The Laplacian spectral radius of unicyclic graphs with k pendent vertices*

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Abstract

Let $\mathcal{U}_n(k)$ denote the set of all unicyclic graphs on n vertices with k ($k \geqslant 1$) pendant vertices. Let $\Diamond_4^k$ be the graph on n vertices obtained from C_4 by attaching k paths of almost equal lengths at the same vertex. In this paper, we prove that $\Diamond_4^k$ is the unique graph with the largest Laplacian spectral radius among all the graphs in $\mathcal{U}_n(k)$, when $n \geqslant k+4$.

Key words: Laplacian spectral radius; Unicyclic graph

AMS subject classification: 05C50; 15A18

1 Introduction

The graphs in this paper are simple and undirected. Let $G = (V, E)$ be a graph on n vertices. The *Laplacian matrix* is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacent matrix of G. The Laplacian characteristic polynomial of G is just det($xI - L(G)$), which is denoted by $\Phi(G, x)$, or simply $\Phi(G)$. From the fact that $L(G)$ is a real symmetric matrix and Geršgorin's theorem [4], it follows that its eigenvalues are nonnegative real numbers, and 0 is the smallest eigenvalue of $L(G)$. Hence its eigenvalues can be denoted by

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0,$$

*This work is supported by NSFC.

in a non-increasing order. The largest eigenvalue $\mu_1(G)$ is the *Laplacian spectral radius* of graph G. Similarly, the *spectral radius* of graph G is the largest eigenvalue of $A(G)$, which is denoted by $\rho(G)$.

Up to now, there are many results about Laplacian spectral radius of graphs. Some are about bounds (see [7, 12, 13, 16]), and others are about extremal graphs (see [2, 9, 19, 20, 21]). In this paper, we restrict our consideration to the Laplacian spectral radius of unicyclic graphs on n vertices with k ($k \geqslant 1$) pendant vertices, and prove that $\lozenge_4^k$ is the unique graph with the largest Laplacian spectral radius among all these graphs when $n \geqslant k+4$.

2 Prelimaries

We first give some lemmas that will be used in the main results.

Let G be a graph and let $G' = G + e$ be the graph obtained from G by inserting a new edge e into G. It follows by the well-known Courant-Weyl inequalities (see, e.g., [1], Theorem 2.1) that the following is true.

Lemma 2.1. $\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0$.

Let $Gu : vH$ denote the graph formed by identifying the vertex u of G with the vertex v of H (see Fig. 1). If u is a vertex of G, let $L_u(G)$ denote the principal submatrix of $L(G)$ formed by deleting the row and the column corresponding to vertex u. In the following, we always use $\Phi(L_u(G))$ to denote the characteristic polynomial of $L_u(G)$. The *line graph* L^G of a graph G is constructed by taking the edges of G as vertices of L^G, and joining two vertices in L^G whenever the corresponding edges in G have a common vertex. The set of neighbors of a vertex v_i in G is denoted by $N_G(v_i)$, or briefly by $N(v_i)$.

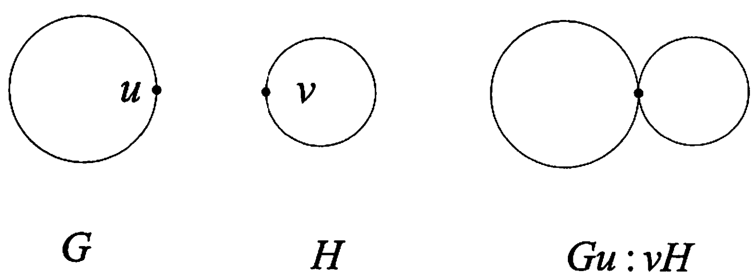

Fig. 1. Graph $Gu : vH$.

