ALTERNATING DOMINATION IN ARC-COLORED DIGRAPHS

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ABSTRACT. An arc-colored digraph D is called alternating whenever $\{(u,v),(v,w)\} \subseteq A(D)$ implies that the color assigned to (u,v) is different from the color of (v,w). In arc-colored digraphs a set of vertices N is said to be a kernel by alternating paths whenever it is an independent and dominating set by alternating directed paths (there is no alternating directed path between every pair of its vertices and for every vertex not in N there exists an alternating path from it to some vertex in N). With this new concept we generalize the concept of kernel in digraphs. In this paper we prove the existence of alternating kernels in possibly infinite arc-colored digraphs with some coloration properties. We also state a bilateral relation between the property of every induced subdigraph of an arc-colored digraph D of having a kernel by alternating paths and the property of every induced subdigraph of the non colored digraph D of having a kernel, with this we enounce several sufficient conditions for D to have an alternating kernel. Previous results on kernels are generalized.

1. INTRODUCTION

1.1. General concepts and notation. Let D be a **digraph**, **V(D)** and **A(D)** will denote the set of vertices and arcs of D, respectively. If $S \subseteq V(D)$ is a nonempty set then the **subdigraph** of D **induced by the vertex set** S, $D[S]$, is that digraph having vertex set S and whose arc set consist of all those arcs of D joining vertices of S. Also, if $F \subseteq A(D)$ is a nonempty set then $D[F]$, the **subdigraph** of D **induced by the arc set** F, is the digraph with F as the arc set and whose vertices are the end points of the arcs in F. An arc $z_1z_2 \in A(D)$ is called an **asymmetrical arc** (symmetrical) if $z_2z_1 \notin A(D)$ ($z_2z_1 \in A(D)$); the **asymmetrical part of** D (the symmetrical part of D) denoted by $Asym(D)$ ($Sym(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (symmetrical) arcs of D. The arc $(z_1, z_2) \in A(D)$ is called as a S_1S_2**-arc** whenever

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$z_1 \in S_1 \subseteq V(D)$ and $z_2 \in S_2 \subseteq V(D)$. Let $z \in V(D)$, the set $N^+(z) = \{x \in V(D)$ such that $(z,x) \in F(D)\}$ is called the **exterior neighborhood of z**, meanwhile the set $A^+(z) = \{(z,x) \in A(D)$ such that $x \in V(D)\}$ will be called the **arc-exterior neighborhood of z.** $I \subseteq V(D)$ is an **independent set** in D whenever $A(D[I]) = \emptyset$. If W is a directed path or cycle in D then $\ell(W)$ will denote its length. If $\{z_1, z_2\} \subseteq V(W)$ then (z_1, W, z_2) will denote the z_1z_2**- walk** contained in W. If $I \subseteq V(D)$ and $z \in V(D)$ then a zI**-walk** is a zx-walk for some $x \in I$. By C_n we will denote the directed cycle of length n. Let $C = (0, 1, \ldots, m, 0)$ be a directed cycle of D, a **pseudodiagonal** of C is an arc $f = (i,j) \in A(D) - A(C)$ such that $i \neq j$, $\{i,j\} \subseteq V(C)$ and $\ell(i,C,j) \leq \ell(C) - 1$. A **pole** of the cycle C is the terminal vertex y of a pseudodiagonal (x,y) of C. Along the paper all the walks, paths and cycles considered are directed ones.
D will be called a m**-colored digraph** if its arcs are colored with m colors. We will denote by $color(x,y)$ the **color of the arc** $(x,y) \in A(D)$ in a m-coloration of D. $F \subseteq A(D)$ is a **monochromatic set** if all of its elements are colored alike. Then $H \subseteq D$ is a **monochromatic subdigraph** of a m-colored digraph if $A(H)$ is a monochromatic set. Let us define the **shadow** of D as the digraph obtained from D by deleting the colors of the arcs of D, this digraph will be denoted by S_D.
For more of this general concepts on digraphs we refer the reader to [5] and [3].

