S_{10.2} \cup S_{11.3}\\
& \cup S_{9.1} \cup S_{12.2} \cup S_{12.1} \cup S_{7.1} \cup S_{7.2} \cup S_{10.1} \cup S_{9.2} \cup S_{8.2} \cup S_{11.1} \cup S_{12.3}\\
& \cup S_5 \cup S_3 \cup S_1\\
=\ & \{\ 0,\quad 1, 3, \ldots, 8k-1,\quad 2, 4, \ldots, 8k,\quad 8k+1, 8k+2, \ldots, 10k,\\
& 10k+1, 10k+2, \ldots, 12k,\quad 12k+1,\quad 12k+2, 12k+3, \ldots, 14k+1,\\
& 14k+2,\quad 14k+3, 14k+5, \ldots, 16k-1,\quad 16k+1, 16k+2, \ldots, 18k,\\
& 18k+2, 18k+4, \ldots, 20k-2,\quad 20k, 20k+1, \ldots, 22k-1,\\
& 22k, 22k+2, \ldots, 26k-4,\quad 22k+1, 22k+3, \ldots, 26k-1,\\
& 26k+1, 26k+2, \ldots, 28k,\quad 28k+1, 28k+2, \ldots, 30k,\\
& 32k, 32k+1, \ldots, 34k-1,\quad 34k, 34k+1, \ldots, 36k-1,\\
& 36k, 36k+1, \ldots, 38k-1,\quad 38k, 38k+1, \ldots, 40k-1,\quad 40k,\\
& 40k+1, 40k+2, \ldots, 44k,\quad 44k+1, 44k+2, \ldots, 48k,\\
& 48k+1, 48k+2, \ldots, 52k\}.
\end{aligned}$$

It is clear that the labels of each vertex are different, and $Max\{f(v_j^i)|1 \le i \le 4k\} = 52k = |E|$.

Denote by

$$D_j = \{g(v_j^i, v_{(j+1) \bmod 13}^i)|\ 1 \le i \le 4k\},\ 0 \le j \le 12,$$

$$g(v_j^i, v_{(j+1) \bmod 13}^i) = |f(v_{(j+1) \bmod 13}^i) - f(v_j^i)|,\ 1 \le i \le 4k,\ 0 \le j \le 12.$$

Then

$$\begin{aligned}
D_0 &= \{|f(v_1^i) - f(v_0^i)|\ 1 \le i \le 4k\} = \{52k+1-i|1 \le i \le 4k\} \\
&= \{52k, 52k-1, \ldots, 48k+1\}, \\
D_1 &= \{44k+i|1 \le i \le 4k\} = \{44k+1, 44k+2, \ldots, 48k\}, \\
D_2 &= \{40k+i|1 \le i \le 4k\} = \{40k+1, 40k+2, \ldots, 44k\}, \\
D_3 &= D_{3.1} \cup D_{3.2} \\
&= \{36k-2i|1 \le i \le 2k\} \cup \{40k+1-2i|2k+1 \le i \le 4k\} \\
&= \{36k-2, 36k-4, \ldots, 32k\} \cup \{36k-1, 36k-3, \ldots, 32k+1\}, \\
D_4 &= D_{4.1} \cup D_{4.2} \\
&= \{32k-2i|1 \le i \le 2k\} \cup \{36k+1-2i|2k+1 \le i \le 4k\} \\
&= \{32k-2, 32k-4, \ldots, 28k\} \cup \{32k-1, 32k-3, \ldots, 28k+1\}, \\
D_5 &= \{36k-1+i|1 \le i \le 4k\} = \{36k, 36k+1, \ldots, 40k-1\}, \\
D_6 &= D_{6.1} \cup D_{6.2} \\
&= \{20k-1+i|1 \le i \le 2k\} \cup \{24k-1+i|2k+1 \le i \le 4k\} \\
&= \{20k, 20k+1, \ldots, 22k-1\} \cup \{26k, 26k+1, \ldots, 28k-1\}, \\
D_7 &= D_{7.1} \cup D_{7.2} \cup D_{7.3} \\
&= \{20k+1-2i|1 \le i \le 2k\} \cup \{2k-2+2i|2k+1 \le i \le 3k\} \\
&\quad \cup \{2k-2+2i|3k+1 \le i \le 4k\} \\
&= \{20k-1, 20k-3, \ldots, 16k+1\} \cup \{6k, 6k+2, \ldots, 8k-2\} \\
&\quad \cup \{8k, 8k+2, \ldots, 10k-2\}, \\
D_8 &= D_{8.1} \cup D_{8.2} \\
&= \{14k-2i|1 \le i \le 2k\} \cup \{2i-4k-1|2k+1 \le i \le 4k\} \\
&= \{14k-2, 14k-4, \ldots, 10k\} \cup \{1, 3, \ldots, 4k-1\}, \\
D_9 &= D_{9.1} \cup D_{9.2} \\
&= \{10k-1+2i|1 \le i \le 2k\} \cup \{24k-2i|2k+1 \le i \le 4k\} \\
&= \{10k+1, 10k+3, \ldots, 14k-1\} \cup \{20k-2, 20k-4, \ldots, 16k\},
\end{aligned}$$