Lemma 2.2. *[19] Let $G = G_1u : vG_2$. Then*

$$\Phi(G) = \Phi(G_1)\Phi(L_v(G_2)) + \Phi(L_u(G_1))\Phi(G_2) - x\Phi(L_u(G_1))\Phi(L_v(G_2)).$$

If $h(x)$ is a polynomial in the variable x, let $\lambda(h)$ denote the largest real root of equation $h(x) = 0$.

Lemma 2.3. *[20] Let $h(x)$ and $g(x)$ be monic polynomials with real roots. If $h(x) < g(x)$ for all $x \geq \lambda(g)$, then $\lambda(h) > \lambda(g)$.*

In the following lemma, we assume that X, Y are two unit eigenvectors of H_1, H_2 corresponding to $\mu_1(H_1)$, $\mu_1(H_2)$, respectively.

Lemma 2.4. *[19] Let G_1 and G_2 be shown as in Fig. 2, $G_1 = H_1u^* : uGv : v^*H_2$ and $G_2 = H_1u^* : vGv : v^*H_2$. If $\Phi(L_u(G)) \leqslant \Phi(L_v(G))$ for all $x \geqslant \mu_1(G_1)$. Then $\mu_1(G_1) \leqslant \mu_1(G_2)$. In particular, inequality is strict if H_1 and H_2 are both bipartite graphs.*

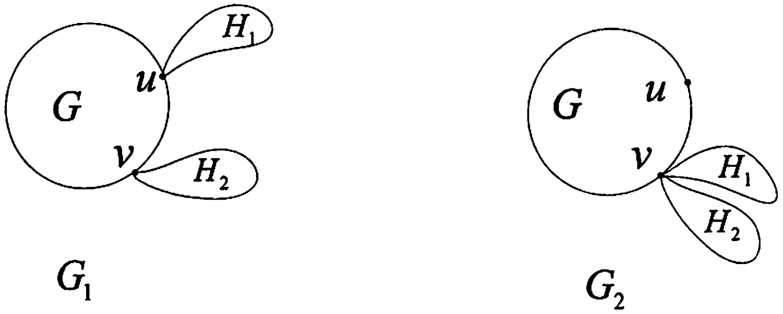

Fig. 2. Graphs G_1 and G_2.

Lemma 2.5. *[19] Let G be a connected graph of order n and $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T$ a unit eigenvector of G correspoding to $\mu_1(G)$. If there exist three different vertices v_i, v_{i+1}, v_j such that $x_{v_i} \leqslant x_{v_{i+1}} \leqslant x_{v_j}$ and v_i is adjacent to v_{i+1}, v_j is nonadjacent to v_i. Then $\mu_1(G) \leqslant \mu_1(G - v_iv_{i+1} + v_iv_j)$. Especially, inequality is strict if $x_{v_{i+1}} \neq x_{v_j}$.*

Lemma 2.6. *[8] Let u, v be two vertices of a connected bipartite graph $G = (V_1, V_2, E)$. Suppose that $v_1, v_2, \ldots, v_s$ $(1 \leqslant s \leqslant d(v))$ are some vertices of $N_G(v) \setminus N_G(u)$ different from u. Let X be a unit eigenvector of G correspoding to $\mu_1(G)$, and let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leqslant i \leqslant s)$. If $|x_u| \geqslant |x_v|$ and G^* is also a bipartite graph, then $\mu_1(G^*) > \mu_1(G)$.*

Let G be a connected graph, and $uv \in E(G)$. The graph $G_{u,v}$ is obtained from G by subdividing the edge uv, i. e., adding a new vertex w and edges wu, wv in $G - uv$. Hoffman and Smith define an internal path of G as a walk $v_0 v_1 \cdots v_s$ ($s \geqslant 1$) such that the vertices $v_0, v_1, \ldots, v_s$ are distinct, $d(v_0) > 2$, $d(v_s) > 2$, and $d(v_i) = 2$, whenever $0 < i < s$. And s is called the length of the internal path. An internal path is closed if $v_0 = v_s$. They proved the following result.