1.2. **Kernels and alternating paths.** Let D be a digraph. $N \subseteq D$ is a dominating set (also called an absorbent one by Berge and Duchet in [6]) whenever from every vertex $x \in V(D) - N$ there exists an $xN-$arc in D (it is important to mention that many authors differentiate the concept of absorbent set from dominating set, however they are close related and it can be showed by taking the reversal digraph, see [19] and [15]). A **kernel** in D is an independent and an absorbent set of vertices of D. D is called a kernel-perfect digraph whenever every induced subdigraph of D has a kernel, and it is called a kernel-critical-imperfect digraph if every proper induced subdigraph of D has a kernel but D does not have one. The following claims will be used in the last section of this paper.

Theorem 1. *D is a kernel-perfect digraph if one of the following conditions holds:*

(1) *D has no cycles of odd length*
(2) *Every directed cycle of odd length in D has at least two symmetric arcs*
(3) *$Asym(D)$ is acyclic*
(4) *Every directed cycle of odd length in D has at least two consecutive poles*
(5) *Every directed cycle in D has at least one symmetrical arc*

These claims were proved respectively by Richardson [24], Duchet [11], Duchet and Meyniel [12], Galeana-Sánchez and Neumann-Lara [13], and by Berge and Duchet [6]. Related results can be found in [25], [5], [22], [14], [7], as well as in the survey [8].
Let D be a m-colored digraph. A subdigraph H of D is called alternating whenever $\{(u,v),(v,w)\} \subseteq A(D)$ implies that the color assigned to (u,v) is different from the color of (v,w). If H is a path of length one (an arc) then H is consider to be alternating.
Alternating paths and cycles have been studied by several authors in [4], [1], [2], [16], [20], [21] and [26]. Applications of this concept appears from topics of graph theory and algorithms (see [18], [23], [27]), to genetics (see [10]) and social sciences (see [9]). In specific J. Bang-Jensen and G. Gutin survey in [1] theoretic and algorithmic results about alternating cycles and paths in edge-colored graphs. They also shown useful connections between the theory of paths and cycles in digraphs and alternating paths and cycles in edge-colored graphs. In [17], G. Gutin, B. Sudakov and A. Yeo consider the problem of the existence of an alternating cycle in 2-arc-colored digraphs as a generalization of the alternating cycle problem in 2-edge-colored graphs and the problem of the existence of a dicycle (an even length cycle) in a digraph (which are, both of them, polynomial time solvable), they actually proved that the alternating dicycle problem is NP-complete.
In this paper we relate this two concepts, the kernel and the alternating coloration. The result is a generalization of the first one. In order to present our mains results, let us first define the following concepts.
$I \subseteq V(D)$ is an **independent set by alternating paths** in D (or simply an **a-independent set**) if for every two different vertices in D there is no alternating path between them in D. A set $S \subseteq V(D)$ will be called an **absorbent set by alternating paths** in D (or simply an **a-absorbent set**) if for every $x \in V(D) - S$ there exists a xS-alternating path in D. According with this definitions we will say that $N \subseteq V(D)$ is an **alternating kernel** of D (**a-kernel**) if N is an a-independent set in D and an a-absorbent set in D as well. An m-colored digraph D will be called an **a-kernel-perfect** digraph or simply an **a-perfect** one if every induced subdigraph of D has an a-kernel. D will be labeled as an **a-kernel-imperfect** or simply an **a-imperfect** digraph if D has no a-kernel but every proper induced subdigraph of D does. Now we present some propositions related with the last definitions:

Proposition 1. *Let D be a 1-colored digraph (a monochromatic one) and let S_D be its shadow. N is an a-kernel of D if and only if N is a kernel of S_D (to prove it only notice that if D is a monochromatic digraph, then the alternating paths of D are just the arcs of S_D).*

Corollary 1. *Let H be a 1-colored digraph. H is an a-kernel-imperfect if and only if its shadow, S_H, is a kernel-imperfect digraph.*

Corollary 2. *Let D be a 1-colored digraph. D is an a-kernel-perfect digraph if and only if its shadow, S_D, is a kernel-perfect digraph.*

Let D be a m-colored digraph. $S \subseteq V(D)$ will be called an **a-semikernel** (or an **alternating semikernel**) of D if it is an a-independent set and for every $x \in V(D) - S$, such that there exists an Sx-alternating path T in D, there exists an xS-alternating path T' in D. Notice that if N is an a-kernel of D then it is an a-semikernel of D, and also observe that the empty set is an a-semikernel of every possibly infinite digraph.

