# Radio Number for Square Paths

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October 2005 (Revised February 2006, March 2007)

**Abstract**

Let $G$ be a connected graph. For any two vertices $u$ and $v$, let $d(u, v)$ denote the distance between $u$ and $v$ in $G$. The maximum distance between any pair of vertices is called the diameter of $G$ and denoted by $\text{diam}(G)$. A *radio-labeling* (or multi-level distance labeling) with span $k$ for $G$ is a function $f$ that assigns to each vertex with a label from the set $\{0, 1, 2, \cdots, k\}$ such that the following holds for any vertices $u$ and $v$: $|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1$. The *radio number* of $G$ is the minimum span over all radio-labelings of $G$. The *square* of $G$ is a graph constructed from $G$ by adding edges between vertices of distance two apart in $G$. In this article, we completely determine the radio number for the square of any path.

## 1 Introduction

Radio-labeling (cf. [3, 4]) is motivated by the channel assignment problem introduced by Hale [9]. Suppose we are given a set of stations or transmitters, the task is to assign to each station (or transmitter) with a channel (non-negative integer) such that the interference is avoided. The interference is closely related to the geographical locations of the stations – the

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closer are the stations the stronger the interference that might occur. To avoid interference, the separation of the channels assigned to nearby stations must be large enough. To model this problem, we construct a graph so that each station is represented by a vertex, and two vertices are adjacent when their corresponding stations are close. The ultimate goal is to find a *valid* labeling such that the span (range) of the channels used is minimized.

Let $G$ be a connected graph. For any two vertices $u$ and $v$, the *distance* between $u$ and $v$, denoted by $d_G(u,v)$ (or $d(u,v)$ when $G$ is understood in the context), is the length of a shortest $(u,v)$-path in $G$. A *distance-two labeling* (or *$\lambda$-labeling*) with span $k$ is a function, $f : V(G) \to \{0,1,2,\cdots,k\}$, such that the following are satisfied:

$$|f(u) - f(v)| \geq \begin{cases} 2, & \text{if } d(u,v) = 1; \\ 1, & \text{if } d(u,v) = 2. \end{cases}$$

The *$\lambda$-number* of $G$ is the smallest $k$ such that $G$ admits a distance-two labeling with span $k$. Since introduced by Griggs and Yeh [8] in 1992, distance-two labeling has been studied extensively (cf. [1, 2, 5 - 8, 10, 11, 14, 15, 17, 18]).

Radio-labeling extends the number of interference level considered in distance-two labeling from two to the largest possible - the diameter of $G$. The *diameter* of $G$, denoted by $\text{diam}(G)$, is the maximum distance among all pairs of vertices in $G$. A *radio-labeling* (or *multi-level distance labeling* [16, 12]) with span $k$ for a graph $G$ is a function, $f : V(G) \to \{0,1,2,\cdots,k\}$, such that the following holds for any $u$ and $v$:

$$|f(u) - f(v)| \geq \text{diam}(G) - d(u,v) + 1.$$

The *radio number* of $G$, denoted by $rn(G)$, is the minimum span of a radio-labeling for $G$. Note that if $\text{diam}(G) = 2$, then radio-labeling and $\lambda$-labeling become identical.

Besides its motivation by the channel assignment, radio labeling itself is an interesting graph labeling problem and has been studied by several authors. The radio numbers for paths and cycles were investigated by Chartrand et al. [3], Chartrand, Erwin and Zhang [4], and Zhang [19], and were completely solved by Liu and Zhu [16]. The radio number for trees was investigated in [12].

The *square* of a graph $G$, denoted by $G^2$, is the graph constructed by adding to $G$ those edges connecting pairs of vertices that are distance two apart in $G$. We call the square of a path (or a cycle, respectively) a *square path* (or *square cycle*, respectively). After the complete solution for the radio numbers of paths and cycles was obtained as mentioned in the above, it becomes natural to consider the square paths and square cycles. The radio number for square cycles has been studied in [13], in which the exact values were determined for most of the square cycles, while bounds were given for others.

Comparing to the complicated results on the radio number for square cycles for which some cases still remain open [13], to our surprise, the radio number for square paths can be completely determined by a very simple result. The aim of this article is to prove such a result.