Lemma 2.7. *[3] Let uv be an edge of the connected graph G on n vertices.*

(i) If uv does not belong to an internal path of G, and $G \neq C_n$, then $\rho(G_{u,v}) > \rho(G)$.

(ii) If uv belongs to an internal path of G, and $G \neq W_n$, where W_n is shown in Fig. 3 , then $\rho(G_{u,v}) < \rho(G)$.

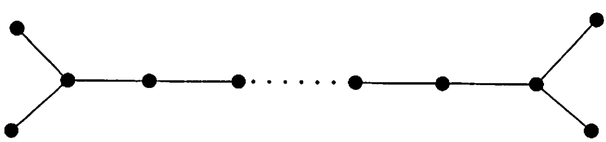

Fig. 3. W_n.

Lemma 2.8. *[19] Let $e = uv$ be an arbitrary edge of a bipartite graph $G = (V, E)$ ($|V| = n$), $G^{uv}(w)$ be the graph obtained from G by contracting the edge e into a new vertex w. Then*

(1) If uv does not belong to an interval path, then $\mu_1(G^{uv}(w)) < \mu_1(G)$.

(2) If uv belongs to an interval path and $G^{uv}(w)$ is still a bipartite graph, then $\mu_1(G^{uv}(w)) > \mu_1(G)$.

Lemma 2.9. *[17] $\mu_1(G) \leqslant 2 + \rho(L^G)$, the equality holds if and only if G is a bipartite graph.*

Lemma 2.10. *[5] Let G be a connected bipartite graph and H a subgraph of G. Then $\mu_1(H) \leqslant \mu_1(G)$, and equality holds if and only if $G = H$.*

Lemma 2.11. *[8] Let G be a connected graph on n vertices and v be a vertex of G. Let $G_{k,l}$ be the graph defined as in Fig. 4. If $l \geqslant k \geqslant 1$, then*

$$\mu_1(G_{k-1,l+1}) \leqslant \mu_1(G_{k,l}),$$

with equality if and only if there exists a unit eigenvector of $G_{k,l}$ corresponding to $\mu_1(G_{k,l})$ taking the value 0 on vertex v. Especially, inequality is strict if G is a bipartite graph.

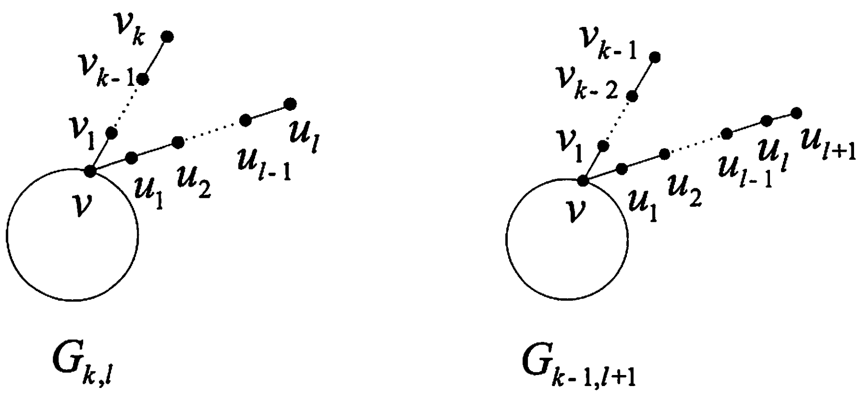

Fig. 4. Graphs $G_{k,l}$ and $G_{k-1,l+1}$.

