Paving matroids of arbitrary cardinality *

Hua Mao

[1](Department of Mathematics, Hebei University, Baoding 071002, China)

[2](Mathematical Research Center of Hebei Province, Shijiazhuang 050016, China)

[3](Key Lab. in Mach. Learn. and Comp. Intel. of Hebei Prov., Baoding 071002, China)

e-mail: yushengmao@263.net

Abstract

This paper extends the concept of paving from finite matroids to matroids of arbitrary cardinality. Afterwards, a paving matroid of arbitrary cardinality is characterized in terms of its collection of closed sets, independent sets and circuits respectively.

Key words paving; matroid of arbitrary cardinality; circuit; independent set

2002 AMSC 05B35

1 Introduction and Preliminaries

There is no single class of structures that one calls infinite matroids. This paper will adopt the concept of infinite matroids presented in [1]. In addition, one notices that the area of combinatorics in which matroid theory has not been fruitfully extended to infinite sets seems to be finite paving matroid–which is an important class of finite matroids. The purpose of this paper is to extend paving from finite matroids to matroids of arbitrary cardinality and characterize the new paving matroids. For this purpose,

*This research is supported by NSF of Edu. Dept. of Hebei Province(2006105) and NSF of Mathematics Research Special Fund of Hebei Province(08M005)

one first establishes the independent axioms for matroids of arbitrary cardinality, followed by giving three characterizations of paving matroids of arbitrary cardinality.

Next it starts by reviewing the definitions and properties of matroids of arbitrary cardinality from [1]. In what follows, we assume that E is some arbitrary–possibly infinite–set; for a set $\{A\}, max\{A\}$ denotes the maximum element in $\{A\}$ and $Max\{A\}$ denotes a maximal element in $\{A\}$.

Definition 1 [1] Assume $m \in \mathbb{N}_0$ and $\mathcal{F} \subseteq \mathcal{P}(E) = 2^E$. Then the pair $M := (E, \mathcal{F})$ is called a *matroid of rank m with $\mathcal{F}$ as its closed sets*, if the following axioms hold:

(F1) $E \in \mathcal{F}$;

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;

(F3) Assume $F_0 \in \mathcal{F}$ and $x_1, x_2 \in E \setminus F_0$. Then one has either $\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$ or $F_1 \cap F_2 = F_0$ for certain $F_1, F_2 \in \mathcal{F}$ containing $F_0 \cup \{x_1\}$ or $F_0 \cup \{x_2\}$, respectively;

(F4) $m = max\{n \in \mathbb{N}_0 |$ there exist $F_0, F_1, ..., F_n \in \mathcal{F}$ with $F_0 \subset F_1 \subset ... \subset F_n = E\}$.

The *closure operator* $\sigma = \sigma_M : \mathcal{P}(E) \to \mathcal{F}$ of M is defined by $\sigma(A) = \sigma_M(A) := \bigcap_{\substack{F \in \mathcal{F} \\ A \subseteq F}} F$. The *rank function* $\rho : \mathcal{P}(E) \to \{0, 1, \ldots, m\}$ of M is defined by $\rho(A) := max\{k \in \mathbb{N}_0 |$ there exist $F_0, F_1, \ldots, F_k \in \mathcal{F}$ with $F_0 \subset F_1 \subset \ldots \subset F_k = \sigma(A)\}$.

Lemma 1 [1] Let $M = (E, \mathcal{F})$ be a matroid definied as in definition 1 with σ_M as its closure operator. Then

(1) For any family $(F_i)_{i \in I}$ of closed sets in M, one has $F := \bigcap_{i \in I} F_i \in \mathcal{F}$.

(2) For $\forall A \subseteq E$, one has $\sigma_M(A) = A \Leftrightarrow A \in \mathcal{F}$. Besides σ_M fits the following conditions:

$A \subseteq \sigma_M(A) = \sigma_M(\sigma_M(A))$ for all $A \subseteq E$; for $A \subseteq B \subseteq E$, one has $\sigma_M(A) \subseteq \sigma_M(B)$;

for $A \subseteq E$ and $x, y \in E \setminus \sigma_M(A)$, one has $y \in \sigma_M(A \cup \{x\}) \Leftrightarrow x \in \sigma_M(A \cup \{y\})$.

In this paper, a matroid $M = (E, \mathcal{F})$ defined as in definition 1 is called a *matroid of arbitrary cardinality.* One calls $A \in \mathcal{I} = \{A \subseteq E | x \in A, x \notin \sigma_M(A \setminus \{x\})\}$ an *independent set* of M. A subset of E not belonging to $\mathcal{I}$ is called *dependent.* A *circuit* of M is to be a minimal dependent set. M is called a *paving* if it has no circuits of size less than $\rho(E) = m$.

