On the choosability of bipartite graphs *

Guoping Wang and Qiongxiang Huang
The College of Mathematics and Systems Sciences,
Xinjiang University, Urumqi, Xinjiang 830046, P.R.China
Email: wanggpxd@163.com

Abstract. Let u be an odd vertex of a bipartite graph B and suppose that $f : V(B) \to \mathbb{N}$ is a function such that $f(u) = \lceil d_B(u)/2 \rceil$ and $f(v) = \lceil d_B(v)/2 \rceil + 1$ for $v \in V(B) \backslash u$, where $d_B(v)$ is the degree of v in B. In this paper, we prove that B is f-choosable.

Key words: Kernel, L-coloring, f-critical, f-choosable

MR classification: 05C15

1. Introduction

Let $G = (V, E)$ be a simple graph. A *list assignment* L of G is a mapping that assigns to each $v \in V$ a set $L(v)$ of colors. An *L-coloring* of G is a proper coloring c of the vertices such that $c(v) \in L(v)$ for each $v \in V$. Let N denote the set of positive integers, and let $f : V \to \mathbb{N}$ be a function. G is *f-choosable* if, for any list assignment L of G such that $|L(v)| \geq f(v)$ for each $v \in V$, G has an L-coloring. For a positive integer k, G is *k-choosable* if G is f-choosable when $f(v) = k$ for each $v \in V$.

Let B be a bipartite graph. N.Alon and M.Tarsi in [1] showed that B is $(\lceil \Delta(B)/2 \rceil + 1)$-choosable, where $\Delta(B)$ is the maximum degree of B. Let $u \in V(B)$ be a vertex of odd degree and suppose that $f : V(B) \to \mathbb{N}$ is the function such that $f(u) = \lceil d_B(u)/2 \rceil$ and $f(v) = \lceil d_B(v)/2 \rceil + 1$ for $v \in V(B) \backslash u$, where $d_B(v)$ is the degree of v in B. In this paper, we prove that B is f-choosable.

*Project 10171085 supported by Natural Science Foundation of China

2. The main results

A *kernel* in a digraph D is a set K of nonadjacent vertices such that every vertex in $V(D)\backslash K$ is joined by an arc to at least one vertex in K. The following lemma is a special case of a result of Galvin [2].

Lemma 1. *Let D be an orientation of a graph G and suppose that $f : V(G) \to \mathbb{N}$ is a function such that $f(v) \geq d_D^+(v) + 1$ $(v \in V(G))$, where $d_D^+(v)$ denotes the outdegree of v in D. If every induced subdigraph of D has a kernel, then G is f-choosable.*

Since an Eulerian cycle of an Eulerian graph G naturally gives one of its orientations, G has an orientation D such that $d_D^+(v) = \frac{d_G(v)}{2}$ for each $v \in V(G)$. It is also well known that any orientation of a bipartite graph has a kernel. Hence, combining Lemma 1, we have

Lemma 2. *Let B be an Eulerian bipartite graph and suppose that $f : V(B) \to \mathbb{N}$ is the function such that $f(v) = d_B(v)/2+1$ for each $v \in V(B)$. Then B is f-choosable.*

A vertex u of a graph G is *odd* if $d_G(u)$ is odd and *even* otherwise. For a bipartite graph $B = (X, Y)$, we denote by $O(X)$ and $O(Y)$ the sets of the odd vertices in X and Y, respectively. Since $\sum_{x \in X} d(x) = \sum_{y \in Y} d(y) = |E(B)|$, $|O(X)|$ and $|O(Y)|$ have the same parity. If both $|O(X)|$ and $|O(Y)|$ are even, then we add two new vertices x_0 and y_0 and let $X_0 = X \cup \{x_0\}$ and $Y_0 = Y \cup \{y_0\}$. Construct a new bipartite graph $B_0 = (X_0, Y_0)$ from B by adding new edges (x_0, y) for every $y \in O(Y)$ and (y_0, x) for every $x \in O(X)$. B_0 is Eulerian since each vertex of B_0 has even degree. If $|O(X)|$ and $|O(Y)|$ are both odd, then the same construction works if we add a further edge (x_0, y_0).

