Degree-type Conditions for Bipartite Matching Extendability *

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Abstract: Let G be a simple connected graph containing a perfect matching. G is said to be BM-extendable if every matching M whose induced subgraph is a bipartite graph extends to a perfect matching of G. In this paper, for recognizing BM-extendable graphs, we present some conditions in terms of vertex degrees, including the degree sum conditions, the minimum degree conditions and the Fan-type condition. Furthermore, we show that all these conditions are best possible in some sense.

Keywords: matching; bipartite matching; bipartite matching extendable; degree

1 Introduction

Graphs considered in this paper are finite and simple. Let $G = (V(G), E(G))$ be a graph. For $V' \subseteq V(G)$, we denote by $G[V']$ the subgraph induced by V'. For $M \subseteq E(G)$, set

$$V(M) = \{v \in V(G) : \text{ there is an } x \in V(G) \text{ such that } vx \in M\}.$$

$M \subseteq E(G)$ is a *matching* of G if $V(e) \cap V(f) = \emptyset$ for every two distinct edges e, $f \in M$. A matching M of G is *perfect* if $V(M) = V(G)$. The matching extendability is a significant topic in matching theory [3]. Plummer [5] first proposed the notion of n-extendability: A graph G is said to be n-*extendable* if every matching M with n edges extends to a perfect matching. There has been an extensive study on the characterizations of n-extendable

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graphs in the literature [6, 9, 10]. On the other hand, Yuan [11] suggested a variant of n-extendability: A graph G is said to be *induced matching extendable* (IM-extendable in short) if every induced matching M extends to a perfect matching. From a viewpoint of linking the matchings of bipartite graphs to the ones of non-bipartite graphs, we investigate another variant as follows. We say that a matching M is a *bipartite matching* if $G[V(M)]$ is a bipartite graph. We further say that G is *bipartite-matching extendable* (BM-extendable in short) if every bipartite matching M of G extends to a perfect matching of G. When G itself is bipartite, this concept coincides with that of n-extendability (for all n). In our previous paper [8], we proved that the recognition of BM-extendable graphs is hard in a computational complexity point of view, and characterized the BM-extendability of cubic graphs, complete r-partite graphs and claw-free graphs; some elementary properties of BM-extendable graphs were also studied.

Plummer[6] developed the degree-sum and neighborhood union conditions for n-extendability. For instance, he presented the following result:

- Let G be a graph on p vertices with p even, and let n be an integer with $1 \le n < \frac{p}{2}$. If $d(u) + d(v) \ge p + 2n - 1$ for each pair of nonadjacent vertices u and v in G, then G is n-extendable.

Xu and Yu [9] established another degree-sum conditions and Fan-type conditions. As an example:

- If $\kappa(G) \ge 2n + 1$ and $\max\{d(u), d(v)\} \ge \frac{p}{2} + n$ for each pair of nonadjacent vertices u and v in G, then G is n-extendable.

Similarly, for IM-extendability, researchers obtained a number of results, e.g.,

- Let G be a graph with $2n$ vertices. If $d(u) + d(v) \ge 2\lceil\frac{4n}{3}\rceil$ for each pair of nonadjacent vertices u and v in G, then G is IM-extendable ([7]).
- If $k \ge \lceil\frac{2n}{3}\rceil$, then any k-regular graph G is IM-extendable ([4]).

In this paper, we study this type of sufficient conditions for BM-extendable graphs. In a condition, we shall derive a lower bound b for some degree-type parameter $\varphi(G)$ such that all graphs with $\varphi(G) \ge b$ are BM-extendable. And we say that the lower bound b (or the condition) is best possible (or sharp) if there exists a non-BM-extendable graph G such that $\varphi(G) = b-1$. That is to say, this b is the minimal value to ensure that all graphs G satisfying $\varphi(G) \ge b$ are BM-extendable.

The paper is organized as follows. In Section 2, we present some notations needed in this paper and some preliminary results. In Section 3, for BM-extendability, we obtain the degree sum conditions and minimum degree conditions for general graphs and claw-free graphs and construct graphs to show that these conditions are best possible. In Section 4, we give a Fan-type condition.