1.3. **Statement of results.** This paper has been structured in two sections. The first one gathers results for infinite arc-colored digraphs and uses the concept of an alternating-semikernel in order to prove the existence of alternating kernels in possibly infinite arc-colored digraphs in the following way: first it is demonstrated for possibly infinite arc-colored digraphs with a certain coloration property, that they have an a-kernel whenever every of their induced subdigraph has a non empty a-semikernel; using this result we prove the existence of an alternating kernel in possibly infinite colored digraphs, by asking for the monochromaticity of the arc-exterior neighborhood of every vertex and the alternating coloration of every cycle in D.
In the final section we state a bilateral relation between the alternating kernel-perfection of an arc-colored digraph and the property of its shadow, S_D, of being a kernel-perfect digraph. To prove it we first prove certain properties of alternating kernel-perfect and kernel-imperfect colored digraphs: the existence of an a-kernel imperfect induced subdigraph in every non a-kernel perfect arc-colored digraph, the strong connectivity of a-kernel imperfect arc-colored digraphs, the absence of a non empty alternating semikernel and also the monochromaticity of a-kernel imperfect arc-colored digraphs such that arc-exterior neighborhood of every vertex is a monochromatic set. This last result allow us to present several corollaries asserting about the a-kernel-perfection of an arc-colored digraph.

2. Main results

2.1. a-semikernels.

Lemma 1. *Let D be a m-colored digraph and $\mathfrak{S}$, the set of a-semikernels of D ordered by the inclusion. Then the hypothesis of Zorn's Lemma holds.*

Proof. Let $\mathfrak{C}$ be a chain in $\mathfrak{S}$ and let us consider

$$\mathfrak{U} = \bigcup \{S \mid S \in \mathfrak{C}\}.$$

We must prove that $\mathfrak{U}$ is an a-semikernel of D:

1: $\mathfrak{U}$ is an a-independent set in D:
Suppose, by the contrary, that there exist $u, v \in \mathfrak{U}$ such that there is an uv-alternating path in D. $u \in S_1$ for some $S_1 \in \mathfrak{C}$ and $v \in S_2$ for some $S_2 \in \mathfrak{C}$, then $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$ (as $S_1, S_2 \in \mathfrak{C}$), without loss of generality let us suppose the first case, so $u, v \in S_2$, a contradiction (S_2 is an a-independent set as a consequence of being an a-semikernel).

2: $\mathfrak{U}$ satisfies the second property of an a-semikernel:
Let $x \in V(D) - \mathfrak{U}$ and suppose that there exists an $\mathfrak{U}x$-alternating path in D, T. Then, because the definition of $\mathfrak{U}$, there exists $u \in S$ for some $S \in \mathfrak{C}$ such that T is an ux-alternating path in D. Since S is an a-semikernel, there exists a xs-alternating path in D for some $s \in S$, let us call such path T'. We conclude that T' is a $x\mathfrak{U}$-alternating path in D, as $s \in S \subseteq \mathfrak{U}$).

□

Lemma 2. *Let D be a m-colored digraph such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set. Let S be a non empty a-semikernel of D, consider $B = \{v \in V(D) - S \mid$ there is no vS-alternating path in $D\}$ and take $H = D[B]$. If S_H is an a-semikernel of H, then $S \cup S_H$ is an a-semikernel of D.*

Proof. Due to the definition of an a-semikernel let us prove the following facts:

1: If $S \cup S_H$ is not an a-independent set in D then there exists a s_0s_n-alternating path in D, with $\{s_0, s_n\} \subseteq S_H$:
Suppose $S \cup S_H$ is not an a-independent set in D, so there exist vertices s_0 and s_n in $S \cup S_H$ such that there exists a s_0s_n-alternating path in D. Now, S is an a-independent set in D then $\{s_0, s_n\} \nsubseteq S$. In the other hand, there is no S_HS-alternating path in D because $S_H \subseteq B$ and because the definition of B. Besides, there is no SS_H-alternating path in D (by the contrary, as S is an a-semikernel, there exists a S_HS-alternating path in D, a contradiction). Then $\{s_0, s_n\} \subseteq S_H$.