Lemma 2.12. *[6, 15] Let G be a connected graph on n vertices with at least one edge, then $\mu_1(G) \geqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the graph G, with equality if and only if $\Delta(G) = n - 1$.*

Lemma 2.13. *[18] Let G be a graph on vertices labelled $1, 2, \ldots, n$, and suppose that vertices 1 and 2 of G are not adjacent. Form $\hat{G}$ from G by adding the edge between vertices 1 and 2. Then the Laplacian spectral integral variation occurs in one place if and only if in G, vertices 1 and 2 have the same set of neighbours. In the case that Laplacian spectral integral variation occurs in one place, the eigenvalue of G that increased by 2 is given by the degree of vertex 1 (equivalently, the degree of vertex 2).*

3 Main results

A *unicyclic* graph is a connected graph in which the number of edges equals the number of vertices. We may use the following notation to represent an unicyclic graph: $G = U(C_l; T_1, T_2, \ldots, T_l)$; where C_l is the unique cycle in G with $V(C_l) = \{v_1, v_2, \ldots, v_l\}$ such that v_i is adjacent to v_{i+1} (mod l) for $1 \leqslant i \leqslant l$. For each i, let T_i be the component of $G - \{V(C_l) - v_i\}$ containing v_i (see Fig. 5). If $|V(T_i)| = 1$, we say T_i is a trivial tree. Let $\mathcal{U}_n(k)$ denote the set of all unicyclic graphs with n vertices and k ($k \geqslant 2$) pendant vertices. Let $\Diamond_4^k$ be the graph on n vertices obtained from C_4 by attaching k paths of almost equal lengths at the same vertex. If U is any vertex set of G, we usually use $G - U$ to denote the graph obtained from G by deleting all the vertices in U and their incident edges.

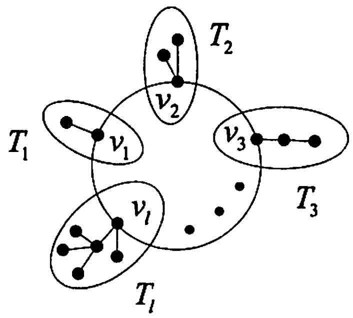

Fig. 5. Graph $U(C_l; T_1, T_2, \ldots, T_l)$.

Let $\mathcal{U}_n(k)$ denote the set of all unicyclic graphs on n ($n \geqslant k+4$) vertices with k ($k \geqslant 1$) pendant vertices. Let

$$\mathcal{U}_o^k = \{G = U(C_l; T_1, \ldots, T_l) \in \mathcal{U}_n(k) \mid l \text{ is odd}\};$$
$$\mathcal{U}_e^k = \{G = U(C_l; T_1, \ldots, T_l) \in \mathcal{U}_n(k) \mid l \text{ is even}\}.$$

Lemma 3.1. *For any graph $G \in \mathcal{U}_e^k$, we have*

$$\mu_1(G) \leqslant \mu_1(\Diamond_4^k),$$

and equality holds if and only if $G \cong \Diamond_4^k$.

Proof: Let $G = U(C_l; T_1, T_2, \ldots, T_l) \in \mathcal{U}_e^k$ and $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector of G corresponding to $\mu_1(G)$, where x_i corresponds to the vertex v_i ($1 \leqslant i \leqslant n$).

Choose $G \in \mathcal{U}_e^k$ such that the Laplacian spectral radius of G is as large as possible. We first show some facts.

Fact 1. G has a unique nontrivial attached tree.

Proof. Suppose not, we may assume that $|V(T_i)| \neq 1$ and $|V(T_j)| \neq 1$, where $i \neq j$. Denote $N(v_i) \setminus V(C_l) = \{u_1, \ldots, u_s\}$, $N(v_j) \setminus V(C_l) = \{w_1, \ldots, w_t\}$.

If $|x_i| \geqslant |x_j|$, let $G^* = G - v_jw_1 - \cdots - v_jw_t + v_iw_1 + \cdots + v_iw_t$.

If $|x_i| \leqslant |x_j|$, let $G^* = G - v_iu_1 - \cdots - v_iu_s + v_ju_1 + \cdots + v_ju_s$.

Then, in either case, $G^* \in \mathcal{U}_e^k$. By Lemma 2.6, we have $\mu_1(G) < \mu_1(G^*)$, which is a contradiction.