2 Preliminaries

In this paper, we follow the graph-theoretic terminology and notation of [1, 3]. We shall write $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$, the *union* of two graphs G and H, and kG for the union of k disjoint copies of G. The *join* $G + H$ is obtained from $G \cup H$ by adding all possible edges between G and H. For $V' \subseteq V(G)$, denote by $G - V'$ the subgraph obtained from G by deleting all the vertices in V' together with their incident edges. For $M \subseteq E(G)$, let $G - M$ denote the spanning subgraph of G with edge set $E(G) \setminus M$. Let K_n denote the complete graph with n vertices and $\overline{K_n}$ denote an empty graph with n vertices. Let $K_{n,m}$ denote the complete bipartite graph that the cardinality of the two maximal independent sets are n and m respectively. The *neighbor set* of a vertex u in graph G, denoted by $N(u)$, is the set of vertices adjacent to u. Let $o(G)$ denote the number of odd components of graph G, and $\delta(G)$ the minimum degree of G. A graph G is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph.

The following preliminary results are important to our work.

Lemma 1 (Tutte's Theorem) [1]. A graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for any $S \subset V(G)$.

Lemma 2 (Ore's Theorem) [1]. If G is a graph with $d(u) + d(v) \geq |V(G)| - 1$ for each pair of nonadjacent vertices u and v in G, then G has a hamiltonian path.

Lemma 3 (Fan's Theorem) [2]. Let G be a 2-conncected graph with $|V(G)| \geq 3$. If $\max\{d(u), d(v)\} \geq \frac{|V(G)|}{2}$ for each pair of nonadjacent vertices u and v with $d(u, v) = 2$, then G is hamiltonian.

Lemma 4 [3]. Every connected claw-free graph with an even number of vertices contains a perfect matching.

Lemma 5. If M is a bipartite matching of a claw-free graph G, then $|N(u) \cap V(M)| \leq 4$ for each vertex $u \in V(G) \setminus V(M)$.

This is so because if $|N(u) \cap V(M)| \geq 5$, then there would be a claw formed by vertex u and three adjacent vertices in $V(M)$.

For a subset S of $V(G)$, let $m_b(S)$ be the number of the edges of the maximum bipartite matching in $G[S]$. By Lemma 1, we obtain the following necessary and sufficient condition for BM-extendable graphs.

Theorem 1. A graph G is BM-extendable if and only if

$$o(G-S) \leq |S| - 2m_b(S) \quad \text{for any } S \subseteq V(G).$$

Proof. Suppose that G is a BM-extendable graph and S is an arbitrary subset of $V(G)$. Let M_S be a maximum bipartite matching in $G[S]$. Clearly, $G - V(M_S)$ has a perfect matching. Let $S' = S \setminus V(M_S) \subseteq V(G - V(M_S))$, by Lemma 1, we have

$$o(G-S) = o(G - V(M_S) - S') \leq |S'| = |S| - 2m_b(S).$$

Conversely, suppose that $o(G-S) \leq |S| - 2m_b(S)$ for any $S \subseteq V(G)$. We will prove that G is BM-extendable. Let M be a bipartite matching in G and S' an arbitrary subset of $V(G - V(M))$. Then $S = S' \cup V(M)$ is a subset of $V(G)$, and so $|M| \leq m_b(S)$. Thus

$$o(G - V(M) - S') = o(G-S) \leq |S| - 2m_b(S) \leq |S| - 2|M| = |S'|.$$

By Lemma 1 again, $G - V(M)$ has a perfect matching. The proof is completed. □

Corollary 2. A graph G is BM-extendable if and only if for any $S \subseteq V(G)$,

(i) $o(G-S) \leq |S|$, and

(ii) $o(G-S) = |S| - 2k$ $(0 \leq k \leq \frac{|S|}{2})$ implies that $m_b(S) \leq k$.

3 Degree Sum and Minimum Degree Conditions

We first introduce the degree sum condition of BM-extendability for general graphs.

Theorem 3. Let G be a graph on $2n$ vertices. If $d(u)+d(v) \geq 2\lceil \frac{3n}{2} \rceil - 1$ for each pair of nonadjacent vertices u and v in G, then G is BM-extendable.