2: $S \cup S_H$ is an a-independent set of D:
By contradiction, let us suppose that $S \cup S_H$ is not an a-independent set of D. Then it follows from the previous point that there exists $T = (s_0, s_1, s_2, ..., s_n) \subseteq D$, a s_0s_n-alternating path in D, with $\{s_0, s_n\} \subseteq S_H$. Consider $A = V(D) - (B \cup S)$:

Claim 1: $\{j \mid 1 \leq j \leq n-1 \text{ y } s_j \in A\} \neq \emptyset$:
By the contrary, as $V(D) = A \cup B \cup S$, we have that for every j such that $1 \leq j \leq n-1$ it holds that $s_j \in S \cup B$. If there exists some j such that $1 \leq j \leq n-1$ and $s_j \in S$, then (s_0, T, s_j) is a S_HS-alternating path in D, in contradiction with the definition of

B, so we have that for every j such that $1 \leq j \leq n-1$, it holds that $s_j \in B$. Then $T \subseteq H$ which means that T is a $S_H S_H$-alternating path in H, a contradiction (because S_H is an a-independent set in H). So Claim 1 holds.

Now, let

$$i = \min_{1 \leq j \leq n-1} \{j \mid s_j \in A\}$$

Notice that i is well defined (Claim 1). From the definition of A we have that there exists a $s_i S$-alternating path in D, let $P = (s_i = r_0, r_1, ..., r_m) \subseteq D$ such path. Now, $color(s_{i-1}, s_i) \neq color(s_i, s_{i+1})$ as T is an alternating path; besides $(s_i, s_{i+1}) \in F^+(s_i)$ and such set is a monochromatic one by hypothesis, then $color(s_i, s_{i+1}) = color(s_i = r_0, r_1)$; and so $(s_0, T, s_i) \cup (s_i, P, r_m)$ is a $S_H S$-alternating path in D (it is certainly a path because of the choice of i and because $P \subseteq A$), contradicting the definition of B. We conclude that $S \cup S_H$ is an a-independent set in D.

3: Let $x \in V(D) - (S \cup S_H)$. If there exists $Q = (x_0, x_1, x_2, \ldots, x_k = x) \subseteq D$, a $(S \cup S_H)x$-alternating path in D, then there exists some $x(S \cup S_H)$-alternating path in D:
If $x \notin B$ then it follows from the definition of B that there exists some xS-alternating path in D and the affirmation holds. Then let us suppose that $x \in B$:

Claim 3.1: $x_0 \in S_H$: if it is not the case then $x_0 \in S$, then Q is a Sx-alternating path in D and so there is a xS-alternating path in D (because S is an a-semikernel of D) in contradiction with the assumption ($x \in B$).

Claim 3.2: For every i, $0 \leq i \leq k$, it holds that $x_i \notin S \cup S_H$ ($S \cup S_H$ is an a-independent set due to the point 2).

Claim 3.3: For every i, $1 \leq i \leq k-1$, we have that $x_i \notin A$: By the contrary, there exists

$$t = \min_{1 \leq i \leq k-1} \{i \mid x_i \in A\},$$

and then there exists R, a $x_t s$-alternating path in D, with $s \in S$ (because of the definition of B and A). In other hand, as Q is an alternating path we have that $color(x_{t-1}, x_t) \neq color(x_t, x_{t+1})$. Considering that $(x_t, x_{t+1}) \in F^+(x_t)$ and that such set is a monochromatic one we haver that $(x_0, Q, x_t) \cup (x_t, R, s)$ is a $S_H S$-alternating path in D (it is a path because the choice of t and because $R \subseteq A$), in contradiction with point 2. So Claim 3.3 holds.

We conclude that $Q \subseteq H$ (consequence of the previous point and because $A = V(D) - (B \cup S)$ and the point 2) and the affirmation holds as S_H is an a-semikernel in H.