Definition 2 ([2,pp.385-387 & 3,p.74]) An *independence space* $M_p(E)$ is a set E together with a collection $\mathcal{I}$ of subsets of E(called *independent sets*) such that

(i1) $\mathcal{I} \neq \emptyset$;

(i2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;

(i3) If $A, B \in \mathcal{I}$ and $|A|, |B| < \infty$ with $|A| = |B| + 1$, then $\exists a \in A \setminus B$ fits $B \cup \{a\} \in \mathcal{I}$;

(i4) If $A \subseteq E$ and every finite subset of A is a member of $\mathcal{I}$, then $A \in \mathcal{I}$.

$X \subseteq E$ is *dependent* if $X \notin \mathcal{I}$. A *circuit* is to be a minimal dependent. The *closure operator* σ of $M_p(E)$ is defined by $x \in \sigma(A)$ if $x \in A$ or there exists a circuit C with $x \in C \subseteq A \cup \{x\}$. A set X is *closed* if $\sigma(X) = X$.

Lemma 2([2,pp.387-389&3,p.75]) A function $\sigma : 2^E \to 2^E$ is the closure operator of an independence space on E if and only if for X, Y subsets of E, and $x, y \in E$:

(s1) $X \subseteq \sigma(X)$;

(s2) $Y \subseteq X \Rightarrow \sigma(Y) \subseteq \sigma(X)$;

(s3) $\sigma(X) = \sigma(\sigma(X))$;

(s4) $y \in \sigma(X \cup \{x\}) \setminus \sigma(X) \Rightarrow x \in \sigma(X \cup \{y\})$;

(s5) $a \in \sigma(X) \Rightarrow a \in \sigma(X_f)$ for some finite subset X_f of X.

2 Independent axioms

This section mainly gives the independent axioms for a matroid of arbitrary cardinality.

Theorem 1 (Independent axioms) A collection $\mathcal{I}$ of subsets of E is the set of independent sets of a matroid of arbitrary cardinality on E if and only if $\mathcal{I}$ satisfies (i1)-(i4) and (i5): $max\{k \in \mathbb{N}_0 |$ there exists $I_0, I_1, \ldots, I_k \in \mathcal{I}$ such that $I_0 \subset I_1 \subset \ldots \subset I_k\} < \infty$.

Proof ($\Longrightarrow$) Let $\mathcal{I}$ be the set of independent sets of a matroid of arbitrary cardinality M on E, m be the rank of M and σ be the closure operator of M. By lemma 1, σ satisfies (s1)-(s4). By contradiction, one proves that σ satisfies (s5) as follows. Suppose $\sigma(X_f) \not\ni a \in \sigma(X)$ holds for some $X \subseteq E$ and any finite subset X_f of X. Then $|X| \not< \infty$. Let $X_0 = \emptyset \subset X$. One has $a \notin \sigma(X_0) \subset \sigma(X)$. If $X \setminus \sigma(X_0) = \emptyset$, then $X \subseteq \sigma(X_0)$, and so by (s2) and (s3), $\sigma(X) \subseteq \sigma(\sigma(X_0)) = \sigma(X_0)$, say, $\sigma(X) = \sigma(X_0)$. Thus $a \in \sigma(X_0)$, a contradiction with the supposition. Let $x_1 \in X \setminus \sigma(X_0)$ and $X_1 = \{x_1\} \cup X_0$. Then $X_1 \subseteq X, |X_1| < \infty$ and $a \notin \sigma(X_0) \subseteq \sigma(X_1)$. Certainly, $\sigma(X_0) \subset \sigma(X_1)$. Repeated application of this process yields that there exists $X_i \subseteq E$ with $X_i = X_{i-1} \cup \{x_i\}, x_i \in X \setminus \sigma(X_{i-1})$ $(i = 1, 2, \ldots, m, m+1, \ldots)$ satisfying $\sigma(X_0) \subset \sigma(X_1) \subset \ldots \subset \sigma(X_m = \{x_1, x_2, \ldots, x_m\}) \subset \sigma(X_m \cup \{x_{m+1}\}) = \sigma(X_{m+1} = \{x_1, x_2, \ldots, x_{m+1}\}) \subset \ldots \subset \sigma(X) \subseteq E$ and $a \notin \sigma(X_i)$ $(i = 0, 1, \ldots, m+1, \ldots)$, a contradiction with (F4). Hence σ satisfies (s5). By lemma 2, this implies that $\mathcal{I}$ satisfies (i1)-(i4). Next is to prove that $\mathcal{I}$ satisfies (i5).

Let $I_0, I_1, \ldots, I_k \in \mathcal{I}$ with $I_0 \subset I_1 \subset \ldots \subset I_k$. Then $\sigma(I_0) \subset \sigma(I_1) \subset \ldots \subset \sigma(I_k)$ by (s2) and the definition of $\mathcal{I}$. Furthermore, by (F4), we have that (i5) holds.