Suppose that v_1 is the root of the nontrivial attached tree.

Fact 2. For any vertex $v \in V(T_1) \setminus \{v_1\}$, $d_G(v) \leqslant 2$.

Proof. On the contrary, there exists a vertex v_r of $T_1 - v_1$ such that $d_G(v_r) > 2$. Since T_1 is a tree, there is a unique path P connecting vertices v_1 and v_r. Denote $P = v_1v_m \cdots v_{r-1}v_r$. Let $G^* = G^{v_{r-1}v_r}(w) + v_su$, where v_s is a pendant vertex of $T_1 - v_1$ and u is a new vertex different from the vertices of $G^{v_{r-1}v_r}(w)$. Then $G^* \in \mathcal{U}_e^k$. By Lemma 2.8, we have

$\mu_1(G) < \mu_1(G^{v_{r-1}v_r}(w))$. Since $G^{v_{r-1}v_r}(w)$ is a proper subgraph of G^*, by Lemma 2.10, we get $\mu_1(G^{v_{r-1}v_r}(w)) < \mu_1(G^*)$. So $\mu_1(G) < \mu_1(G^*)$, which is a contradiction.

Fact 3. $l = 4$.

Proof. On the contrary, $l \geqslant 6$. Let G^* be the graph obtained from G by contracting the edges v_1v_2 and v_2v_3. Suppose that u is a pendent vertex of G, and v, w are two new vertices different from the vertices of G^*. Let $G^{**} = G^* + uv + vw$. Then $G^{**} \in \mathcal{U}_e^k$. By Lemma 2.7 and Lemma 2.10, we have $\rho(L^G) < \rho(L^{G^*}) < \rho(L^{G^{**}})$. Since G, G^*, G^{**} are all bipartite graphs, by Lemma 2.9, $\mu_1(G) < \mu_1(G^*) < \mu_1(G^{**})$, which is a contradiction.

Fact 4. The k paths attached to v_1 have almost equal lengths.

Proof. It is obvious by Lemma 2.11.

Up to now, we have proved the result. □

Lemma 3.2. *Let G_1 and G_2 be shown as in Fig. 6. Then $\mu_1(G_1) < \mu_1(G_2)$.*

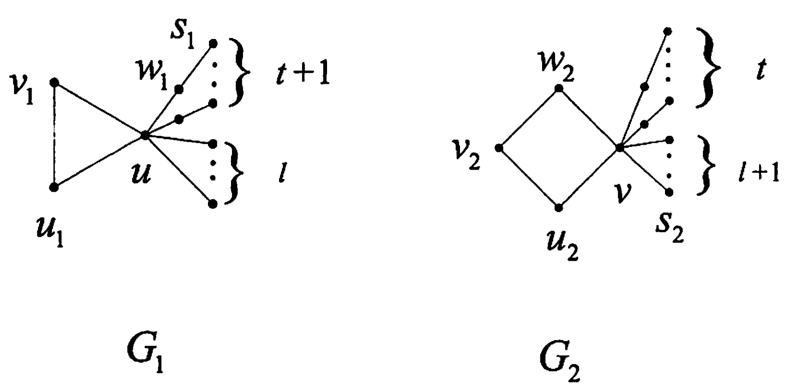

Fig. 6. Graphs G_1 and G_2.

Proof: Let $H_1 = G - \{u_1, v_1\}$, $H_2 = G - \{u_2, v_2, w_2\}$, $H = G - \{u_1, v_1, w_1, s_1\} = G - \{u_2, v_2, w_2, s_2\}$. By Lemma 2.2, we have

$$\begin{aligned} \Phi(C_3) &= x(x-3)^2; \\ \Phi(C_4) &= x(x-2)^2(x-4); \\ \Phi(L_u(C_3)) &= (x-1)(x-3); \\ \Phi(L_v(C_4)) &= (x-2)(x^2-4x+2); \\ \Phi(L_u(H_1)) &= (x-1)^l(x^2-3x+1)^{t+1}; \end{aligned}$$