It follows from point 2 and 3 that $S \cup S_H$ is an a-semikernel of D. □

Lemma 3. *Let D be a possibly infinite m-colored digraph such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set. If every induced subdigraph of D has a non empty a-semikernel, then D has an a-kernel.*

Proof. Take $(\mathfrak{S}, \subseteq)$, the set of a-semikernels of D ordered with the inclusion as the relation. It follows from Lemma 1 and Zorn's Lemma that $(\mathfrak{S}, \subseteq)$ has a maximal element $S^\star$. We will prove that $S^\star$ is an alternating kernel of D:
By contradiction, suppose that $S^\star$ is not an a-kernel of D. So due to the definition of an a-kernel (recall that $S^\star$, as an a-semikernel, is an a-independent set) we have that there exists $x \in V(D) - S^\star$ such that there is no $xS^\star$-alternating path in D. Now consider the set

$$B = \{v \in (V(D) - S^\star) \mid \text{there is no } vS^\star\text{-alternating path in } D\}$$

and let $H = D[B]$. Now take S_H, a non empty a-semikernel of H (it exists because of the hypothesis). In consequence of the Lemma 2 we have that $S^\star \cup S_H$ is an a-semikernel of D, in contradiction with the choice of $S^\star$. Then the Lemma holds. □

Corollary 3. *[22] Let D be a possibly infinite digraph. If every induced subdigraph of D has a non empty semikernel then D has a kernel.*

Proof. Let D_C be the monochromatic digraph obtained from coloring D with one only color. Then the hypothesis of the Lemma holds and as a consequence D_C has an a-kernel which is also a kernel of its shadow D, as the Proposition 1 asserts. □

Theorem 2. *Let D be a possibly infinite m-colored digraph such that:*
a) For every $z \in V(D)$ $A^+(z)$ is monochromatic,
b) Every cycle in D is alternating, and
c) There are no infinite exterior paths in D.
Then D has an alternating kernel.

Proof. Take $U \subseteq V(D)$. Due to Lemma 2.1 we only must to prove that $H = D[U]$ has a non empty a-semikernel. Suppose, by the contrary, that H has not a non empty a-semikernel. The following affirmations will allow us to get a contradiction:

1: Consider $\{u, v\} \subseteq V(D)$. Then every uv-alternating walk in D contains as a subsequence an uv-alternating path in D:
Let W be an uv-alternating walk in D. By induction over $\ell(W)$, the length of W. If $\ell(W) = 1$ then W is an uv-alternating path. Assume that the affirmation holds for every uv-alternating walk with length $\ell < n$ and let $W = (u = z_0, z_1, z_2, ..., z_n = v) \subseteq D$ be an uv-alternating

walk with length n. If $z_i \neq z_j$ for every $i \neq j$ then W is an uv-alternating path in D. Suppose then that there exist i and j, $i \neq j$, such that $z_i = z_j$. Without loss of generality let us suppose that $i < j$. Now, W is an alternating walk, so $color(z_{i-1}, z_i) \neq color(z_i, z_{i+1})$. Even more, as $\{(z_i, z_{i+1}), (z_j = z_i, z_{j+1})\} \subseteq F^+(z_i = z_j)$, which is a monochromatic set by hypothesis, then $color(z_{i-1}, z_i) \neq color(z_j = z_i, z_{j+1})$. Notice that $W' = (u = z_0, W, z_i) \cup (z_i = z_j, W, z_n = v) \subset W \subseteq D$ is an uv-alternating walk with length $\ell(W') < \ell(W)$, and it follows from the inductive hypothesis that W' contains as a subsequence an uv-alternating path. We conclude the proof by noticing that $T \subset W' \subset W$.

2: Every closed walk in D is alternating:
Let W be a closed walk in D. We will proceed by induction over $\ell(W)$. If $\ell(W) = 2$ then W is a cycle and it is an alternating one by hypothesis. Suppose that the affirmation holds for every walk with length $\ell < n$ and let $W = (u = z_0, z_1, z_2, ..., z_n = u) \subseteq D$ be a closed walk in D with length n. If $z_i \neq z_j$ for every $i \neq j$, then W is a cycle and it is an alternating one by hypothesis again. Let us assume then that there exist i and j, $i \neq j$, such that $z_i = z_j$, without loss of generality suppose $i < j$. Consider $W_1 = (z_i, W, z_j) \subset W$ and $W_2 = (z_0, W, z_i) \cup (z_j, W, z_n) \subset W$. Both of them, W_1 and W_2, are walks with length strictly less that $\ell(W)$, so it follows from the induction hypothesis that W_1 and W_2 are alternating walks and then $W_1 \cup W_2 = W$ is also alternating (as $A^+(z_i) = F^+(z_j)$ is a monochromatic set).