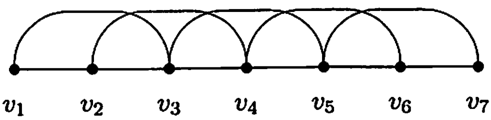


Figure 1: A square path on 7 vertices, denoted by $P_7^2$.

**Theorem 1** *Let $P_n^2$ be a square path on $n$ vertices and let $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$rn(P_n^2) = \begin{cases} k^2+2, & \text{if } n \equiv 1 \pmod 4 \text{ and } n \geq 9; \\ k^2+1, & \text{otherwise.} \end{cases}$$

# 2 Lower Bound

In this section, we establish the lower bound for Theorem 1. Throughout, we denote a path with $n$ vertices by $P_n$, where $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i = 1, 2, \cdots, n-1\}$. Hence, $V(P_n^2) = V(P_n)$ and $E(P_n^2) = E(P_n) \cup \{v_i v_{i+2} : i = 1, 2, 3, \cdots, n-2\}$. The diameter of $P_n^2$ is $\lfloor \frac{n}{2} \rfloor$. Figure 1 shows a square path on 7 vertices.

We denote the distance between two vertices $u$ and $v$ in $P_n^2$ by $d(u, v)$. Observe,

**Proposition 2** *For any $u, v \in V(P_n^2)$, we have*

$$d(u, v) = \left\lceil \frac{d_{P_n}(u, v)}{2} \right\rceil.$$

A *center* of $P_n$ is defined as a "middle" vertex of $P_n$. An odd path $P_{2k+1}$ has only one center $v_{k+1}$, while an even path $P_{2k}$ has two centers $v_k$ and $v_{k+1}$. For each vertex $u \in V(P_n)$, the *level* of $u$, denoted by $L(u)$, is the smallest distance in $P_n$ from $u$ to a center of $P_n$. For instance, if $n = 2k+1$, then $L(v_1) = k$ and $L(v_{k+1}) = 0$. Denote the levels of a sequence of vertices $A$ by $L(A)$. If $n = 2k+1$, then

$$L(v_1, v_2, \cdots, v_{2k+1}) = (k, k-1, \cdots, 3, 2, 1, 0, 1, 2, 3, \cdots, k-1, k)$$

If $n = 2k$, then

$$L(v_1, v_2, \cdots, v_{2k}) = (k-1, k-2, \cdots, 2, 1, 0, 0, 1, 2, \cdots, k-2, k-1).$$

Define the *left- and right-vertices* by: If $n = 2k+1$, then the left- and right-vertices, respectively, are

$$\{v_1, v_2, \cdots, v_k, v_{k+1}\} \text{ and } \{v_{k+1}, v_{k+2}, v_{k+3}, \cdots, v_{2k+1}\}.$$

(Note here the center $v_{k+1}$ is both a left- and right-vertex). If $n = 2k$, then the left- and right-vertices, respectively, are

$$\{v_1, v_2, \cdots, v_k\} \text{ and } \{v_{k+1}, v_{k+2}, v_{k+3}, \cdots, v_{2k}\}.$$

If two vertices are both right (or left)-vertices, then we say they are on the same side; otherwise, they are on the opposite sides. Observe

**Lemma 3** *If $n = 2k+1$, then for any $u, v \in V(P_n^2)$, we have:*

$$d(u,v) = \begin{cases} \lceil \frac{L(u)+L(v)}{2} \rceil, & \text{if } u \text{ and } v \text{ are on the opposite sides;} \\ \lceil \frac{|L(u)-L(v)|}{2} \rceil, & \text{otherwise.} \end{cases}$$

*If $n = 2k$, then for any $u, v \in V(P_n^2)$, we have:*

$$d(u,v) = \begin{cases} \lceil \frac{L(u)+L(v)+1}{2} \rceil, & \text{if } u \text{ and } v \text{ are on the opposite sides;} \\ \lceil \frac{|L(u)-L(v)|}{2} \rceil, & \text{otherwise.} \end{cases}$$

**Lemma 4** *Let $P_n^2$ be a square path on $n$ vertices and let $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$rn(P_n^2) \geq \begin{cases} k^2+2, & \text{if } n \equiv 1 \pmod 4 \text{ and } n \geq 9; \\ k^2+1, & \text{otherwise.} \end{cases}$$

**Proof.** Let $f$ be a radio-labeling for $P_n^2$. Re-arrange $V(P_n^2) = \{x_1, x_2, \cdots, x_n\}$ with $0 = f(x_1) < f(x_2) < f(x_3) < \ldots < f(x_n)$. We first claim that $rn(P_n^2) \geq k^2 + 1$, by showing $f(x_n) \geq k^2 + 1$.