($\Longleftarrow$) Let $\mathcal{I} \subseteq 2^E$ satisfy (i1)-(i5). One asserts that for all $I \in \mathcal{I}$, $|I| < \infty$ holds. Otherwise by (i2) and $|I| \not< \infty$ for some $I = \{x_1, x_2, \ldots\} \in \mathcal{I}$, it induces $\emptyset \subset I_1 = \{x_1\} \subset I_2 = \{x_1, x_2\} \subset \ldots \subset I_{k+1} = \{x_1, \ldots, x_{k+1}\} \subset \ldots \subset I_t = \{x_1, x_2, \ldots\}$, and so $t = |I| \not< \infty$, a contradiction to (i5).

Since $\mathcal{I}$ satisfies (i1)-(i4), by lemma 1, $M_p(E) = (E, \mathcal{I})$ is an independence space. Let σ be the closure operator of $M_p(E)$ and $\mathcal{F}$ be the set of closed sets of $M_p(E)$. Besides, one asserts $\mathcal{I} = \{A \subseteq E | x \in A, x \notin \sigma(A \setminus \{x\})\}$. Otherwise for some $A \in \mathcal{I}$ and $x \in A, x \in \sigma(A \setminus \{x\})$ holds. Since $x \notin (A \setminus \{x\})$, one gets that there exists a circuit C with $x \in C \subseteq ((A \setminus \{x\}) \cup \{x\}) = A$, a contradiction.

Before proceeding, we prove $\sigma(X) = \bigcap_{X \subseteq F \in \mathcal{F}} F$. Firstly, since $\bigcap_{X \subseteq F \in \mathcal{F}} F \subseteq \sigma(\bigcap_{X \subseteq F \in \mathcal{F}} F) \subseteq \sigma(F) = F$, (for all $F \in \mathcal{F}$ and $X \subseteq F$) by (s1) and (s2), one gets $\bigcap_{X \subseteq F \in \mathcal{F}} F = \sigma(\bigcap_{X \subseteq F \in \mathcal{F}} F)$. Secondly by (s2), $\sigma(X) \subseteq \sigma(\bigcap_{X \subseteq F \in \mathcal{F}} F)$. However, (s1) tells $X \subseteq \sigma(X)$ and besides, (s3) shows $\sigma(X) \in \mathcal{F}$, and hence $\bigcap_{X \subseteq F \in \mathcal{F}} F \subseteq \sigma(X)$. Say $\sigma(X) = \bigcap_{X \subseteq F \in \mathcal{F}} F$.

Next to prove the hold of (F1)-(F4) for $(E, \mathcal{F})$. (F1) holds obviously.

Let $F_1, F_2 \in \mathcal{F}$. Since $\sigma(F_1 \cap F_2) \subseteq \sigma(F_j)$ $(j = 1, 2)$ by (s2), one gets $F_1 \cap F_2 \subseteq \sigma(F_1 \cap F_2) \subseteq \sigma(F_1) \cap \sigma(F_2) = F_1 \cap F_2$. Thus $F_1 \cap F_2 = \sigma(F_1 \cap F_2)$. Say, (F2) holds.

To prove the hold of (F3) and (F4) for $\mathcal{F}$, we need the following results (I)-(III).

(I) Let $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$ be a maximal one of $\mathcal{I}$ in X. Then $\sigma(I) = \sigma(X)$.

Otherwise, by (s2), $\sigma(I) \subset \sigma(X)$, and so $X \setminus \sigma(I) \neq \emptyset$. Let $y \in X \setminus \sigma(I)$. Then $y \notin \sigma(I) = \sigma((I \cup \{y\}) \setminus \{y\})$. By the selection of I, it follows $I \cup \{y\} \notin \mathcal{I}$. This means that there exists $a \in I \cup \{y\}$ with $a \in \sigma((I \cup \{y\}) \setminus \{a\}) = \sigma((I \setminus \{a\}) \cup \{y\}) = \sigma(\sigma(I \setminus \{a\}) \cup \{y\})$. Hence $a \neq y$, i.e. $a \in I \in \mathcal{I}$, and so $a \notin \sigma(I \setminus \{a\})$. $y \notin \sigma(I)$ tells us $y \notin \sigma(I \setminus \{a\})$. Thus by (s4), $y \in \sigma(\sigma(I \setminus \{a\}) \cup \{a\}) = \sigma((I \setminus \{a\}) \cup \{a\}) = \sigma(I)$, a contradiction. Say, $\sigma(X) = \sigma(I)$.

(II) Let $I_1, I_2 \in \mathcal{I}$ with $I_1 \subset I_2$. Then $\sigma(I_1) \subset \sigma(I_2)$.