$$\begin{aligned}
\Phi(L_v(H_2)) &= (x-1)^{l+1}(x^2-3x+1)^t; \\
\Phi(G_1) &= \Phi(H_1)\Phi(L_u(C_3)) + \Phi(L_u(H_1))(\Phi(C_3) - x\Phi(L_u(C_3))); \\
\Phi(G_2) &= \Phi(H_2)\Phi(L_v(C_4)) + \Phi(L_v(H_2))(\Phi(C_4) - x\Phi(L_v(C_4))); \\
\Phi(H_1) &= \Phi(H)B_2 + (P_3 - xB_2)B_2^t B_1^l; \\
&= \Phi(H)(x^2-3x+1) - x(x-2)(x-1)^l(x^2-3x+1)^t; \\
\Phi(H_2) &= \Phi(H)B_1 + (P_2 - xB_1)B_2^t B_1^l; \\
&= \Phi(H)(x-1) - x(x-1)^l(x^2-3x+1)^t. \\
\Phi(G_1) - \Phi(G_2) &= \Phi(H_1)\Phi(L_u(C_3)) - \Phi(H_2)\Phi(L_v(C_4)) \\
&\quad + \Phi(L_u(H_1))(\Phi(C_3) - x\Phi(L_u(C_3))) \\
&\quad - \Phi(L_v(H_2))(\Phi(C_4) - x\Phi(L_v(C_4))) \\
&= \Phi(H)(x-1)(x-3)(x^2-3x+1) \\
&\quad - x(x-2)(x-3)(x-1)^{l+1}(x^2-3x+1)^t \\
&\quad + (x-1)^l(x^2-3x+1)^{t+1}(-2x)(x-3) \\
&\quad - \Phi(H)(x-1)(x-2)(x^2-4x+2) \\
&\quad + x(x-2)(x^2-4x+2)(x-1)^l(x^2-3x+1)^t \\
&\quad - (x-1)^{l+1}(x^2-3x+1)^t(-2x)(x-2)(x-3) \\
&= \Phi(H)(x-1)[(x-3)(x^2-3x+1) - (x-2)(x^2-4x \\
&\quad + 2)] - x(x-2)(x-1)^l(x^2-3x+1)^t[(x-1)(x-3) \\
&\quad - (x^2-4x+2)] - 2x(x-3)(x-1)^l(x^2-3x+1)^t \\
&\quad [(x^2-3x+1) - (x-1)(x-2)] \\
&= \Phi(H)(x-1) - x(x-2)(x-1)^l(x^2-3x+1)^t \\
&\quad + 2x(x-3)(x-1)^l(x^2-3x+1)^t \\
&= \Phi(H)(x-1) + x(x-4)(x-1)^l(x^2-3x+1)^t.
\end{aligned}$$

Since H and C_4 are both proper subgraphs of bipartite graph G_2, by Lemma 2.10, we get $\mu_1(G_2) > \mu_1(H)$, $\mu_1(G_2) > \mu_1(C_4) = 4$. As we know $\lambda(G_2) = \mu_1(G_2)$, we get $\Phi(G_1) - \Phi(G_2) > 0$, for all $x \geqslant \lambda(G_2)$. By Lemma 2.3, we get $\lambda(G_1) < \lambda(G_2)$, i.e., $\mu_1(G_1) < \mu_1(G_2)$. □

Lemma 3.3. *For any graph $G \in \mathcal{U}_o^k$, we have $\mu_1(G) < \mu_1(\lozenge_4^k)$.*

Proof: Let $G = U(C_l; T_1, T_2, \ldots, T_l) \in \mathcal{U}_0^k$. We can prove the result by induction on l.