By definition, $f(x_{i+1}) - f(x_i) \geq k + 1 - d(x_{i+1}, x_i)$ for $1 \leq i \leq n-1$. Summing up these $n-1$ in-equalities, we have

$$f(x_n) \geq (n-1)(k+1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}).$$

Thus, to minimize $f(x_n)$ it bounds to maximize the sum $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$.

Assume $n = 2k$. By Lemma 3, we have

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \left\lceil \frac{L(x_{i+1}) + L(x_i) + 1}{2} \right\rceil.$$

Observe from the above in-equality we have:

1) For each $i$, the equality for $d(x_i, x_{i+1}) \leq \lceil \frac{L(x_{i+1})+L(x_i)+1}{2} \rceil$ holds only when $x_i$ and $x_{i+1}$ are on the opposite sides, unless one of them is a center; and

2) in the last summation term on the right-hand-side each vertex of $P_n^2$ occurs exactly twice, except $x_1$ and $x_n$, for which each occurs only once.

Note, we have

$$\lceil (L(x_{i+1}) + L(x_i) + 1)/2 \rceil \le (L(x_{i+1}) + L(x_i) + 2)/2,$$

and the equality holds only if $L(x_{i+1})$ and $L(x_i)$ have the same parity. Combining this with 1), there exist at most $n-2$ of the $i$'s such that $d(x_i, x_{i+1}) = (L(x_{i+1}) + L(x_i) + 2)/2$. Moreover, among all the vertices only the two centers are of level 0, we conclude that

$$\begin{aligned}
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) &\le \sum_{i=1}^{n-1} \tfrac{L(x_{i+1})+L(x_i)+2}{2} - \tfrac{1}{2} \\
&= [\sum_{i=1}^{n} L(x_i)] - \tfrac{L(x_1)+L(x_n)}{2} + n - 1 - \tfrac{1}{2} \\
&\le 2(1 + 2 + \cdots + (k-1)) + n - \tfrac{3}{2} \ (\ L(x_1) = L(x_n) = 0) \\
&= (k-1)k + 2k - \tfrac{3}{2} \\
&= k^2 + k - \tfrac{3}{2}.
\end{aligned}$$

Hence,

$$rn(P_n^2) \ge (2k-1)(k+1) - (k^2 + k - 2) = k^2 + 1.$$

Assume $n = 2k+1$. By Lemma 3, we have

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \le \sum_{i=1}^{n-1} \left\lceil \frac{L(x_{i+1}) + L(x_i)}{2} \right\rceil.$$

Similar to the case $n = 2k$, we observe from the above in-equality and get:

1) For each $i$, the equality for $d(x_i, x_{i+1}) \le \lceil \frac{L(x_{i+1})+L(x_i)}{2} \rceil$ holds only when $x_i$ and $x_{i+1}$ are on the opposite sides, unless one of them is the center; and

2) in the last summation term on the right-hand-side each vertex of $P_n^2$ occurs exactly twice, except $x_1$ and $x_n$, for which each occurs only once.

Note, we have

$$\lceil (L(x_{i+1}) + L(x_i))/2 \rceil \le (L(x_{i+1}) + L(x_i) + 1)/2,$$

and the equality holds only if $L(x_{i+1})$ and $L(x_i)$ have different parities. Combining this with 1), there are two possible cases to consider.

**Case 1.** There exist at most $n-2$ of the $i$'s such that $d(x_i, x_{i+1}) = (L(x_{i+1}) + L(x_i) + 1)/2$. Then as $L(x_1) + L(x_n) \geq 1$, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} d(x_i, x_{i+1}) &\leq \sum_{i=1}^{n-1} \frac{L(x_{i+1})+L(x_i)+1}{2} - \frac{1}{2} \\
&= \left(\sum_{i=1}^{n} L(x_i)\right) - \frac{L(x_1)+L(x_n)}{2} + \frac{n-1}{2} - \frac{1}{2} \\
&\leq 2(1+2+\cdots+k) + \frac{n-1}{2} - 1 \\
&= (k+1)k + k - 1 \\
&= k^2 + 2k - 1.
\end{aligned}$$