3: For every $u \in U$ it holds that $A^+(u) \neq \emptyset$:
Assume by contradiction that there exists $u \in U$ such that $A^+(u) = \emptyset$. Clearly $\{u\}$ is an a-semikernel of H, a contradiction.

4: There exists a sequence of vertices

$$S = (u_i)_{i \in \mathbb{N}},$$

defined as follows: for every i there exists an $u_i u_{i+1}$-alternating path

$$T_i = (u_i = x_0^i, x_1^i, ..., x_{n_i}^i = u_{i+1}) \subseteq D,$$

and there is no $u_{i+1}u_i$-alternating path in D (as a consequence of the previous point and the fact that $\{u_i\}$ is not an a-semikernel).

5: For every i such that $i \geq 0$, and for every $j \notin \{i-1, i+1\}$, it holds that $T_i \cap T_j = \emptyset$:
Proceeding by contradiction, suppose that there exist i and j as the statement and such that there exists $w \in T_i \cap T_j$. Without loss of generality assume that $i < j$. Then the walk

$$W = (w, T_i, u_{i+1}) \cup (\bigcup_{k=i+1}^{j-1} T_k) \cup (u_j, T_j, w)$$

is alternating because of (2). So, in particular, $W' = (u_j, W, u_{j-1})$ is an $u_j u_{j-1}$-alternating walk and in consequence, by (1), we have that W' has an $u_j u_{j-1}$-alternating path, in contradiction with point (3).

6: S is not an infinite sequence of different vertices:
Suppose it is. Now, if for every $i \neq j$ it holds that T_i and T_j are internally disjoint, then $\bigcup T_i$ contains an infinite exterior path in D, a contradiction. In other case, there exist i and j, $i \neq j$, such that T_i and T_j intersect each other in something more than end points. If follows from the previous point that $j \in \{i-1, i+1\}$. Then consider, for each t, the vertex $x_t \in V(T_{t+1})$ as the last vertex in T_{t+1} which is also in T_t. Finally notice that the walk

$$\bigcup[(u_i = x_0^i, T_i, x_i) \cup (x_i, T_{i+1}, u_{i+2})]$$

contains again an infinite exterior path in D, a contradiction.

7: From the previous point we can conclude the existence of two different natural numbers m and r such that $m < r$ and $u_m = u_r$.

8: Take the closed walk

$$W = \bigcup_{t=m}^{r} T_t$$

and consider

$$G = D[W].$$

By using (2) we know that G is an alternating digraph.

9: So

$$W' = (u_{m+1}, W, u_m)$$

is an alternating walk in D and because of point (1) we conclude that W' contains an $u_{m+1}u_m$-alternating path in D, a contradiction (recall that $u_m \in S$).

□

2.2. a-perfect and a-imperfect digraphs.

Lemma 4. *Let D be a finite m-colored digraph. If D is not an a-perfect digraph then D contains an a-imperfect induced subdigraph.*

Proof. It follows directly from the facts that D is finite and digraphs with at most one vertex are a-kernel-perfect digraphs. □

Lemma 5. *Let D be a m-colored finite digraph and such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set. If D is an a-kernel-imperfect digraph then D does not have a non empty a-semikernel.*

Proof. Suppose that D has a non empty a-semikernel S. If for every $v \in V(D) - S$ there exists a vS-alternating path in D, then S is an a-kernel in D, a contradiction, so

$$B = \{v \in V(D) - S \mid \text{there is no } vS\text{-alternating path in } D\}$$

is a non empty set and it holds $H = D[B]$ has an a-kernel N_H (as D is an a-imperfect digraph) which is also an a-semikernel of H. Now, because of the Lemma 2 we know that $N_H \cup S$ is an a-independent set, even more, S is an a-absorbent set in the subdigraph induced by $V(D) - B$, meanwhile N_H is an a-absorbent set in H. So $N_H \cup S$ is an a-kernel of D, a contradiction. □

Lemma 6. *If D is an a-kernel-imperfect m-colored digraph then D is a strong digraph.*