If $l = 3$, let G^* be the graph obtained from G by attaching T_2 and T_3 to vertex v_1. Then by Lemma 2.4, we have $\mu_1(G) \leqslant \mu_1(G^*)$. Denote by T_1^* the unique attached tree of G^*. Let $G' = G^* - v_2v_3$. Then Lemma 2.13, we get $\mu_1(G^*) = \mu_1(G')$. Since $n \geqslant k+4$, there must exist a vertex $v_j \in V(T_1^*)$ such that $d_{G^*}(v_1, v_j) = 2$. Let $G'' = G' + v_2v_j$. Then $G'' \in \mathcal{U}_e^k$.

If $G^* \cong G_1$, then $G'' \cong G_2$, where G_1, G_2 are shown as in Fig. 6. Then by Lemma 3.2, we have $\mu_1(G') < \mu_1(G'')$. Otherwise, $G^* \not\cong G_1$. By Lemma 2.1, we have $\mu_1(G') \leqslant \mu_1(G'')$ and the unique cycle C_q of G'' is of length 4. Then by Lemma 3.1, we get $\mu_1(G'') < \mu_1(\Diamond_4^k)$. So $\mu_1(G) < \mu_1(\Diamond_4^k)$.

Suppose the result is true for each graph belonging to $\mathcal{U}_o^k$ with a cycle of length smaller than l. In the following, we always assume that $l \geqslant 5$.

Case 1. There exists some $1 \leqslant i \leqslant l$ such that $d_G(v_i) = d_G(v_{i+1(mod\ l)}) = 2$.

Let $G^{**} = G^{v_iv_{i+1}}(w) + vu$, where u is a pendant vertex of $G^{v_iv_{i+1}}(w)$ and v is a new vertex different from the vertices of $G^{v_iv_{i+1}}(w)$. Then $G^{**} \in \mathcal{U}_e^k$ and by Lemma 2.7, we have $\rho(L^G) < \rho(L^{G^{v_{i+1}v_i}(w)})$. By Lemma 2.9, we have $\mu_1(G) < 2 + \rho(L^G)$ and $\mu_1(G^{v_{i+1}v_i}(w)) = 2 + \rho(L^{G^{v_{i+1}v_i}(w)})$. So $\mu_1(G) < \mu_1(G^{v_{i+1}v_i}(w))$. Since $G^{v_{i+1}v_i}(w)$ is a proper subgraph of G^{**}, by Lemma 2.10, we get $\mu_1(G^{v_{i+1}v_i}(w)) < \mu_1(G^{**})$. Since $G^{**} \in \mathcal{U}_e^k$, by Lemma 3.1, we get $\mu_1(G^{**}) \leqslant \mu_1(\Diamond_4^k)$. So $\mu_1(G) < \mu_1(\Diamond_4^k)$.

Case 2. Otherwise.

Since l is odd, there must exist some $1 \leqslant j \leqslant l$ such that $d_G(v_j) \geqslant 3$, $d_G(v_{j+1(mod\ l)}) \geqslant 3$. For convenience, we may assume that $j = 1$.

If $l = 5$ and two vertices of v_3, v_4, v_5 are with degree 2, say $d_G(v_3) = d_G(v_5) = 2$. Then $d_G(v_4) \geqslant 3$. Denote $N_G(v_1) = \{u_1, \ldots, u_s\}$. Let $G_1 = G - v_1u_1 - \cdots - v_1u_s + v_2u_1 + \cdots + v_2u_s$. Then by Lemma 2.4, $\mu_1(G) < \mu_1(G_1)$. Since $d_{G_1}(v_1) = d_{G_1}(v_5) = 2$, we can deal with G_1 in a similar way to case 1, and prove the result.