$$
\begin{aligned}
D_{10} = {} & D_{10.1} \cup D_{10.2} \cup D_{10.3} \cup D_{10.4} \cup D_{10.5} \cup D_{10.6} \cup D_{10.7} \\
= {} & \{8k+1-2i | 1 \le i \le k\} \cup \{8k+1-2i | i = k+1\} \\
& \cup \{8k+1-2i | k+2 \le i \le 2k\} \\
& \cup \{2i-4k | 2k+1 \le i \le 4k-3 \ \wedge \ i = 1 \bmod 2\} \\
& \cup \{8k-2i | 2k+2 \le i \le 4k-2 \ \wedge \ i = 0 \bmod 2\} \\
& \cup \{6k-2 | i = 4k-1\} \cup \{4k-2 | i = 4k\} \\
= {} & \{8k-1, 8k-3, \ldots, 6k+1\} \cup \{6k-1\} \\
& \cup \{6k-3, 6k-5, \ldots, 4k+1\} \cup \{2, 6, \ldots, 4k-6\} \\
& \cup \{4k-4, 4k-8, \ldots, 4\} \cup \{6k-2\} \cup \{4k-2\}, \\
D_{11} = {} & D_{11.1} \cup D_{11.2} \cup D_{11.3} \cup D_{11.4} \cup D_{11.5} \\
= {} & \{14k-1+i | 1 \le i \le 2k\} \\
& \cup \{12k-2-i | 2k+1 \le i \le 4k-3 \ \wedge \ i = 1 \bmod 2\} \\
& \cup \{8k-2-i | 2k+2 \le i \le 4k-2 \ \wedge \ i = 0 \bmod 2\} \\
& \cup \{10k-1 | i = 4k-1\} \cup \{26k-2 | i = 4k\} \\
= {} & \{14k, 14k+1, \ldots, 16k-1\} \cup \{10k-3, 10k-5, \ldots, 8k+1\} \\
& \cup \{6k-4, 6k-6, \ldots, 4k\} \cup \{10k-1\} \cup \{26k-2\}, \\
D_{12} = {} & D_{12.1} \cup D_{12.2} \cup D_{12.3} \cup D_{12.4} \\
= {} & \{26k-1 | i = 1\} \cup \{26k+1-2i | 2 \le i \le 2k\} \\
& \cup \{30k-2-2i | 2k+1 \le i \le 4k-1\} \cup \{40k | i = 4k\} \\
= {} & \{26k-1\} \cup \{26k-3, 26k-5, \ldots, 22k+1\} \\
& \cup \{26k-4, 26k-6, \ldots, 22k\} \cup \{40k\}.
\end{aligned}
$$

