The local bases of primitive non-powerful signed symmetric digraphs with loops*

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Abstract

Let S be a primitive non-powerful signed digraph of order n. The base of a vertex u, denoted by $l_S(u)$, is the smallest positive integer l such that there is a pair of $SSSD$ walks of length t from u to each vertex $v \in V(S)$ for any integer $t \geq l$. We choose to order the vertices of S in such a way that $l_S(1) \leq l_S(2) \leq \ldots \leq l_S(n)$, and call $l_S(k)$ the kth local base of S for $1 \leq k \leq n$. In this work, we use $PNSSD$ to denote the class of all primitive non-powerful signed symmetric digraphs of order n with at least one loop. Let $l(k)$ be the largest value of $l_S(k)$ for $S \in PNSSD$, and $L(k) = \{l_S(k) \mid S \in PNSSD\}$. For $n \geq 3$ and $1 \leq k \leq n-1$, we show $l(k) = 2n-1$ and $L(k) = \{2, 3, \ldots, 2n-1\}$. Further, we characterize all primitive non-powerful signed symmetric digraphs whose kth local bases attain $l(k)$.

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1 Introduction

Let D be a digraph (permits loops but no multiple arcs). Digraph D is called *primitive* if there is a positive integer k such that for all ordered pairs of vertices u and v (not necessarily distinct) in D, there exists a walk of length k from u to v([1]).

A *signed digraph* S is a digraph where each arc of S is assigned a sign 1 or -1. The *sign* of the walk W (in a signed digraph), denoted by sgn$W$,

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is defined to be the product of signs of all arcs in W. Two walks W_1 and W_2 in a signed digraph is called *a pair of SSSD walks*, if they have the same initial vertex, same terminal vertex, same length, but different signs. A signed digraph S is called *powerful* if S contains no pair of $SSSD$ walks.

Let S be a primitive non-powerful signed digraph. For any $v_i, v_j \in V(S)$, the *base from v_i to v_j*, denoted by $l_S(v_i, v_j)$, is defined to be the smallest positive integer l_1 such that for each integer $t \geq l_1$, there exists a pair of SSSD walks of length t from v_i to v_j. The *base of vertex* $v_i \in V(S)$, denoted by $l_S(v_i)$, is defined to be the smallest positive integer l_2 such that for each integer $t \geq l_2$ there exists a pair of $SSSD$ walks of length t from v_i to each vertex $v_j \in V(S)$. The *base* of S, denoted by $l(S)$, is defined to be the smallest positive integer l_3 such that for all ordered pairs of vertices v_i and v_j, there is a pair of $SSSD$ walks of length t from v_i to v_j for each integer $t \geq l_3$. Clearly, $l_S(v_i) = \max\{l_S(v_i, v_j) \mid v_j \in V(S)\}$ and $l(S) = \max\{l_S(v_i) \mid v_i \in V(S)\}$. We choose to order the vertices of S in such a way that

$$l_S(1) \leq l_S(2) \leq \ldots \leq l_S(n),$$

and call $l_S(k)$ the *kth local base* of S for $1 \leq k \leq n$.

A digraph D is *symmetric* if for any $v_i, v_j \in V(D)$, (v_i, v_j) is an arc if and only if (v_j, v_i) is an arc. A *signed symmetric digraph* S is a symmetric digraph where each arc of S is assigned a sign 1 or -1, and the sign of (v_i, v_j) may be different from the sign of (v_j, v_i).

We use $PNSSD$ to denote the class of all primitive non-powerful signed symmetric digraphs of order n with at least one loop. For $n \geq 3$, and $1 \leq k \leq n$, let $l(k)$ be the largest value of $l_S(k)$ for $S \in PNSSD$, and $L(k) = \{l_S(k) \mid S \in PNSSD\}$. Since $l(n)$ and $L(n)$ have been determined in [2], in this work, we shall show $l(k) = 2n-1$ and $L(k) = \{2, 3, \ldots, 2n-1\}$ for $1 \leq k \leq n-1$. Further, we characterize all primitive non-powerful signed symmetric digraphs whose kth local bases attain $l(k)$.

2 Some preliminaries

Lemma 2.1 ([3]) *Let S be a primitive signed digraph. Then S is non-powerful if and only if S contains a pair of cycles C_1 and C_2 (of lengths p_1 and p_2, respectively) satisfying one of the following two conditions:*

(1) *p_1 is odd, p_2 is even and* $\text{sgn}C_2 = -1$*;*

(2) *Both p_1 and p_2 are odd and* $\text{sgn}C_1 = -\text{sgn}C_2$.

For convenience, we call a pair of cycles C_1 and C_2 satisfying (1) or (2) in Lemma 2.1 a *distinguished cycle pair*. Suppose C_1 and C_2 form a distinguished cycle pair of lengths p_1 and p_2, respectively. Then the closed walks $W_1 = p_2C_1$ (walk around C_1 p_2 times) and $W_2 = p_1C_2$ have the same length p_1p_2 but with different signs since $(\text{sgn}C_1)^{p_2} = -(\text{sgn}C_2)^{p_1}$.