Hence,

$$rn(P_n^2) \geq 2k(k+1) - (k^2+2k-1) = k^2+1.$$

**Case 2.** The equality $d(x_i, x_{i+1}) = (L(x_{i+1}) + L(x_i) + 1)/2$ holds for all $i = 1, 2, \cdots, n-1$. Then neither $x_1$ nor $x_n$ is the center, and $L(x_1) \equiv L(x_n) \equiv k \pmod 2$. These imply that $L(x_1) + L(x_n)$ is at least 2 if $k$ is odd, and at least 4 if $k$ is even. A similar calculation to Case 1, starting with $\sum_{i=1}^{n-1} d(x_i, x_{i+1}) = \sum_{i=1}^{n-1} \frac{L(x_{i+1})+L(x_i)+1}{2}$, will lead to $rn(P_n^2) \geq k^2+1$ if $k$ is odd; and $rn(P_n^2) \geq k^2+2$ if $k$ is even.

Now, assume $n \equiv 1 \pmod 4$ and $n \geq 9$, that is, $k$ is even, $k \geq 4$. Assume to the contrary that $f(x_n) = k^2+1$. Then only Case 1 is possible. So all the following must hold:

1) $\{x_1, x_n\} = \{v_{k+1}, v_k\}$ or $\{x_1, x_n\} = \{v_{k+1}, v_{k+2}\}$.

2) $f(x_{i+1}) = f(x_i) + k + 1 - d(x_{i+1}, x_i)$ for all $i$.

3) For any $i \geq 1$, the two vertices $x_i$ and $x_{i+1}$ belong to opposite sides unless one of them is the center.

4) There exists some $1 \leq t \leq n-1$ such that $L(x_t) \equiv L(x_{t+1}) \pmod 2$, while $L(x_i) \not\equiv L(x_{i+1}) \pmod 2$ for all other $i \neq t$.

By 1) and by symmetry, we may assume $x_1 = v_{k+1}$. Note, $v_{k+1}$ is the only center of $P_{2k+1}^2$. Since $n = 2k+1$ for some even $k$, $k \geq 4$, there are $k/2$ vertices of even levels and $k/2$ vertices of odd levels on each side, excluding the center. Since $x_n$ is either $v_{k+2}$ or $v_k$ and both of them are level 1, by 2) and 3), the only $t$ in 4) must be $k+1$. For otherwise, there will be at least two $t$'s with $L(x_t) \equiv L(x_{t+1}) \pmod 2$. Hence, we have

5) $L(x_1)$, $L(x_3)$, $L(x_5)$, $\cdots$, $L(x_{k+1})$, $L(x_{k+2})$, $L(x_{k+4})$, $\cdots$, $L(x_{2k})$ are all even, while the levels of other vertices are all odd.

**Claim.** $\{v_1, v_{2k+1}\} = \{x_{k+1}, x_{k+2}\}$.
**Proof)** Suppose $v_1 \notin \{x_{k+1}, x_{k+2}\}$. As $L(v_1) = k$, by 5), we have $v_1 = x_a$ for some $a$ where $x_{a-1}$ and $x_{a+1}$ are vertices on the right side and both $L(x_{a-1})$ and $L(x_{a+1})$ are odd. Let $L(x_{a-1}) = y$ and $L(x_{a+1}) = z$. By 2),

$$f(x_a) - f(x_{a-1}) = k + 1 - (k + y + 1)/2$$

and

$$f(x_{a+1}) - f(x_a) = k + 1 - (k + z + 1)/2.$$

This implies

$$f(x_{a+1}) - f(x_{a-1}) = k + 1 - (y + z)/2,$$

contradicting that $f(x_{a+1}) - f(x_{a-1}) \geq k + 1 - |y - z|/2$ (as $z, y$ are odd so $z, y \neq 0$). Therefore, $v_1 \in \{x_{k+1}, x_{k+2}\}$. Similarly, one can show that $v_{2k+1} \in \{x_{k+1}, x_{k+2}\}$. ∎