Proof. Clearly, the result is true when $n = 2$, so we suppose $n \geq 3$ in the sequel. Let M be a bipartite matching in G and let $G' = G - V(M)$. It is sufficient to show that G' has a perfect matching. If $|M| = 1$, then for each pair of nonadjacent vertices u and v in G',

$$d_{G'}(u) + d_{G'}(v) \geq d_G(u) + d_G(v) - 4 \geq 2n - 2 = |V(G')|.$$

By Lemma 2, G' has a hamiltonian path, and so has a perfect matching. If $|M| \geq 2$, there are two nonadjacent vertices x and y in $V(M)$. Then

$$2\lceil\frac{3n}{2}\rceil - 1 \leq d_G(x) + d_G(y) \leq 2(2n - |M|),$$

implying $|M| \leq \lfloor\frac{n}{2}\rfloor$. So we have, for any nonadjacent vertices u and v in G',

$$d_{G'}(u) + d_{G'}(v) \geq d_G(u) + d_G(v) - 4|M| \geq 2\lceil\frac{3n}{2}\rceil - 4|M| - 1$$

$$\geq 2n - 2|M| - 1 = |V(G')| - 1.$$

By Lemma 2, G' has a perfect matching. The proof is completed. □

Corollary 4. Let G be a connected graph on $2n$ vertices. If $\delta(G) \geq \lceil\frac{3n}{2}\rceil$, then G is BM-extendable.

Remark. The lower bounds in the above two results are sharp. To see this, we construct a family of graphs as follows: when $n = 2$, let $G = K_2 + \overline{K_2}$; when $n = 3$, let $G = C_4 + \overline{K_2}$; when $n \geq 4$, let $G = K_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil} + K_{\lfloor\frac{n}{2}\rfloor-1,\lfloor\frac{n}{2}\rfloor+1}$. Denote by (X, Y) the bipartition of $K_{\lfloor\frac{n}{2}\rfloor-1,\lfloor\frac{n}{2}\rfloor+1}$ with $|X| = \lfloor\frac{n}{2}\rfloor + 1$ and $|Y| = \lfloor\frac{n}{2}\rfloor - 1$. Clearly, $|V(G)| = 2n$; $\delta(G) = \lceil\frac{3n}{2}\rceil - 1$; $d(u) + d(v) \geq 2\lceil\frac{3n}{2}\rceil - 2$ for each pair of nonadjacent vertices u and v in G; $d(u) + d(v) = 2\lceil\frac{3n}{2}\rceil - 2$ for any two vertices u and v in X. However, G is not BM-extendable. For, according to Theorem 1, we can choose $S = V(G)\backslash X$ so that $|S| = \lceil\frac{3n}{2}\rceil - 1$, $m_b(S) = \lceil\frac{n}{2}\rceil$, and $o(G - S) = |X| = \lfloor\frac{n}{2}\rfloor + 1 > \lfloor\frac{n}{2}\rfloor - 1 = |S| - 2m_b(S)$, as required.

If we restrict graphs to be k-regular, the following theorem shows that the lower bound of minimum degree can be slightly improved. We first prove the following lemma.

Lemma 6. Let G be a k-regular connected graph with n vertices and n is even. If $k \geq \frac{n}{2} - 1$, then G has a perfect matching.

Proof. Suppose that G is a k-regular connected graph on n vertices and n is even. When $k \geq \frac{n}{2}$, by Lemma 2, G has a perfect matching. When $k = \frac{n}{2} - 1$, suppose the result is false. Then, by the fact that $o(G - S)$ and $|S|$ have the same parity, there must be a subset S of $V(G)$ such that $o(G - S) \geq |S| + 2$. Denote by $G_1, ..., G_l$ the odd components of $G - S$. Then $l \geq |S| + 2$, and so $n \geq 2|S| + 2$, implying $1 \leq |S| \leq \frac{n}{2} - 1$. Since $k \leq |V(G_i)| + |S| - 1$ for each $1 \leq i \leq l$, we have