Otherwise by (s2), $\sigma(I_1) = \sigma(I_2)$. Since $|I_j| < \infty$ $(j = 1, 2)$, $|I_1| < |I_2|$

and (i3), there exists $a \in I_2 \setminus I_1$ satisfying $I_1 \cup \{a\} \in \mathcal{I}$. Furthermore $\sigma(I_1) \subseteq \sigma(I_1 \cup \{a\}) \subseteq \sigma(I_2) = \sigma(I_1)$, say, $\sigma(I_1) = \sigma(I_1 \cup \{a\})$. Besides, $a \in I_1 \cup \{a\} \in \mathcal{I}$ shows $a \notin \sigma(I_1 \cup \{a\} \setminus \{a\}) = \sigma(I_1)$, a contradiction to $a \in \sigma(I_1 \cup \{a\}) = \sigma(I_1)$. That is to say, $\sigma(I_1) \subset \sigma(I_2)$.

(III) Let $F_1, F_2 \in \mathcal{F}, F_1 \subset F_2$ and I_1 be a maximal one in $\{I \subseteq E | \mathcal{I} \ni I \subseteq F_1\}$. Then there exists $I_2 \in \mathcal{I}$ such that $\sigma(I_2) = F_2$ and $I_1 \subset I_2$.

By the finiteness of all $I \in \mathcal{I}$, (i3), (I) and (II), the needed is obtained.

Now one comes back to prove the (F3).

Assume $F_0 \in \mathcal{F}$ and $x_1, x_2 \in E \setminus F_0$. Then it is not difficult to get "$\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} = \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$" $\Leftrightarrow$ "$\sigma(F_0 \cup \{x_1\}) = \sigma(F_0 \cup \{x_2\})$". Let $\{F \in \mathcal{F} | F_0 \cup \{x_1\} \subseteq F\} \neq \{F \in \mathcal{F} | F_0 \cup \{x_2\} \subseteq F\}$ and $F_j = \sigma(F_0 \cup \{x_j\})$ $(j = 1, 2)$. Hypothesize $F_0 \neq (F_1 \cap F_2)$. In virtue of (s1)-(s3), $F_0 \subset (F_1 \cap F_2)$, one has that for any maximal I_0 in $\{I \subseteq E | \mathcal{I} \ni I \subseteq F_0\}$, by (F2) and (III), it has an $I_{12} \in \mathcal{I}$ and $I_{12} \subseteq F_1 \cap F_2$ as a maximal set in $\{I \subseteq E | \mathcal{I} \ni I \subseteq F_1 \cap F_2\}$ such that $I_0 \subset I_{12}$. Let $a \in I_{12} \setminus I_0$. Because $\sigma(I_{12}) = \sigma(F_1 \cap F_2) \subseteq \sigma(F_j) = F_j$ $(j = 1, 2)$, one has $a \in \sigma(I_{12}) \subseteq F_j = \sigma(F_0 \cup \{x_j\}) = \sigma(\sigma(I_0) \cup \{x_j\})$ and $a \notin (\sigma(I_0) = \sigma(I_0 \setminus \{a\})) \subseteq \sigma(I_{12} \setminus \{a\}$. By (s4), $x_j \in \sigma(\sigma(I_0) \cup \{a\})$ $(j = 1, 2)$. Thus $\sigma(F_0 \cup \{x_j\}) \subseteq \sigma(F_0 \cup \{a\}) \subseteq \sigma(F_1 \cap F_2) = F_1 \cap F_2$, and hence $F_j = F_1 \cap F_2$ $(j = 1, 2)$, a contradiction. Therefore (F3) holds.

Let $F_0, F_1, \ldots, F_k \in \mathcal{F}$ satisfy $F_0 \subset F_1 \subset \ldots \subset F_k = E$. Then by (I)-(III), there exists $I_j \in \mathcal{I}$ as a maximal set of $\mathcal{I}$ in F_j $(j = 1, 2, \ldots, k)$ satisfying $F_0 = \sigma(I_0) \subset F_1 = \sigma(I_1) \subset \ldots \subset F_k = \sigma(I_k) = E$, and $I_0 \subset I_1 \subset \ldots \subset I_k$. In light of (i5), one has $k \leq m = max\{t \in \mathbb{N}_0 |$ there exist $I'_j \in \mathcal{I}$ $(j = 0, 1, \ldots, t)$ with $I'_0 \subset I'_1 \subset \ldots \subset I'_t\} < \infty$. Therefore, by the above result, (i5) and (II), $m = max\{k \in \mathbb{N}_0 | \exists F_0, F_1, \ldots, F_k$ such that $F_0 \subset F_1 \subset \ldots \subset F_k = E\}$. Say (F4) holds.

Hence $(E, \mathcal{F})$ is a matroid of arbitrary cardinality. It is easy to check that $\mathcal{I}$ is the set of independent sets of $(E, \mathcal{F})$.

Corollary 1 $\mathcal{I} \subseteq 2^E$ is the collection of independent sets of a matroid of arbitrary cardinality M on E if and only if $\mathcal{I}$ satisfies (i1),(i2),(i4),(i5) and (i3)′: For $X \subseteq E$, if $I_1, I_2 \in Max\{I \subseteq X \mid I \in \mathcal{I}\}$, then $|I_1| = |I_2|$.