By the Claim, we may assume that $v_1 = x_{k+1}$ and $v_{2k+1} = x_{k+2}$ (the proof for the other case is symmetric). By 5), $L(x_{k+3}) = b$ for some odd $b$. By 3) and 2), we have $f(v_{2k+1}) - f(v_1) = 1$ and

$$f(x_{k+3}) - f(v_{2k+1}) = k + 1 - (b + k + 1)/2.$$

So,

$$f(x_{k+3}) - f(v_1) = (k - b + 3)/2.$$

By definition and Lemma 3,

$$f(x_{k+3}) - f(v_1) \geq k + 1 - (k - b + 1)/2.$$

Hence, we have $b = 1 = L(x_{k+3})$. This implies that $x_{k+3} = v_k$. Similarly, we can get $x_k = v_{k+2}$. By 1), it must be $k = 2$, contradicting the assumption $k \geq 4$. Therefore, $rn(P^2_{2k+1}) \geq k^2 + 2$. ∎

# 3 Upper Bound and Optimal Radio-Labelings

By Lemma 4, to establish Theorem 1, it suffices to give radio-labelings achieving the desired spans. To this end, we will use the following lemma.

**Lemma 5** *Let $P^2_n$ be a square path on $n$ vertices with $k = \lfloor n/2 \rfloor$. Let $\{x_1, x_2, \cdots, x_n\}$ be a permutation of $V(P^2_n)$ such that for any $1 \leq i \leq n-2$,*

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq k + 1,$$

*and if $k$ is even and the equality in the above holds, then $d_{P_n}(x_i, x_{i+1})$ and $d_{P_n}(x_{i+1}, x_{i+2})$ have different parities. Let $f$ be a function, $f : V(P^2_n) \to \{0, 1, 2, \cdots\}$ with $f(x_1) = 0$, and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n - 1$. Then $f$ is a radio-labeling for $P^2_n$.*

**Proof.** Recall, $\text{diam}(P_n^2) = k$. Let $f$ be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k + 1 - d(x_i, x_j)$ for any $j \geq i+2$. For $i = 1, 2, \cdots, n-1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

For any $j \geq i+2$, it follows that

$$f(x_j) - f(x_i) = f_i + f_{i+1} + \cdots + f_{j-1}.$$

Suppose $j = i+2$. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$. (The proof for $d(x_{i+1}, x_{i+2}) \geq d(x_i, x_{i+1})$ is similar.) Then, $d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$. Let $x_i = v_a$, $x_{i+1} = v_b$, and $x_{i+2} = v_c$. It suffices to consider the following cases:

- $\underline{b < a < c \text{ or } c < a < b}$. Since $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, we obtain $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$ and $d(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k+1-d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &\geq 2k+2-2\left(\frac{k+2}{2}\right) \\ &= k+1-d(x_i, x_{i+2}). \end{aligned}$$

- $\underline{a < b < c \text{ or } c < b < a}$. This implies

$$d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) - 1.$$

  Similar to the above, easy calculation shows that $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i, x_{i+2})$.

- $\underline{a < c < b \text{ or } b < c < a}$.
  Assume $k$ is odd or $\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq k$, then we have $d(x_{i+1}, x_{i+2}) \leq (k+1)/2$ and

$$d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}).$$

  Hence, $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i, x_{i+2})$.
  If $k$ is even and $\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} = k+1$, then by our assumption, it must be that $d_{P_n}(x_{i+1}, x_{i+2}) = k+1$ and $d_{P_n}(x_i, x_{i+1})$ is even. Hence, we have

$$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) + 1.$$

  This implies

$$\begin{aligned} f(x_{i+2}) - f(x_i) &\geq 2k+2-2d(x_{i+1}, x_{i+2}) - d(x_i, x_{i+2}) + 1 \\ &= 2k+2-2\left(\tfrac{k+2}{2}\right) - d(x_i, x_{i+2}) + 1 \\ &= k+1-d(x_i, x_{i+2}). \end{aligned}$$

Let $j = i+3$. First, we assume that the sum of some pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, $d(x_{i+2}, x_{i+3})$ is at most $k+2$. Then

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k+3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \\ &\geq k+1 > k+1-d(x_i, x_{i+3}). \end{aligned}$$

Next, we assume that the sum of every pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$ and $d(x_{i+2}, x_{i+3})$ is greater than $k+2$. Then, by our hypotheses it follows that

$$d(x_i, x_{i+1}), d(x_{i+2}, x_{i+3}) > \frac{k+2}{2} \text{ and } d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}. \quad (*)$$