Let $R = \{C_1, \ldots, C_r\}$ be the set of some distinct cycles of signed digraph S. For any $u, v \in V(S)$, $d_R(u, v)$ denote the length of the shortest walk from u to v which meets C_i for each $i = 1, \ldots, r$. The following lemma is clear.

Lemma 2.2 *Let S be a primitive non-powerful signed digraph with at least one loop, and C_1 and C_2 be a distinguished cycle pair of lengths p_1 and p_2, respectively. Denote $R = \{C_1, C_2\}$. If $\min\{p_1, p_2\} = 1$, then $l_S(v_i, v_j) \le d_R(v_i, v_j) + p_1p_2$ for any $v_i, v_j \in V(S)$.*

3 Main results

Theorem 3.1 *Let $n \ge 3$, $1 \le k \le n-1$, and $S \in PNSSD$. Then $l_S(k) \le 2n-1$, and the equality can occur.*

Proof Let C_1 be a loop of S. Since S is primitive non-powerful, by Lemma 2.1, there is a cycle C_2 of length m (m-cycle for short) in S such that C_1 and C_2 form a distinguished cycle pair. Denote $R = \{C_1, C_2\}$. Let D be the underlying digraph of S. For any $v_i, v_j \in V(S)$, we consider the following cases.

Case 1. $m = 1$. Then $d_R(v_i, v_j) \le 2(n-1)$. So $l_S(v_i, v_j) \le 2(n-1)+1 = 2n-1$ by Lemma 2.2 and $l_S(v_i) \le 2n-1$ for any vertex v_i.

Case 2. $m \ge 2$. If m is odd, then $d_R(v_i, v_j) \le 2(n - \frac{m+1}{2})$. So $l_S(v_i, v_j) \le 2(n - \frac{m+1}{2}) + m = 2n-1$ by Lemma 2.2 and $l_S(v_i) \le 2n-1$ for any vertex v_i. If m is even, then the diameter of D $d(D) \le n - \frac{m}{2}$. Suppose $d(D) \le n - \frac{m}{2} - 1$. Then $d_R(v_i, v_j) \le 2(n - \frac{m}{2} - 1)$ and $l_S(v_i, v_j) \le 2(n - \frac{m}{2} - 1) + m = 2n-2$ by Lemma 2.2. So $l_S(v_i) \le 2n-2$ for any vertex v_i. Suppose $d(D) = n - \frac{m}{2}$. Without loss of generality, let $d(v_1, v_{n-\frac{m}{2}+1}) = n - \frac{m}{2}$, and the shortest path in D from v_1 to $v_{n-\frac{m}{2}+1}$ is $v_1 \to v_2 \to \cdots \to v_{n-\frac{m}{2}+1}$. If C_1 is (not) at vertex v_1, then $\max\{d_R(v_i, v_j) \mid v_j \in V(S)\} \le 2(n - \frac{m}{2}) - 1$ for $v_i \ne v_{n-\frac{m}{2}+1}(v_1)$. So $l_S(v_i, v_j) \le 2(n - \frac{m}{2}) - 1 + m = 2n-1$ by Lemma 2.2 and $l_S(v_i) \le 2n-1$ for $v_i \ne v_{n-\frac{m}{2}+1}(v_1)$.

Combining the above cases, we have $l_S(k) \le 2n-1$ for $1 \le k \le n-1$.

On the other hand, take $S_1 \in PNSSD$ such that its underlying digraph is D_1 (as given in Figure 1), the loop at vertex v_1 is negative, and the other arcs are positive.

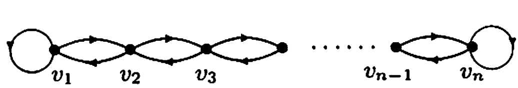

Fig. 1 Digraph D_1

For any $v_i \in V(S_1)$, since each walk of length $2n-2$ from v_i to v_i is positive. Thus there is no pair of $SSSD$ walks of length $2n-2$ from v_i to v_i. So $l_{S_1}(v_i) = 2n-1$ and $l_{S_1}(k) = 2n-1$ for $1 \le k \le n-1$. □

Corollary 3.2 *For $n \ge 3$ and $1 \le k \le n-1$, $l(k) = 2n-1$.*

Lemma 3.3 *For $n \ge 3$, $1 \le k \le n-2$, and $2 \le t \le n-1$, $2t \in L(k)$.*

Proof Let $2 \le t \le n-1$. Take $S \in PNSSD$ such that its underlying digraph is the digraph obtained from D_2 (as given in Figure 2) by adding loops at vertices $v_{t+1}, v_{t+2}, \ldots, v_n$, respectively, the arc (v_1, v_2) is negative, and the other arcs are positive. We shall show $l_S(k) = 2t$ for $1 \le k \le n-2$.