$$|V(G_i)| \geq k - |S| + 1 = \frac{n}{2} - |S|.$$

Thus

$$n \geq \sum_{i=1}^{l} |V(G_i)| + |S| \geq (|S|+2)(\frac{n}{2} - |S|) + |S|.$$

It follows that $|S|^2 - (\frac{n}{2} - 1)|S| \geq 0$. So we have $|S| = \frac{n}{2} - 1$, and then $l \geq \frac{n}{2} + 1$, implying that $l = \frac{n}{2} + 1$ and $|V(G_i)| = 1$ $(1 \leq i \leq l)$. By the $(\frac{n}{2} - 1)$-regularity of G, each vertex of $V(G_i)$ $(1 \leq i \leq \frac{n}{2} + 1)$ is adjacent to all the vertices in S. Thus the degree of vertex in S is at least $\frac{n}{2} + 1$. This contradiction completes the proof. □

Theorem 5. Let G be a k-regular graph on $2n$ vertices. Then G is BM-extendable if

$$k \geq \begin{cases} \lfloor \frac{3n}{2} \rfloor & \text{if } n \equiv 1 \pmod 4, \\ \lceil \frac{3n}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let G be a k-regular graph on $2n$ vertices. It is easy to check that the result holds for $n \leq 3$, and so we suppose $n \geq 4$. If $k \geq \lceil \frac{3n}{2} \rceil$, by Corollary 4, G is BM-extendable. Thus, we need only consider the case that $n \equiv 1 \pmod 4$ and $k = \frac{3n-1}{2}$. Let M be a bipartite matching of G and put $G' = G - V(M)$. If $|M| \geq k = \frac{3n-1}{2}$, we have $|V(M)| = 2|M| \geq 3n-1 > 2n = |V(G)|$, a contradiction. Thus $|M| < k$. Furthermore, by the fact that $2|M| = |V(M)| \leq 2n - (k - |M|)$, we have $|M| \leq 2n - k = \frac{n+1}{2}$. If $|M| \leq \frac{n-1}{2}$, then $\delta(G') \geq k - 2|M| \geq n - |M| = \frac{|V(G')|}{2}$. By Lemma 2, G' has a perfect matching. If $|M| = \frac{n+1}{2}$, then $|V(G')| = n - 1$. By the k-regularity of G, we can see that $G[V(M)]$ is a complete bipartite graph and every vertex in $V(M)$ is adjacent to every vertex of G'. So G' is $(k-2|M|)$-regular. Note that $k - 2|M| = \frac{n-3}{2} = \frac{|V(G')|}{2} - 1$. If G' is connected, then, by Lemma 6, G' has a perfect matching. If G' is not connected, then by the regular degree $\frac{|V(G')|}{2} - 1$ we see that G' has precisely two components each of which is a complete graph with $\frac{|V(G')|}{2}$ vertices. Since $n \equiv 1 \pmod 4$, $\frac{|V(G')|}{2} = \frac{n-1}{2}$ is even. It follows that G' has a perfect matching. Thus G is BM-extendable, completing the proof. □

Remark. This condition is best possible. To show this, we distinguish the following four cases to construct graph G.

Case 1. $n \equiv 0 \pmod 4$.

Let (X, Y) be the bipartition of a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ with $X = \{u_1, ..., u_{\frac{n}{2}}\}$ and $Y = \{v_1, ..., v_{\frac{n}{2}}\}$ and let $e = u_{\frac{n}{2}}v_{\frac{n}{2}}$. Then let

$$G = (K_{\frac{n}{2},\frac{n}{2}} - e) + (\overline{K}_{\frac{n}{2}+1} + \overline{K}_{\frac{n}{2}-1}) - E',$$

where we set $W = V(\overline{K}_{\frac{n}{2}-1}) = \{w_1, ..., w_{\frac{n}{2}-1}\}$ and

$$E' = \{u_i w_i : 1 \le i \le \frac{n}{2} - 1\} \cup \{v_i w_i : 1 \le i \le \frac{n}{2} - 1\}.$$

We can see that G is $(\lceil \frac{3n}{2} \rceil - 1)$-regular. Let $S = X \cup Y \cup W$. Then $o(G-S) = \frac{n}{2}+1 > \frac{n}{2} - 1 = |S| - 2m_b(S)$, and so G is not BM-extendable by Theorem 1.