Let $x_i = v_a$, $x_{i+1} = v_b$, $x_{i+2} = v_c$, $x_{i+3} = v_d$. Since $\text{diam}(P_n^2) = k$, by (*) and our assumption that the sum of any pair of the distances, $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, $d(x_{i+2}, x_{i+3})$, is great than $k+2$, it must be that $a < c < b < d$ (or $d < b < c < a$). Then

$$d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1.$$

By (*), we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k+3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \\ &\geq 3k+2 - 2d(x_{i+1}, x_{i+2}) - d(x_i, x_{i+3}) \\ &\geq k+1 - d(x_i, x_{i+3}). \end{aligned}$$

Let $j \geq i+4$. Since $\min\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})\} \leq \frac{k+2}{2}$, and $f_i \geq k+1-d(x_i, x_{i+1})$ for any $i$, we have $\max\{f_i,\ f_{i+1}\} \geq \frac{k}{2}$ for any $1 \leq i \leq n-2$. Hence,

$$f(x_j) - f(x_i) \geq f_i + f_{i+1} + f_{i+2} + f_{i+3} > k+1 > k+1 - d(x_i, x_j).$$

∎

To show the existence of a radio-labeling achieving the desired bound, we consider cases separately. For each radio-labeling $f$ given in the following, we shall first define a permutation (line-up) of the vertices $V(P_n^2) = \{x_1, x_2, \cdots, x_n\}$, then define $f$ by $f(x_1) = 0$ and for $i = 1, 2\cdots, n-1$:

$$f(x_{i+1}) = f(x_i) + k + 1 - d(x_i, x_{i+1}).$$

**Case 1: $rn(P_{2k+1}^2) \leq k^2+1$, if $k$ is odd.** We give a radio-labeling with span $k^2+1$. The line-up (permutation) of $V(P_{2k+1}^2) = \{x_1, x_2, x_3, \cdots, x_{2k+1}\}$ is given by the arrows in Table 1. That is, $x_1 = v_k$, $x_2 = v_{2k}$, $\cdots$, $x_{2k+1} = v_{k+1}$. The value above each arrow shows the distance between the two consecutive vertices in $P_n$.

By Lemma 5, $f$ is a radio-labeling for $P_n^2$. As $k$ is odd, observe from Table 1, there are two possible distances in $P_{2k+1}^2$ between consecutive

$$v_k \xrightarrow{k} v_{2k} \xrightarrow{k+2} v_{k-2} \xrightarrow{k} v_{2k-2} \xrightarrow{k+2} \cdots \xrightarrow{k} v_{k+3} \xrightarrow{k+2} v_1$$

$$\xrightarrow{k+1} v_{k+2} \xrightarrow{k} v_2 \xrightarrow{k+2} v_{k+4} \xrightarrow{k} v_4 \xrightarrow{k+2} v_{k+6} \xrightarrow{k} \cdots \xrightarrow{k+2} v_{2k+1} \xrightarrow{k} v_{k+1}.$$

Table 1: Vertex ordering of a labeling for $P_{2k+1}^2$ for odd $k$.

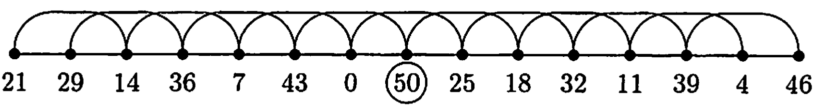


Figure 2: An optimal radio labeling for $P_{15}^2$, with span 50.

vertices, namely, $\frac{k+1}{2}$ and $\frac{k+3}{2}$, with the number of occurences $k+1$ and $k-1$, respectively. It follows by direct calculation that

$$f(x_{2k+1}) = 2k(k+1) - \sum_{i=1}^{2k} d(x_i, x_{i+1}) = k^2 + 1.$$

As an example, Figure 2 shows an optimal radio-labeling (with minimum span) for $P_{15}^2$.

**Case 2: $rn(P_{2k}^2) \leq k^2 + 1$, if $k$ is odd.** Let $G = P_{2k+1}^2$ for some odd $k$. Let $H$ be the subgraph of $G$ induced by the vertices $\{v_1, v_2, \cdots, v_{2k}\}$. Then $H \cong P_{2k}^2$, $\text{diam}(H) = \text{diam}(G) = k$, and $d_G(u,v) = d_H(u,v)$ for every $u, v \in V(H)$. Let $f$ be a radio-labeling for $G$. Then $f$ restricted to $H$ is a radio-labeling for $H$. By Case 1, $rn(P_{2k}^2) \leq rn(P_{2k+1}^2) \leq k^2 + 1$.