Proof. Suppose that D is not a strong digraph then there exists a partition of $V(G)$ in two sets, let us say $\{V_1, V_2\}$, such that there exists no V_1V_2-arc in D. Considering that D is an a-imperfect digraph then we know about the existence of an a-kernel N of $D[V_1]$. The two following affirmations will prove that N is a non empty a-semikernel of D, in contradiction with Lemma 5:
N is an a-independent set in D: if there exists a NN-alternating path in D then such path in not contained in V_1 (as N is an a-independent set in $D[V_1]$) and so the mentioned path has a V_1V_2-arc, a contradiction.
If there exists a Nx-alternating path T in D, with $x \in V(D) - N$, then $T \subseteq D[V_1]$ (as we suppose that there is no V_1V_2-arc in D), so $x \in V_1$ and then there exists a xN-alternating path, because N is an a-kernel in $D[V_1]$. □

Lemma 7. *Let H be an arc-colored a-imperfect digraph. If H is such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set, then H is a monochromatic digraph.*

Proof. For each $z \in V(H)$ consider $H_z = H - \{z\}$. As H is an a-imperfect digraph and it contains H_z as an induced subdigraph, then H_z has an a-kernel N_z. In particular this imply that for every $v \in N_H^+(z)$ ($N_H^+(z) \neq \emptyset$ as H is an a-imperfect digraph) there exists a vN_z-alternating path in H (notice that $v \notin N_z$, in other case N_z absorbs z in H by the alternating path $(z, v) \in A(D)$, in contradiction with the absence of a-kernels in H).

1: For each $z \in H$ consider the following sets:

$$Z_0 = \{z = z_0\},$$

$$Z_i = \{z_i \in V(D) \mid (z_{i-1}, z_i) \in A(D)\},$$

$$A^+(Z_i) = \{(z_i, z_{i+1}) \in A(D) \mid z_i \in Z_i \ \ \textit{and} \ \ z_{i+1} \in V(D)\}$$

(for $i \neq j$, Z_i and Z_j can intersect each other).

2: There exists a natural number n such that

$$A(H) \subseteq \bigcup_{t=0}^{n} A^+(Z_t) :$$

Let $(x, y) \in A(H)$ and $z_0 \in H$. As H is a strong digraph (due to the Lemma 6) then there exists a z_0x-path, let us say $T = (z_0 = x_0, x_1, x_2, ..., x_k = x) \subseteq H$. So $(x, y) \in A^+(Z_k))$.

3: For every p such that $0 \leq p \leq n$, the following digraph is a monochromatic one:

$$J = D[\bigcup_{t=0}^{p} A^+(Z_t)] :$$

By induction over p. If p=0 then $A(J) = A^+(Z_0) = A^+(z_0)$ which is a monochromatic set by hypothesis, and so the affirmation holds. Suppose the veracity of the statement for $p = q < n - 1$, that is, the following digraph is a monochromatic one:

$$\bigcup_{t=0}^{q} A^+(Z_t)],$$

let us say its color is red. Now let $p = q+1$, then we must first recall that for each $z_{q+1} \in Z_{q+1}$ it holds that $A^+(z_{q+1})$ is a monochromatic set by hypothesis, let us label its color with x. Now, for each $z_{q+1} \in Z_{q+1}$ there exists $z_q \in Z_q$ such that $(z_q, z_{q+1}) \in A(H)$ (because of the definition of Z_q). Besides, for each $z_{q+1} \in Z_{q+1}$ there exists a $z_{q+1}N_{z_q}$-alternating path T in H, this because N_{z_q} is a kernel of $H - \{z_q\}$ (notice that $A^+_{H_z}(z_{q+1}) = A^+_{H_{z_q}}(z_{q+1}))$. We can prove that $x = color(z_q, z_{q+1})$ (if this equality does not hold then $(z_q, z_{q+1}) \cup T$ is a $z_qN_{z_q}$-alternating path in H, in contradiction with the a-imperfection of H), even more, we know that $color(z_q, z_{q+1}) = red$ (induction hypothesis), so for every $z_{q+1} \in Z_{q+1}$ every arc in $A^+(z_{q+1})$ is red. In conclusion: J is a monochromatic digraph.

It follows from the previous point that H is a monochromatic digraph. □

Theorem 3. *Let D be a digraph. Its shadow, S_D, is a kernel-perfect digraph if and only if for every m-coloration of D and such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set, we have that D is an a-kernel-perfect digraph.*

Proof. In order to prove the sufficiency let us suppose that for every m-coloration of D we have that D is an a-perfect digraph, whenever it holds that for every $z \in V(D)$ the set $A^+(z)$ is monochromatic. Now, take a monochromatic coloration $\mathfrak{C}$ of D, clearly $\mathfrak{C}$ satisfies the hypothesis of the Theorem (because for every $z \in V(D)$ the set $A^+(z)$ is a monochromatic one), so D with this coloration is an a-perfect digraph, which means that every induced subdigraph H of D has an a-kernel. Then, due to the Corollary 2 we can conclude that S_D is a kernel perfect digraph.