Case 2. $n \equiv 1 \pmod 4$.

Let $G = (K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil} - M') + ((K_{\lfloor \frac{n}{2} \rfloor+1} - E(C)) \cup K_{\lfloor \frac{n}{2} \rfloor -1})$, where M' is a perfect matching in $K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ and C is a hamiltonian cycle in $K_{\lfloor \frac{n}{2} \rfloor+1}$. We can see that G is $(\lfloor \frac{3n}{2} \rfloor - 1)$-regular. Since, let $S = V(K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil} - M')$, $o(G-S) = 2 > 0 = |S| - 2m_b(S)$, G is not BM-extendable by Theorem 1.

In case 3 that $n \equiv 2 \pmod 4$, let

$$G = K_{\frac{n}{2}+1, \frac{n}{2}+1} + ((K_{\frac{n}{2}} - E(C)) \cup K_{\frac{n}{2}-2}),$$

where C is a hamiltonian cycle in $K_{\frac{n}{2}}$. In case 4 that $n \equiv 3 \pmod 4$, let $G = K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil} + 2K_{\lfloor \frac{n}{2} \rfloor}$. Clearly, in both cases, G is $(\lceil \frac{3n}{2} \rceil - 1)$-regular, and G is not BM-extendable by a similar argument as above.

If we restrict graph G to be claw-free, the lower bounds for degree sum and minimum degree can be improved greatly when $|V(G)|$ is large.

Theorem 6. Let G be a claw-free graph on $2n$ vertices. If $d(u)+d(v) \ge 2n+3$ for each pair of nonadjacent vertices u and v in G, then G is BM-extendable.

Proof. Let G be a claw-free graph on $2n$ vertices and M a bipartite matching of G. Put $G' = G - V(M)$. If we can prove that G' is connected, then by Lemma 4 G' has a perfect matching. Suppose to the contrary that G' is disconnected. Let G_1 and G_2 be two components of G', and let $u \in V(G_1)$ and $v \in V(G_2)$. By Lemma 5, we have

$$2n + 3 \le d_G(u) + d_G(v) \le |V(G_1)| + 3 + |V(G_2)| + 3 \le 2n - |V(M)| + 6,$$

implying $|M| = 1$. Then, for any nonadjacent vertices x and y in G',

$$d_{G'}(x) + d_{G'}(y) \ge d_G(x) + d_G(y) - 4 \ge 2n - 1 = |V(G')| + 1.$$

By Lemma 2, G' has a hamiltonian path, and thus G' is connected. This contradiction completes the proof. □

Theorem 7. Let G be a claw-free graph on $2n$ vertices and $\delta(G) \ge 2\lceil \frac{n}{2} \rceil + 1$. Then G is BM-extendable.

Proof. If $\delta(G) \geq n+2$, then for any nonadjacent vertices u and v in G, $d(u)+d(v) \geq 2n+4$ and by Theorem 6 G is BM-extendable. Thus, we need only consider that n is even and $\delta(G) = n+1$ in the sequel. We can see that the result is true when $n = 2$, so we suppose $n \geq 4$. Let M be a bipartite matching of G and put $G' = G - V(M)$. Denote by $G_1, ..., G_l$ the components of G'. If $l = 1$, then G' has a perfect matching by Lemma 4. If $|M| = 1$, then $\delta(G') \geq \delta(G) - 2 = n - 1 = \frac{|V(G')|}{2}$, and so by Lemma 2 G' also has a perfect matching. Now, we suppose that $l \geq 2$ and $|M| \geq 2$. Since, by Lemma 5, $n + 1 = \delta(G) \leq |V(G_i)| + 3$ $(1 \leq i \leq l)$, we have $|V(G_i)| \geq n - 2$, and so

$$2n = \sum_{i=1}^{l} |V(G_i)| + |V(M)| \geq l(n-2) + 2|M|. \qquad (1)$$

If $l \geq 3$, then $n \leq 2$, a contradiction. Thus $l = 2$, and then by (1), $|M| = 2$ and $|V(G_1)| = |V(G_2)| = n - 2$. Since n is even, G_1 and G_2 are even components and again by Lemma 4, G' has a perfect matching. Therefore, G is BM-extendable and the proof is completed. □