Let $D$ be the set of labels of all edges, then we have

$$
\begin{aligned}
D = {} & D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \\
& \cup D_7 \cup D_8 \cup D_9 \cup D_{10} \cup D_{11} \cup D_{12} \\
= {} & D_{8.2} \cup D_{10.4} \cup D_{10.5} \cup D_{10.7} \cup D_{11.3} \cup D_{10.3} \cup D_{10.6} \cup D_{10.2} \\
& \cup D_{7.2} \cup D_{10.1} \cup D_{7.3} \cup D_{11.2} \cup D_{11.4} \cup D_{8.1} \cup D_{9.1} \cup D_{11.1} \\
& \cup D_{9.2} \cup D_{7.1} \cup D_{6.1} \cup D_{12.3} \cup D_{12.2} \cup D_{11.5} \cup D_{12.1} \cup D_{6.2} \\
& \cup D_{4.1} \cup D_{4.2} \cup D_{3.1} \cup D_{3.2} \cup D_5 \cup D_{12.4} \cup D_2 \cup D_1 \cup D_0 \\
= {} & \{1, 3, \ldots, 4k-1, \quad 2, 6, \ldots, 4k-6, \quad 4, 8, \ldots, 4k-4, \quad 4k-2, \\
& 4k, 4k+2, \ldots, 6k-4, \quad 4k+1, 4k+3, \ldots, 6k-3, \quad 6k-2, \\
& 6k-1, \quad 6k, 6k+2, \ldots, 8k-2, \quad 6k+1, 6k+3, \ldots, 8k-1, \\
& 8k, 8k+2, \ldots, 10k-2, \quad 8k+1, 8k+3, \ldots, 10k-3, \quad 10k-1, \\
& 10k, 10k+2, \ldots, 14k-2, \quad 10k+1, 10k+3, \ldots, 14k-1, \\
& 14k, 14k+1, \ldots, 16k-1, \quad 16k, 16k+2, \ldots, 20k-2, \\
& 16k+1, 16k+3, \ldots, 20k-1, \quad 20k, 20k+1, \ldots, 22k-1, \\
& 22k, 22k+2, \ldots, 26k-4, \quad 22k+1, 22k+3, \ldots, 26k-3,
\end{aligned}
$$

$$
\begin{aligned}
&26k-2, \quad 26k-1, \quad 26k, 26k+1, \ldots, 28k-1,\\
&28k, 28k+2, \ldots, 32k-2, \quad 28k+1, 28k+3, \ldots, 32k-1,\\
&32k, 32k+2, \ldots, 36k-2, \quad 32k+1, 32k+3, \ldots, 36k-1,\\
&36k, 36k+1, \ldots, 40k-1, \quad 40k, \quad 40k+1, 40k+2, \ldots, 44k,\\
&44k+1, 44k+2, \ldots, 48k, \quad 48k+1, 48k+2, \ldots, 52k\}\\
= &\{1, 2, \ldots, 52k\}.
\end{aligned}
$$

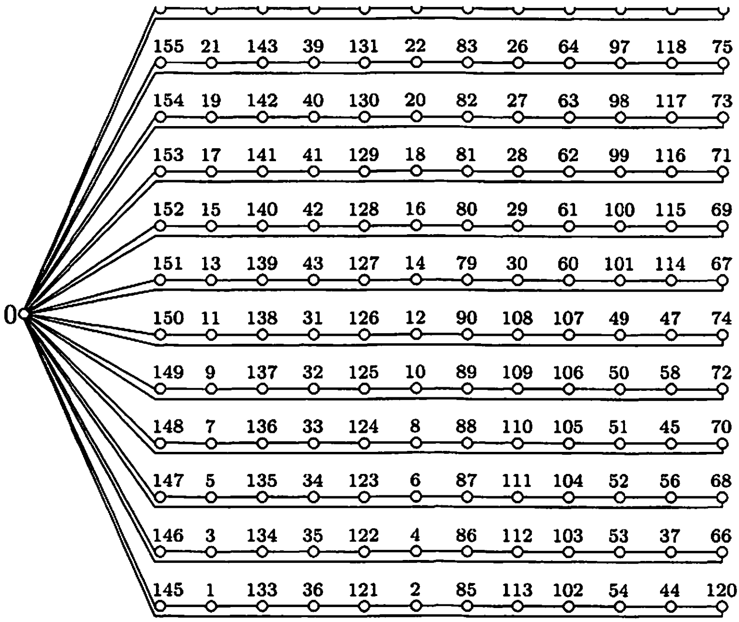
It is clear that the labels of each edge are different. According to the definition of graceful graph, we thus conclude that $C_{13}^{(4k)}$ is graceful.

Case 2. $t \equiv 3 \pmod 4$, say $t = 4k-1$, i.e. $C_{13}^{(4k-1)}$
We define a vertex labeling $f$ as follows.