Proof ($\Longrightarrow$) By theorem 1, $\mathcal{I}$ satisfies (i1)-(i5) and every member in $\mathcal{I}$ is finite. Suppose (i3)′ does not hold, i.e. there exist $X \subseteq E, I_1, I_2 \in Max\{I \subseteq X|I \in \mathcal{I}\}$ but $|I_1| \neq |I_2|$. No harming to assume $|I_1| < |I_2|$. Then by (i3), $\exists a_1 \in I_2 \setminus I_1$ satisfies $I_1 \cup \{a_1\} \in \mathcal{I}$, a contradiction with the selection of I_1.

($\Longleftarrow$) One asserts that for all $I \in \mathcal{I}, |I| < \infty$ holds. Otherwise by (i2) and $|I| \not< \infty$ for some $I = \{x_1, x_2, \ldots\} \in \mathcal{I}$, it induces $\emptyset \subset I_1 = \{x_1\} \subset I_2 = \{x_1, x_2\} \subset \ldots \subset I_{m+1} = \{x_1, \ldots, x_{m+1}\} \subset \ldots \subset I = \{x_1, x_2, \ldots\}$, a contradiction to (i5).

Let $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ and $X = I_1 \cup I_2$. Then there must exist maximal elements $I'_j \in \mathcal{I}$ $(j = 1, 2)$ in X satisfying $I_j \subseteq I'_j\,(j = 1, 2)$. Hence $|I_1| < |I_2| \leq |I'_1| = |I'_2|$. Moreover by $I_1 \subset I'_1$ and (i2), $\exists a \in I'_1 \setminus I_1 \subseteq X \setminus I_1 = I_2 \setminus I_1$ satisfying $I_1 \cup \{a\} \in \mathcal{I}$. Namely, (i3) holds.

Corollary 2 Let $M = (E, \mathcal{F})$ be a matroid of arbitrary cardinality with rank m and $\rho, \mathcal{C}, \mathcal{I}$ as its rank function, families of circuits and independent sets respectively. Then

(1) $|I| \leq m$ holds for all $I \in \mathcal{I}$. Let $X \subseteq E$ and $I_X \in \mathcal{I}$ be a maximal independent set in X. Then $\rho(I_X) = \rho(X)$. Especially, $\rho(I) = |I|$ for any $I \in \mathcal{I}$.

(2) For any $C \in \mathcal{C}$, it has $|C| \leq m + 1$ and $\rho(C) = |C| - 1$.

(3) If $C_1, C_2 \in \mathcal{C}$ and $z \in C_1 \cap C_2$, then it has $C_3 \in \mathcal{C}$ fitting $C_3 \subseteq (C_1 \cup C_2) \setminus \{z\}$.

(4) If $A \in \mathcal{I}$, then for $x \in E, A \cup \{x\}$ contains at most one circuit.

Proof (1) By definition 1, the proof of theorem 1 and theorem 1, one gets the needed .

(2) Let I_C be a maximal independent set contained in C. Then one

has $I_C \cup \{a\} \notin \mathcal{I}$ for $\forall a \in C \setminus I_C$. By the minimality of dependence of C, one has $I_C \cup \{a\} = C$. Thus by (1), $|C| = |I_C| + 1 \leq m + 1$ and $\rho(C) = \rho(I_C) = |C| - 1$.

(3) Let $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $z \in C_1 \cap C_2$. Suppose no such C_3 exists. Then $(C_1 \cup C_2) \setminus \{z\} \in \mathcal{I}$, and so $|(C_1 \cup C_2) \setminus \{z\}| = |(C_1 \setminus \{z\}) \cup (C_2) \setminus \{z\}| = |C_1 \cup C_2| - 1$. Besides, $C_1 \cap C_2 \subset C_1$ shows $C_1 \cap C_2 \in \mathcal{I}$, and so $\rho(C_1 \cap C_2) = |C_1 \cap C_2|$. If $C_1 \cup C_2 \in \mathcal{I}$, then $C_1 \subset C_1 \cup C_2$ induces $C_1 \in \mathcal{I}$, a contradiction. Namely, $C_1 \cup C_2 \notin \mathcal{I}$. However, $(C_1 \cup C_2) \setminus \{z\} \subset C_1 \cup C_2$, $|(C_1 \cup C_2) \setminus \{z\}| = |C_1 \cup C_2| - 1$ and $(C_1 \cup C_2) \setminus \{z\} \in \mathcal{I}$ taken together implies that $(C_1 \cup C_2) \setminus \{z\}$ is a maximal independent subset in $C_1 \cup C_2$.