Remark. The above two results are best possible. Indeed, let $G = C_4 + (K_{n-2} \cup K_{n-2})$ when n is odd; $G = C_4 + (K_{n-1} \cup K_{n-3})$ when n is even. We can see that G is claw-free with $|V(G)| = 2n$ and $\delta(G) = 2\lceil \frac{n}{2} \rceil$. For each pair of nonadjacent vertices u and v in G, $d(u) + d(v) \geq 2n + 2$, and there are two nonadjacent vertices whose degree sum equals $2n + 2$. But G is not BM-extendable for the reason that $G - V(M)$ has no perfect matching, where M is a perfect matching of the C_4.

If we restrict a claw-free graph to be k-regular, the lower bound of minimum degree can also be improved (see [8]).

4 Fan-type Condition

We present another condition in terms of degree of two nonadjacent vertices for BM-extendable graphs. The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum k that G has a vertex set V' with $|V'| = k$ and $G - V'$ is disconnected.

Theorem 8. Let G be a graph on $2n$ vertices. If $\kappa(G) \geq 2\lfloor \frac{n}{2} \rfloor + 1$ and $\max\{d(u), d(v)\} \geq \lceil \frac{3n}{2} \rceil$ for any nonadjacent vertices u and v in G, then G is BM-extendable.

Proof. Let G be a graph satisfying the hypotheses of the theorem. It is obvious that the result holds for $n \leq 3$, and thus we may suppose $n \geq 4$ in the sequel. Let M be a bipartite matching of G, and $G' = G - V(M)$.

Claim. For any nonadjacent vertices u and v in G',

$$\max\{d_{G'}(u), d_{G'}(v)\} \geq \frac{|V(G')|}{2}.$$

In fact, if $|M| = 1$, then

$$\max\{d_{G'}(u), d_{G'}(v)\} \geq \max\{d_G(u), d_G(v)\} - 2 \geq \lceil\frac{3n}{2}\rceil - 2 \geq \frac{|V(G')|}{2}.$$

If $|M| \geq 2$, let x and y be two nonadjacent vertices in $V(M)$, we have

$$\lceil\frac{3n}{2}\rceil \leq \max\{d_G(x), d_G(y)\} \leq 2n - |M|,$$

and so $|M| \leq 2n - \lceil\frac{3n}{2}\rceil = \lfloor\frac{n}{2}\rfloor$. Thus, for any nonadjacent vertices u and v of G',

$$\max\{d_{G'}(u), d_{G'}(v)\} \geq \max\{d_G(u), d_G(v)\} - 2|M| \geq \lceil\frac{3n}{2}\rceil - 2|M|$$

$$\geq n - |M| = \frac{|V(G')|}{2}.$$

In the following, we will show that G' has a perfect matching by this claim.

If $1 \leq |M| \leq \lfloor\frac{n}{2}\rfloor - 1$, then $\kappa(G') \geq \kappa(G) - 2|M| \geq 3$, and so by Lemma 3 there is a perfect matching in G'. If $|M| = \lfloor\frac{n}{2}\rfloor$, then we have $|V(G')| = 2n - 2|M| = 2\lceil\frac{n}{2}\rceil$; for any nonadjacent vertices u and v of G',

$$\max\{d_{G'}(u), d_{G'}(v)\} \geq \frac{|V(G')|}{2} = n - |M| = \lceil\frac{n}{2}\rceil. \qquad (2)$$

And since $\kappa(G) \geq 2\lfloor\frac{n}{2}\rfloor + 1$, G' is connected. If G' is 2-connected, by Lemma 3 again, G' has a perfect matching. Otherwise, let u_0 be a cut vertex of G' and denote by $G_1, ..., G_l$ the components of $G' - u_0$. We shall prove $l = 2$. Suppose to the contrary that $l \geq 3$, and thus there are three vertices u_1, u_2 and u_3 with $u_i \in V(G_i), 1 \leq i \leq 3$, among which at least two vertices, say u_1 and u_2, have degree at least $\lceil\frac{n}{2}\rceil$ by (2). Then the number of vertices of the component G_i containing u_i ($i = 1, 2$), is at least $\lceil\frac{n}{2}\rceil$, and thus