**Case 3: $rn(P_{2k}^2) \leq k^2 + 1$, if $k$ is even.** Similar to Case 1, we line-up the vertices according to Table 2.

By Lemma 5, $f$ is a radio-labeling for $P_n^2$. Indeed, observe from Table 2, as $k$ is even, there are three possible distances between any consecutive vertices in $P_{2k}^2$, namely, $\frac{k}{2}$, $\frac{k+2}{2}$ and $k-2$, with the number of occurences $k-2$, $k$ and 1, respectively. By some calculation, the span of $f$ is $k^2+1$. As an example, Figure 3 gives an optimal radio-labeling for $P_{12}^2$.

We now consider the case $n = 2k+1$ for even $k$. If $k = 2$, the labeling $(3, 5, 0, 2, 4)$ is a radio-labeling for $P_5^2$. Hence, we assume $k \geq 4$.

$$v_k \xrightarrow{k-1} v_{2k-1} \xrightarrow{k+1} v_{k-2} \xrightarrow{k+2} v_{2k} \xrightarrow{k+1} v_{k-1} \xrightarrow{k-1} v_{2k-2} \xrightarrow{k+1} v_{k-3} \cdots \xrightarrow{k+1} v_1$$

$$\xrightarrow{2k-4} v_{2k-3} \xrightarrow{k+1} v_{k-4} \xrightarrow{k-1} v_{2k-5} \xrightarrow{k+1} v_{k-6} \xrightarrow{k-1} \cdots \xrightarrow{k+1} v_2 \xrightarrow{k-1} v_{k+1}.$$

Table 2: Vertex ordering of a labeling for $P_{2k}^2$, for some even $k$.

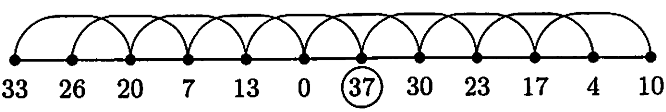


Figure 3: An optimal radio labeling for $P_{12}^2$, with span 37.

**Case 4: If $k$ is even and $k \geq 4$, then $rn(P_{2k+1}^2) \leq k^2 + 2$.** Similarly, we arrange the vertices according to Table 3.

$$v_k \xrightarrow{k+1} v_{2k+1} \xrightarrow{2k} v_1 \xrightarrow{k+1} v_{k+2} \xrightarrow{k-1} v_3 \xrightarrow{k+1} v_{k+4} \cdots \xrightarrow{k-1} v_{k-1} \xrightarrow{k+1} v_{2k}$$

$$\xrightarrow{2k-2} v_2 \xrightarrow{k+1} v_{k+3} \xrightarrow{k-1} v_4 \xrightarrow{k+1} v_{k+5} \xrightarrow{k-1} \cdots \xrightarrow{k-1} v_{k-2} \xrightarrow{k+1} v_{2k-1} \xrightarrow{k-2} v_{k+1}$$

Table 3: Vertex ordering of a labeling for $P_{2k+1}^2$, for even $k$.

By Lemma 5, $f$ is a radio-labeling for $P_{2k+1}^2$. Observe from Table 3, there are five possible distances between any consecutive vertices $x_i$ and $x_{i+1}$ in $P_n^2$, namely, $k$, $k-1$, $\frac{k-2}{2}$, $\frac{k+2}{2}$ and $\frac{k}{2}$, with the number of occurences 1, 1, 1, $k$ and $k-3$, respectively. By some calculation, one can show that the span of $f$ is $k^2+2$. As an example, Figure 4 gives an optimal radio-labeling for $P_{13}^2$. This completes the proof of Theorem 1.

**Acknowledgment.** The authors would like to thank an anonymous referee for immediate report and careful reading of the manuscript. Special thanks go to Maria Nogin for valuable comments.

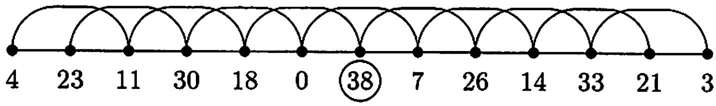


Figure 4: An optimal radio labeling for $P_{13}^2$, with span 38.

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