Since $C_1 \neq C_2$ induces $C_1 \setminus C_2 \neq \emptyset$. Let $e_1 \in C_1 \setminus C_2$. Then $\mathcal{I} \ni C_1 \setminus \{e_1\} \subseteq C_1 \cup C_2$. By (i3) and $|C_1 \setminus \{e_1\}| < |(C_1 \cup C_2) \setminus \{z\}|$, one has that $C_1 \setminus \{e_1\}$ can be augmented to be a maximal independent set I in $C_1 \cup C_2$. Because $C_2 \in \mathcal{C}$ implies $C_2 \setminus I \neq \emptyset$. Let $e_2 \in C_2 \setminus I$. Then $e_2 \notin C_1 \setminus \{e_1\}$ and $I \subseteq (C_1 \setminus \{e_1\}) \cup (C_2 \setminus \{e_2\}) = (C_1 \cup C_2) \setminus \{e_1, e_2\}$ and hence $|I| \leq |(C_1 \cup C_2) \setminus \{e_1, e_2\}| \leq |C_1 \cup C_2| - 2 < |C_1 \cup C_2| - 1 = |(C_1 \cup C_2) \setminus \{z\}|$, a contradiction to (i3)$'$. Hence such C_3 is existed.

(4) Suppose $A \in \mathcal{I}$ satisfies that there exists $x \in E$ with two distinct circuits C_1, C_2 satisfying $C_1 \cup C_2 \subseteq A \cup \{x\}$. Then $x \in C_1 \cap C_2$, and hence by the above (3), there exists $C_3 \in \mathcal{C}$ satisfies $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\} \subseteq A$, contradiction to $A \in \mathcal{I}$.

3 Characterizations

In this section, a paving matroid of arbitrary cardinality will be characterized in terms of its closed sets, independent sets and circuits.

Theorem 2 Let $M = (E, \mathcal{F})$ be a matroid of arbitrary cardinality with rank r. Then M is a paving if and only if every subset of E with at most $r - 2$ elements is a closed set.

Proof ($\Longrightarrow$) Let $\mathcal{I}$ be the family of independent sets of M. Then $|I| \leq r$ for all $I \in \mathcal{I}$ by corollary 2, and besides, any circuit C fits to $|C| \geq r$. Let σ be the closure operator of M and $X \subseteq E$ with $|X| \leq r-2$. Then $X \in \mathcal{I}$. Suppose $X \notin \mathcal{F}$. In virtue of (s1), $X \subset \sigma(X)$ holds. By the maximality of $X \in \mathcal{I}$ in $\sigma(X)$, one has that $X \cup \{a\} \in \mathcal{I}$ for $a \in \sigma(X) \setminus X$ is a contradiction, and so $\sigma(X) \notin \mathcal{I}$ and $|\sigma(X)| \geq r$. $r \leq |\sigma(X)|$ follows that for any $a \in \sigma(X) \setminus X, X \subset X \cup \{a\} \subset \sigma(X)$ and $|X \cup \{a\}| \leq r-1$ hold. By the maximality of independence of X in $\sigma(X)$, $X \cup \{a\} \notin \mathcal{I}$, and so there exists a circuit $C_X \subseteq X \cup \{a\}$, and hence $|C_X| \leq r-1$, a contradiction. Say, $X = \sigma(X)$, equivalently, $X \in \mathcal{F}$.

($\Longleftarrow$) Let σ be the closure operator of M. Suppose $X \subseteq E$ with $|X| \leq r-2$ is not an independent set of M. Then $I_X \subset X \subseteq \sigma(X)$ where I_X is a maximal independent set in X, and besides by corollary 2 and definition 1, $\sigma(I_X) = \sigma(X)$. However, $|I_X| < |X| \leq r-2$ implies $I_X \in \mathcal{F}$, and so $\sigma(I_X) = I_X \subset X \subseteq \sigma(X) = \sigma(I_X) = I_X$, a contradiction. Therefore X is an independent set of M. Suppose $Y \subseteq E$ with $|Y| = r-1$ is not an independent set of M. Considering the above discussion, one has that Y is a circuit of M. By corollary 2 and definition 1, $\sigma(Y) = \sigma(I_Y)$ and $Y = I_Y \cup \{a\}$, where I_Y is a maximal independent set in Y and $a \in Y \setminus I_Y$. On the other hand, $|I_Y| < |Y| = r-1$ induces $I_Y = \sigma(I_Y)$. Furthermore, $a \notin I_Y$ shows $a \notin (\sigma(I_Y) = \sigma(Y)) \supseteq Y$, a contradiction. That is to say, the size of any circuits of M is at least r. Hence M is a paving.

Theorem 3 $\mathcal{I} \subseteq 2^E$ is the collection of independent sets of a paving matroid of arbitrary cardinality on E if and only if there is a positive integer r such that $\mathcal{I}$ satisfies (i3) and the following (PI1)-(PI3). For $X \subseteq E$

(PI1) $|X| \leq r-1 \Rightarrow X \in \mathcal{I}$;

(PI2) There is at least X with $|X| = r$ satisfying $X \in \mathcal{I}$.