$$|V(G')| > |V(G_1)| + |V(G_2)| + 1 \geq 2\lceil\frac{n}{2}\rceil + 1 = |V(G')| + 1,$$

a contradiction. Hence, G' has just two components G_1 and G_2. By (2) we may assume that

$$\lceil\frac{n}{2}\rceil \leq |V(G_1)| \leq 2\lceil\frac{n}{2}\rceil - 2 \text{ and } |V(G_2)| = |V(G')| - |V(G_1)| - 1 \leq \lceil\frac{n}{2}\rceil - 1.$$

Furthermore, G_2 must be a complete graph. Otherwise, let u and v be two nonadjacent vertices of G_2. Then

$$\max\{d_{G_2}(u), d_{G_2}(v)\} \geq \max\{d_{G'}(u), d_{G'}(v)\} - 1 \geq \lceil\frac{n}{2}\rceil - 1 \geq |V(G_2)|,$$

a contradiction. Thus for any vertex $v \in V(G_2)$, $d_{G'}(v) \leq \lceil\frac{n}{2}\rceil - 1$, and so by (2), for any vertex $u \in V(G_1)$, $d_{G'}(u) \geq \lceil\frac{n}{2}\rceil$. Moreover,

$$d_{G_1}(u) \geq d_{G'}(u) - 1 \geq \lceil\frac{n}{2}\rceil - 1 \geq \frac{|V(G_1)|}{2}.$$

This implies $\delta(G_1) \geq \frac{|V(G_1)|}{2}$, and then by Lemma 2 G_1 is hamiltonian. Since G' has even number of vertices, we see that only one of G_1 and G_2 is an odd component. So, the cut vertex u_0 can be matched to a vertex v of the odd component. By the fact that $G_1 - v$ has an even hamiltonian path if $v \in V(G_1)$ and $G_2 - v$ is a complete graph with even number of vertices if $v \in V(G_2)$, we assert that $G' - \{u_0, v\}$ has a perfect matching M'. So, $M' \cup \{u_0v\}$ is a perfect matching in G'. The proof is completed. □

Remark. In the above result, neither of the two lower bounds can be reduced. To see this, we first consider graph $G = K_{\lfloor\frac{n}{2}\rfloor+1,\lfloor\frac{n}{2}\rfloor+1} + (K_{2\lceil\frac{n}{2}\rceil-3} \cup K_1)$, where $n \geq 3$. We can see $\kappa(G) = 2\lfloor\frac{n}{2}\rfloor + 2$; for any two nonadjacent vertices u and v of G, $\max\{d(u), d(v)\} \geq \lceil\frac{3n}{2}\rceil - 1$; for any two nonadjacent vertices in $V(K_{\lfloor\frac{n}{2}\rfloor+1,\lfloor\frac{n}{2}\rfloor+1})$, the lower bound $\lceil\frac{3n}{2}\rceil - 1$ can be obtained. Let $S = V(K_{\lfloor\frac{n}{2}\rfloor+1,\lfloor\frac{n}{2}\rfloor+1})$. Then $o(G - S) = 2 > |S| - 2m_b(S))$, and so by Theorem 1 G is not BM-extendable. The following graph G show that in this condition the lower bound of $\kappa(G)$ can not be reduced. Let $G = K_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor} + (K_{2\lceil\frac{n}{2}\rceil-1} \cup K_1)$. Clearly, $\kappa(G) = 2\lfloor\frac{n}{2}\rfloor$ and $\max\{d(u), d(v)\} \geq \lceil\frac{3n}{2}\rceil$ for any two nonadjacent vertices u and v of G. By a similar discussion as before, G is not BM-extendable.

For further study on degree-type conditions, we may consider to combine with other parameters such as connectivity and independent number to ensure the BM-extendability of graphs. Noting that the degree-sum and Fan-type conditions are of independent sets of two vertices, we may consider the case of independent sets of m ($m \geq 3$) vertices. Moreover, we may study all these conditions on other graph classes besides claw-free graphs, such as series-parallel graphs.

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