Theorem 3. *Let $B = (X, Y)$ be a bipartite graph and suppose that $f : V(B) \to \mathbb{N}$ is the function such that $f(v) = \lceil d_B(v)/2 \rceil + 1$ for each $v \in V(B)$. Then B is f-choosable.*

Proof If B is Eulerian, then the result is clear by Lemma 2. Otherwise we first construct the Eulerian bipartite graph B_0 from B as above. Let $f_0 : V(B_0) \to \mathbb{N}$ be the function such that $f_0(v) = d_{B_0}(v)/2 + 1$ for each $v \in V(B_0)$. By Lemma 2, B_0 is f_0-choosable. Noting that B is a subgraph of B_0 and $d_{B_0}(v)/2 = \lceil d_B(v)/2 \rceil$ for each $v \in V(B)$, we claim that B is f-choosable. □

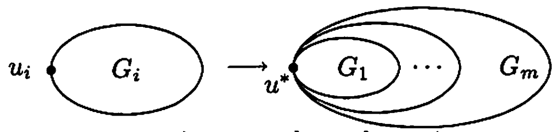

Figure 1: $O_{u_1 \ldots u_m}[G_1 \ldots G_m]$

For each i $(1 \le i \le m)$, let u_i be a vertex of a graph G_i. We denote by $O_{u_1 \ldots u_m}[G_1 \ldots G_m]$ the new graph obtained by overlapping $u_1, u_2, \ldots, u_m$ at a new vertex u^*, as in Fig. 1.

If H is a subgraph of a graph G, and L is a list assignment of G, let $L|H$ denote L restricted to the vertices of H.

Lemma 4. *For each $i \in \{1, \ldots, m\}$, let G_i be a graph and suppose that $f_i : V(G_i) \to \mathbb{N}$ is a function. Let $O^*_m = O_{u_1 \ldots u_m}[G_1 \ldots G_m]$ and suppose that $f^* : V(O^*_m) \to \mathbb{N}$ is the function such that $f^*(u^*) = \Sigma_{1 \le i \le m}(f_i(u_i) - 1) + 1$ and $f^*(v) = f_i(v)$ for $v \in V(G_i) \backslash u_i$ $(1 \le i \le m)$. Then O^*_m is f^*-choosable if G_i is f_i-choosable $(1 \le i \le m)$.*

Proof Suppose that L is a list assignment of O^*_m such that $|L(v)| = f^*(v)$ for each $v \in V(O^*_m)$. We will prove that O^*_m has an L-coloring. For each $i \in \{1, \ldots, m-1\}$, let T_i be the set of colors $\tilde{c} \in L(u^*)$ such that G_i has an $(L|G_i)$-coloring in which u_i is colored with $\tilde{c}$, and let $S_i = L(u^*) \backslash T_i$. Since G_i is f_i-choosable, $|S_i| \le f_i(u_i) - 1$ $(1 \le i \le m-1)$. Define a list assignment L_m of G_m by setting $L_m(u_m) = L(u^*) \backslash \cup_{1 \le i \le m-1} S_i$ and $L_m(v) = L(v)$ for $v \in V(G_m) \backslash u_m$. Noting that $|L_m(u_m)| = |L(u^*)| - |\cup_{1 \le i \le m-1} S_i| \ge f_m(u_m)$ and G_m is f_m-choosable, we can obtain that G_m has an L_m-coloring c_m. Since $u_m = u^*$ is given a color $c_m(u_m)$ that is not in any set S_i, and hence is in every set T_i $(1 \le i \le m-1)$, it follows from the definition of T_i that this coloring can be extended to an L-coloring of O^*_m. □

Let G be a graph, and let f, $g_u : V(G) \to \mathbb{N}$ be such that $g_u(u) = f(u) - 1$ and $g_u(v) = f(v)$ for $v \in V(G) \backslash u$. Suppose that G is f-choosable. Then G is *f-critical at $u \in V(G)$* if G is not g_u-choosable. G is *f-critical* if G is f-critical at each vertex of G. For a positive integer k, G is *k-critical* if G is f-critical when $f(v) = k$ for each $v \in V(G)$. It is easy to see that an even cycle is 2-critical.

Lemma 5. *Let G_i, f_i, O^*_m and f^* be as in Lemma 4. Suppose that G_i is f_i-critical at u_i for each $i \in \{1, \ldots, m-1\}$. Then G_m is f_m-critical at $u \in V(G_m)$ if and only if O^*_m is f^*-critical at u.*

Proof "If" We will prove that if G_m is not f_m-critical at $u \in V(G_m)$ then O^*_m is not f^*-critical at u. Let L be a list assignment of O^*_m such that $|L(u)| = f^*(u) - 1$ and $|L(v)| = f^*(v)$ for each $v \in V(O^*_m)\backslash u$. Then we can obtain that O^*_m has an L-coloring as in the proof of Lemma 4.