(PI3) $r+1 \leq |X| \Rightarrow X \notin \mathcal{I}$.

Proof ($\Longrightarrow$) Let $\mathcal{I}$ be the collection of independent sets of a paving matroid $M = (E, \mathcal{F})$ with rank r and ρ as its rank function. Theorem 1 shows that $\mathcal{I}$ satisfies (i3). By the definition of paving, one has that for any circuit C of M, $|r = \rho(E)| \leq |C|$. Hence for $X \subseteq E$ satisfying $|X| \leq r-1$, $X \in \mathcal{I}$ holds. Namely, (PI1) is correct. Since $\rho(E) = r$ tells us that there is at least a maximal element $B \subseteq E$ in $\mathcal{I}$ satisfying $\rho(B) = \rho(E)$, and hence by corollary 2, $|B| = \rho(B)$, i.e. (PI2) holds. In addition, if $\mathcal{I} \ni X \subseteq E$ with $r+1 \leq |X|$, then by corollary 2, $r = \rho(E) < \rho(X) = |X| = r+1$, a contradiction, i.e. (PI3) holds.

($\Longleftarrow$) By (PI1), one gets $\emptyset \in \mathcal{I}$, and so (i1) holds. (i2) holds because of (PI1), (PI2) and (PI3). In addition, (PI3) means $|X| \leq r < \infty$ for all $X \in \mathcal{I}$. Thus it is straightforward to obtain the hold of (i4) and (i5). By theorem 1, $\mathcal{I}$ is the collection of independent sets of a matroid of arbitrary cardinality M with r as its rank. (PI1) and (PI3) together implies that for a circuit C of M, $r \leq |C|$. Namely, M is a paving.

Theorem 4 Let $\mathcal{D}$ be a collection of non-empty subsets of E. Then $\mathcal{D}$ is the set of circuits of a paving matroid of arbitrary cardinality on E if and only if there is a positive integer k and a subset $\mathcal{D}'$ of $\mathcal{D}$ such that

(PC1) every member of $\mathcal{D}'$ has k elements, and if two distinct members D_1 and D_2 of $\mathcal{D}'$ have $k-1$ common elements, then every k-element subset of $D_1 \cup D_2$ is in $\mathcal{D}'$.

(PC2) $\mathcal{D} - \mathcal{D}'$ consists of all of the $(k+1)$-element subsets of E that contain no member of $\mathcal{D}'$.

Proof ($\Longrightarrow$) Let M be a paving matroid of arbitrary cardinality on E with rank r, $\mathcal{D} = \{C \subseteq E | C \text{ is a circuit of } M\}$ and $k = r$. Then by the definition of paving and corollary 2, one knows $r \leq |C| \leq r+1$ for any $C \in \mathcal{D}$. Setting $\mathcal{D}' = \{C \in \mathcal{D} | \, |C| = r\}$ and $\mathcal{D}'' = \{C \in \mathcal{D} | \, |C| = r+1\}$. Evidently $\mathcal{D}'' = \mathcal{D} - \mathcal{D}'$.

If $D_1, D_2 \in \mathcal{D}'$ with $|D_1 \cap D_2| = k-1$, then there is $a_j \in E \backslash I_j$ satisfying $D_j = I_j \cup \{a_j\}$, where I_j is a maximal independent set in D_j ($j = 1, 2$), and

$|D_1 \cup D_2| = |D_1| + |D_2| - |D_1 \cap D_2| = k+1$. Let $Y \subseteq D_1 \cup D_2, |Y| = k$ and $Y \notin \mathcal{D}'$. Then Y is an independent set of M. For any $x \in (D_1 \cup D_2) \setminus Y$, one has $Y \cup \{x\}$ is a dependent set of M, otherwise it induces that the rank of M is at least $|Y \cup \{x\}| = |Y| + 1 = k + 1 > r$, a contradiction. In light of the independence of Y and corollary 2, one obtains that $Y \cup \{x\}$ contains only one circuit. But $Y \subseteq D_1 \cup D_2, |Y| = k$ and $|D_1 \cup D_2| = k + 1 = |Y| + 1 = |Y \cup \{x\}|$ taken together follows $x = (D_1 \cup D_2) \setminus Y$, and further $D_1, D_2 \subseteq Y \cup \{x\} = D_1 \cup D_2$, contradiction with the unique circuit contained in $Y \cup \{x\}$. Hence (PC1) holds.

Let $X \subseteq E, |X| = r + 1$ and X contain none of members of $\mathcal{D}'$. Then X is dependent in M because the rank of M is r and $Z \notin \mathcal{D}'$ for $Z \subset X$. By the definitions of circuits and $\mathcal{D}'$, one has that $Z \subset X$ is independent in M. Hence X is a circuit of M, i.e. $X \in \mathcal{D}''$.