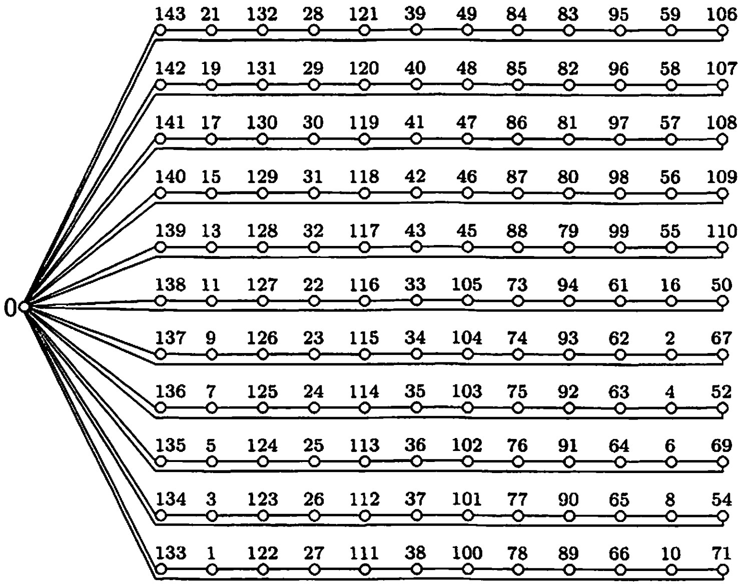
$$
\begin{aligned}
f(v) &= 0,\\
f(v_1^i) &= 52k-12-i, && 1 \le i \le 4k-1,\\
f(v_2^i) &= 8k-1-2i, && 1 \le i \le 4k-1,\\
f(v_3^i) &= 48k-11-i, && 1 \le i \le 4k-1,\\
f(v_4^i) &= \begin{cases} 10k-3+i, & 1 \le i \le 2k-1,\\ 6k-2+i, & 2k \le i \le 4k-1, \end{cases}\\
f(v_5^i) &= 44k-10-i, && 1 \le i \le 4k-1,\\
f(v_6^i) &= \begin{cases} 14k-4+i, & 1 \le i \le 2k-1,\\ 10k-3+i, & 2k \le i \le 4k-1, \end{cases}\\
f(v_7^i) &= \begin{cases} 18k-4-i, & 1 \le i \le 2k-1,\\ 40k-9-i, & 2k \le i \le 4k-1, \end{cases}\\
f(v_8^i) &= \begin{cases} 30k-7+i, & 1 \le i \le 2k-1,\\ 24k-5+i, & 2k \le i \le 4k-1. \end{cases}\\
f(v_9^i) &= \begin{cases} 30k-6-i, & 1 \le i \le 2k-1,\\ 36k-8-i, & 2k \le i \le 4k-1, \end{cases}\\
f(v_{10}^i) &= \begin{cases} 34k-8+i, & 1 \le i \le 2k-1,\\ 20k-5+i, & 2k \le i \le 4k-1. \end{cases}\\
f(v_{11}^i) &= \begin{cases} 22k-6-i, & 1 \le i \le 2k-1,\\ 6k-2, & i = 2k,\\ 2i-4k, & 2k+1 \le i \le 4k-1. \end{cases}\\
f(v_{12}^i) &= \begin{cases} 38k-9+i, & 1 \le i \le 2k-1,\\ 16k-4+i, & 2k \le i \le 4k-2 \wedge i \equiv 0 \pmod 2,\\ 22k-6+i, & 2k+1 \le i \le 4k-1 \wedge i \equiv 1 \pmod 2. \end{cases}
\end{aligned}
$$

Similar to the proof in Case 1, it can be shown that this assignment provides a graceful labeling of $C_{13}^{(4k-1)}$. Hence $C_{13}^{(t)}$ is graceful for $t \equiv 0, 3 \pmod 4$. □

In Figure 1, we illustrate our graceful labeling for $C_{13}^{(12)}$ and $C_{13}^{(11)}$.




$C_{13}^{(12)}$




$C_{13}^{(11)}$

Figure 1: Graceful labelings of $C_{13}^{(12)}$ and $C_{13}^{(11)}$.

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