We will prove the necessity by contradiction. Let D be an m-colored digraph such that its shadow, S_D, is a kernel-perfect digraph. Take an m-coloration of $A(D)$ with the property of the set $A^+(z)$ to be monochromatic for every $z \in V(D)$, and suppose that D is not an a-perfect digraph. It follows from Lemma 4 the existence of a colored a-imperfect induced subdigraph H of D, and we know because of the Lemma 5 that such subdigraph does not have a non empty a-semikernel. Now, from Lemma 7 we know about the monochromaticity of H, so its shadow, S_H (which is also a proper induced subdigraph of S_D) is a kernel-imperfect digraph (consequence of Corollary 1), a contradiction. □

Theorem 4. *Let D be a m-colored digraph such that for every $z \in V(D)$ it holds that $A^+(z)$ is a monochromatic set. D is an a-kernel-perfect digraph if and only if the shadow of every strong and monochromatic induced subdigraph of D is a kernel perfect digraph.*

Proof. (necessity) Assume there exists H, a strong and monochromatic induced subdigraph of D and such that S_H, its shadow, is not a kernel-perfect digraph, then there exists F, a kernel-imperfect induced subdigraph of S_H. Notice that $F = S_{F_H}$, with F_H an induced subdigraph of H (and D). So it follows from Corollary 1 that F_H is an a-kernel-imperfect digraph, in consequence D is not an a-kernel-perfect digraph.
(sufficiency) By contradiction. Suppose D is not an a-kernel perfect digraph. Lemma 4 asserts there exists H, an a-kernel-imperfect induced subdigraph of D. We know H is a strong and monochromatic digraph (Lemmas 7 and 6), so it follows from Corollary 1 that S_H, the shadow of a strong and monochromatic induced subdigraph of D, is a kernel-imperfect digraph, a contradiction. □

Corollary 4. *Let D be a m-colored digraph and consider S_D, its shadow. If S_D has no cycles of odd length, then D is an a-kernel perfect digraph.*

Corollary 5. *Let D be a m-colored digraph and consider S_D, its shadow. If every directed cycle of odd length in S_D has at least two symmetric arcs, then D is an a-kernel perfect digraph.*

Corollary 6. *Let D be a m-colored digraph and consider S_D, its shadow. If Asym (S_D) is acyclic, then D is an a-kernel perfect digraph.*

Corollary 7. *Let D be a m-colored digraph and consider S_D, its shadow. If every directed cycle of odd length in S_D has at least two consecutive poles, then D is an a-kernel perfect digraph.*

Corollary 8. *Let D be a m-colored and complete digraph and consider S_D, its shadow. If every directed cycle in S_D has at least one symmetrical arc, then D is an a-kernel perfect digraph.*

Proof. The sufficient condition of each of this Corollaries imply S_D is a kernel-perfect digraph (see Theorem 1), and then every induced subdigraph of S_D is a kernel-perfect digraph (if H_1 is an induced subdigraph of H_2, which is an induced subdigraph of D, then H_1 is and induced subdigraph of D). So the shadow of every induced subdigraph of D is a kernel-perfect digraph, in particular this happens for the shadow of every strong and monochromatic induced subdigraph of D. We conclude D is an a-kernel-perfect arc-colored digraph as a consequence of Theorem 4. □

Remark 1. In Theorem 2 if we ask only for the monochromaticity of $A^+(z)$ for every $z \in V(D)$ then the result will fail.

Proof. Consider the following digraph D: $V(D) = \{v_0, v_1, v_2, x\}$ and $A(D) = \{(v_0, v_1), (v_1, v_2), (v_2, v_0)\} \cup \{(x, v_i)$ with $0 \leq i \leq 2\}$ and such that the cycle $(v_0, v_1, v_2, v_0) \subseteq D$ is a monochromatic one, let us say colored red, and the arcs from x are all colored blue. □

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