($\Longleftarrow$) Let $\mathcal{I} = \{I \subseteq E | X \notin \mathcal{D}$ for all $X \subseteq I\}$. Then it is easily to know the hold of (i1) and (i2) for $\mathcal{I}$. Besides, by (PC2), for all $X \subseteq E$ with $k + 1 \leq |X|$, one gets that $\exists Y \in \mathcal{D} - \mathcal{D}'$ fits $Y \subseteq X$ or $\exists D \in \mathcal{D}$ satisfies $D \subseteq X$. Say $X \notin \mathcal{I}$, i.e., $I \in \mathcal{I}$ has $|I| \leq k$. Using this result, we prove that (i4) and (i5) hold. Let $I \subseteq E$ and $|I| \leq k$. If for $\forall A \subseteq I, A \in \mathcal{I}$ is correct, then by the definition of $\mathcal{I}$, $I \in \mathcal{I}$, say, (i4) holds. Let $I_0, I_1, \ldots, I_t \in \mathcal{I}$ with $I_0 \subset I_1 \subset \ldots \subset I_t$. By the above result, we have $|I_j| \leq k$ $(1 \leq j \leq t)$, and so $t \leq k$, further, $max\{t \in \mathbb{N}_0|$ there exists $I_0, I_1, \ldots, I_t \in \mathcal{I}$ such that $I_0 \subset I_1 \subset \ldots \subset I_t\} \leq k < \infty$, say, (s5) holds.

By corollary 1, we only need to check the hold of (i3)$'$.

Let $X \subseteq E, I_1, I_2 \in Max\{I | I \in \mathcal{I}, I \subseteq X\}$ and $|I_1| \neq |I_2|$. No harming to suppose $|I_1| < |I_2|$. For $a \in I_2 \setminus I_1, I_1 \cup \{a\} \notin \mathcal{I}$ holds according to the maximality of $I_1 \in \mathcal{I}$ in X and $I_1 \cup \{a\} \subseteq X$. Hence there exists $D \subseteq \mathcal{D}$ satisfying $D \subseteq I_1 \cup \{a\}$. If $D \in \mathcal{D} - \mathcal{D}'$, then $k + 1 \leq |D| \leq |I_1 \cup \{a\}| = |I_1| + 1 \leq |I_2|$, a contradiction with $|I_2 \in \mathcal{I}| \leq k$. If $D \in \mathcal{D}'$, then $k = |D| \leq |I_1| + 1 \leq |I_2| \leq k$. This implies $|I_2| = |I_1 \cup \{a\}| = |D|$, and so $D = I_1 \cup \{a\}$ is a circuit and $I_1 \cup \{a\} \in \mathcal{D}'$, and hence $|I_1| = k - 1$. Furthermore, $|I_2| = k$. If $I_2 \backslash (I_1 \cup \{a\}) = \emptyset$, then it must demand $I_2 = I_1 \cup \{a\}$, a contradiction with the maximality of I_1 in X. If there is $b \in I_2 \backslash (I_1 \cup \{a\})$, by the same discuss as the above, $I_1 \cup \{b\} \in \mathcal{D}'$. Then by (PC1), any k-element subset of $I_1 \cup \{a\} \cup \{b\}$ is in $\mathcal{D}'$. Repeated this augmentation, by (PC1) and $|I_2 \setminus I_1| < |I_2| = k < \infty$, we have for any k-element subset of $I_1 \cup (I_2 \setminus I_1)$ is in $\mathcal{D}'$, especially for $(I_2 \cap I_1) \cup (I_2 \setminus I_1) \subseteq I_1 \cup (I_2 \setminus I_1)$, we obtain $(I_2 \cap I_1) \cup (I_2 \setminus I_1) \in \mathcal{D}'$, say, $I_2 = (I_2 \cap I_1) \cup (I_2 \setminus I_1) \in \mathcal{D}'$, a contradiction with $I_2 \in \mathcal{I}$. Therefore $|I_1| = |I_2|$. Namely (i3)$'$ holds.

References

[1] D.Betten and W.Wenzel, On linear spaces and matroids of arbitrary cardinality, *Algebra Universalis* **49**(2003)259-288.

[2] D.J.A.Welsh, *Matroid Theory.* (London: Academic Press Inc., 1976)

[3] J.Oxley, Infinite Matroid, in *Matroid Application*, ed. by N.White, (Cambridge: Cambridge Universtiy Press, 1992,pp.73-90)

[4] J.Oxley, *Matroid Theory.* (New York: Oxford University Press, 1992)

[5] H.Mao, On geometric lattices and matroids of arbitrary cardinality, *Ars Combinatoria* **81**(2006)23-32.

[6] J.Oxley, http://www.math.lsu.edu/~oxley/errata.03.pdf (February 20,2003).