"Only if" We now prove that if G_m is f_m-critical at $u \in V(G_m)$ then O^* is f^*-critical at u. Since G_i is f_i-choosable ($1 \le i \le m$), O^*_m is f^*-choosable by Lemma 4. Suppose first that $u = u_m$. Then for each $i \in \{1, \ldots, m\}$ we choose a set S_i of colors such that $|S_i| = f_i(u_i) - 1$ and $S_i \cap S_j = \emptyset$ if $i \ne j$. Since G_i is f_i-critical at u_i, there exists a list assignment L_i of G_i with $L_i(u_i) = S_i$ and $|L_i(v)| = f_i(v)$ for $v \in V(G_i)\backslash u_i$ such that G_i has no L_i-coloring ($1 \le i \le m$). Define a list assignment L of O^*_m by setting $L(u^*) = \cup_{1 \le i \le m} S_i$ and $L(v) = L_i(v)$ for $v \in V(G_i)\backslash u_i$ ($1 \le i \le m$). Clearly O^*_m has no L-coloring. This shows that O^*_m is f^*-critical at $u^*(= u)$.

Suppose now that $u \in V(G_m)\backslash u_m$. Then we choose two sets $\overline{S}_1$ and $\overline{S}_2$ of colors such that $|\overline{S}_1| = f_m(u_m)$ and $|\overline{S}_2| = \sum_{1 \le i \le m-1}(f_i(u_i) - 1)$ and $\overline{S}_1 \cap \overline{S}_2 = \emptyset$. Since G_m is f_m-critical at u, we can make a list assignment L_m of G_m with $L_m(u_m) = \overline{S}_1$ and $|L_m(u)| = f_m(u) - 1$ and $|L_m(v)| = f_m(v)$ for $v \in V(G_m)\backslash\{u, u_m\}$ such that G_m has no L_m-coloring. Let $g^* : V(O^*_{m-1}) \to \mathbb{N}$ be the function such that $g^*(u^*) = \Sigma_{1 \le i \le m-1}(f_i(u_i) - 1) + 1$ and $g^*(v) = f_i(v)$ for $v \in V(G_i)\backslash u_i$ ($1 \le i \le m-1$). By the above argument, O^*_{m-1} is g^*-critical at u^*, and so we can make a list assignment L' of O^*_{m-1} with $L'(u^*) = \overline{S}_2$ and $|L'(v)| = f_i(v)$ for $v \in V(G_i)\backslash u_i$ ($1 \le i \le m-1$) such that O^*_{m-1} has no L'-coloring. Define a list assignment L of O^*_m by setting $L(u^*) = \overline{S}_1 \cup \overline{S}_2$ and $L(v) = L_m(v)$ for $v \in V(G_m)\backslash u_m$ and $L(v) = L'(v)$ for $v \in V(O^*_{m-1})\backslash u^*$. Clearly, O^*_m has no L-coloring, and so O^*_m is f^*-critical at u. □

For each i ($1 \le i \le m$), let C_i be an even cycle and u_i be a vertex on C_i. Suppose that $f : V(O_{u_1 \ldots u_m}[C_1 \cdots C_m]) \to \mathbb{N}$ is a function such that $f(u^*) = m + 1$ and $f(v) = 2$ for $v \ne u^*$. Then, by Lemma 5, $O_{u_1 \ldots u_m}[C_1 \cdots C_m]$ is f-critical.

Lemma 6. *Let B be a bipartite graph, and let $f : V(B) \to \mathbb{N}$ be a function such that $f(v) = \lceil d_B(v)/2 \rceil + 1$ for $v \in V(B)$. Suppose that B is f-critical at $u \in V(B)$. Then $d_B(u)$ is even.*

Proof Let $B^*_2 = O_{uu}[BB]$ and suppose that $f^* : V(B^*_2) \to \mathbb{N}$ is the function such that $f^*(u^*) = 2f(u) - 1$ and $f^*(v) = f(v)$ for $v \ne u^*$. By

Lemma 5, B_2^* is f^*-critical at u^*. This implies that B_2^* is not g-choosable, where $g : V(B_2^*) \to \mathbb{N}$ is the function such that $g(u^*) = 2(f(u) - 1)$ and $g(v) = f(v)$ for $v \neq u^*$. If $d_B(u)$ is odd then $g(u^*) = 2(f(u) - 1) = 2\lceil d_B(u)/2 \rceil = d_B(u) + 1 = d_{B_2^*}(u^*)/2 + 1$. Noting that B_2^* is still a bipartite graph, we can obtain that B_2^* is g-choosable by Theorem 3. This contradiction shows that $d_B(u)$ is even. □

As one consequence of Lemma 6, we have

Theorem 7. *Let u be an odd vertex of a bipartite graph B and suppose that $f : V(B) \to \mathbb{N}$ is the function such that $f(u) = \lceil d_B(u)/2 \rceil$ and $f(v) = \lceil d_B(v)/2 \rceil + 1$ for each $v \in V(B)\backslash u$. Then B is f-choosable.*

References

[1] N.Alon and M.Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), 125-134.

[2] F.Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153-158.