If $l = 5$ and only one vertex of v_3, v_4, v_5 are with degree 2, say $d_G(v_5) = 2$. Then $d_G(v_3) \geqslant 3$, $d_G(v_4) \geqslant 3$. We may assume that $x_{v_1} \leqslant x_{v_2}$, since $-X$ is also a unit eigenvector of G corresponding to $\mu_1(G)$. If $x_{v_2} \leqslant x_{v_i}$, let $G_2 = G - v_1v_2 + v_1v_i$ $(i = 3, 4)$; if $x_{v_i} \leqslant x_{v_2}$ $(i = 3, 4)$ and $x_{v_4} \leqslant x_{v_3}$, let $G_2 = G - v_3v_4 + v_2v_4$; if $x_{v_i} \leqslant x_{v_2}$ $(i = 3, 4)$ and $x_{v_5} \geqslant x_{v_4} > x_{v_3}$, let $G_2 = G - v_3v_4 + v_3v_5$; if $x_{v_i} \leqslant x_{v_2}$ $(i = 3, 4)$, $x_{v_5} < x_{v_4}$ and $x_{v_4} > x_{v_3}$, let $G_2 = G - v_4v_5 + v_2v_5$. Then, in either case, $G_2 \in \mathcal{U}_e^k$ or $G_2 \in \mathcal{U}_o^k$ and by lemma 2.5, $\mu_1(G) \leqslant \mu_1(G_2)$. If $G_2 \in \mathcal{U}_e^k$, since $G_2 \not\cong \Diamond_4^k$, by Lemma 3.1, we get $\mu_1(G_2) < \mu_1(\Diamond_4^k)$. If $G_2 \in \mathcal{U}_o^k$, since the unique cycle of G_2 is of length 3, according to the first step of induction hypothesis, we get $\mu_1(G_2) < \mu_1(\Diamond_4^k)$. So, in either case, $\mu_1(G) < \mu_1(\Diamond_4^k)$.

If $l \geqslant 7$, we only consider the case that $d_G(v_{2i+1}) \geqslant 3$ and $d_G(v_{2i}) = 2$ $(i = 1, \ldots, \lfloor \frac{l}{2} \rfloor)$, since the other cases are similar to it and easier than it. For a similar reason to the above, we may assume that $x_{v_l} \leqslant x_{v_1}$. If there exists a vertex v_i $(2 \leqslant i \leqslant l-2)$ such that $x_{v_1} \leqslant x_{v_i}$, let $G_3 = G - v_lv_1 + v_lv_i$. Otherwise, $x_{v_i} < x_{v_1}$ $(2 \leqslant i \leqslant l-2)$. In this case, if $x_{v_i} \geqslant x_{v_4}$, let $G_3 = G - v_4v_i + v_1v_4$ $(i = 3, 5)$. If $x_{v_i} < x_{v_4}$ $(i = 3, 5)$, we consider $-X$. Let $Y = -X$. Then $y_{v_1} \leqslant y_{v_l}$, $y_{v_3} > y_{v_4}$. If $y_{v_l} \leqslant y_{v_3}$, let $G_3 = G - v_1v_l + v_1v_3$; if $y_{v_l} > y_{v_3}$, let $G' = G - v_3v_4 + v_4v_l$. Then, in either case, $G_3 \in \mathcal{U}_e^k$ or $G_3 \in \mathcal{U}_o^k$ with a smaller cycle than that of G, and by Lemma 2.5, $\mu_1(G) \leqslant \mu_1(G_3)$.

If $G_3 \in \mathcal{U}_e^k$, since $G_3 \ncong \Diamond_4^k$, by Lemma 3.1, we get $\mu_1(G') < \mu_1(\Diamond_4^k)$. If $G_3 \in \mathcal{U}_o^k$, then by the hypothesis, we get $\mu_1(G_3) < \mu_1(\Diamond_4^k)$. So in either case, $\mu_1(G) < \mu_1(\Diamond_4^k)$. □

Combining the above three lemmas, we get our main result:

Theorem 3.4. *For any graph $G \in \mathcal{U}_n(k)$, we have*

$$\mu_1(G) \leqslant \mu_1(\Diamond_4^k),$$

and equality holds if and only if $G \cong \Diamond_4^k$.

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