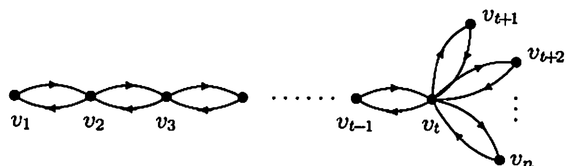

Fig. 2 Digraph D_2

For vertex v_1, since there exists a walk of length $2t$ from v_1 to v_j such that it meets both a positive loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_1, v_j) \le 2t+2$ by Lemma 2.2 and $l_S(v_1) \le 2t+2$. On the other hand, each walk of length $2t+1$ from v_1 to v_1 is negative. Thus there is no pair of $SSSD$ walks of length $2t+1$ from v_1 to v_1 and $l_S(v_1) = 2t+2$.

For vertex v_2, since there exists a walk of length $2t-1$ from v_2 to v_j such that it meets both a positive loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_2, v_j) \le 2t-1+2 = 2t+1$ by Lemma 2.2 and $l_S(v_2) \le 2t+1$. On the other hand, each walk of length $2t$ from v_2 to v_1 is positive. Thus there is no pair of $SSSD$ walks of length $2t$ from v_2 to v_1 and $l_S(v_2) = 2t+1$.

For vertex v_i, where $3 \le i \le n$, since there exists a walk of length $2(t-1)$ from v_i to v_j such that it meets both a positive loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_i, v_j) \le 2(t-1)+2 = 2t$ by Lemma 2.2 and $l_S(v_i) \le 2t$. On the other hand, each walk of length $2t-1$ from v_i to v_i is positive. Thus there is no pair of $SSSD$ walks of length $2t-1$ from v_i to v_i and $l_S(v_i) = 2t$.

Consequently, $l_S(k) = 2t$ for $1 \le k \le n-2$. □

Lemma 3.4 *For $n \ge 3$ and $1 \le t \le n-2$, $2t+2 \in L_n(n-1)$.*

Proof Let $1 \le t \le n-2$. Take $S \in PNSSD$ such that its underlying digraph is the digraph obtained from D_2 by adding a loop at vertex v_1, the arcs $(v_t, v_{t+1}), (v_t, v_{t+2}), \ldots, (v_t, v_n)$ are negative, and the other arcs are positive. We shall show $l_S(n-1) = 2t+2$.

For vertex v_i, where $t+1 \leq i \leq n$, since there exists a walk of length $2t$ from v_i to v_j such that it meets both a positive loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_i, v_j) \leq 2t+2$ by Lemma 2.2 and $l_S(v_i) \leq 2t+2$. On the other hand, each walk of length $2t+1$ from v_i to v_i is negative. Thus there is no pair of $SSSD$ walks of length $2t+1$ from v_i to v_i and $l_S(v_i) = 2t+2$.

For vertex v_i, where $1 \leq i \leq t$, since there exists a walk of length $2t-1$ from v_i to v_j such that it meets both a positive loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_i, v_j) \leq 2t-1+2 = 2t+1$ by Lemma 2.2 and $l_S(v_i) \leq 2t+1$.

Since $t \leq n-2$, consequently $l_S(n-1) = 2t+2$. □

Lemma 3.5 *For* $n \geq 3$, $1 \leq k \leq n-1$, *and* $1 \leq t \leq n-1$, $2t+1 \in L(k)$.

Proof Let $1 \leq t \leq n-1$. Take $S \in PNSSD$ such that its underlying digraph is the digraph obtained from D_2 by adding loops at vertices $v_1, v_{t+1}, v_{t+2}, \ldots, v_n$, respectively, the loop at vertex v_1 is negative, and the other arcs are positive. We shall show $l_S(k) = 2t+1$ for $1 \leq k \leq n-1$.

For any $v_i \in V(S)$, since there exists a walk of length $2t$ from v_i to v_j such that it meets both a negative loop and a positive loop for any vertex $v_j \in V(S)$, so $l_S(v_i, v_j) \leq 2t+1$ by Lemma 2.2 and $l_S(v_i) \leq 2t+1$. On the other hand, each walk of length $2t$ from v_i to v_i is positive. Thus there is no pair of $SSSD$ walks in S of length $2t$ from v_i to v_i and $l_S(v_i) = 2t+1$.

Consequently, $l_S(k) = 2t+1$ for $1 \leq k \leq n-1$. □

Lemma 3.6 *For* $n \geq 3$ *and* $1 \leq k \leq n-1$, $2 \in L(k)$.

Proof Take $S \in PNSSD$ with $V(S) = \{v_1, v_2, \ldots, v_n\}$ such that its underlying digraph is the symmetric complete digraph with a loop at each vertex, the arcs $(v_2, v_1), (v_3, v_1), \ldots, (v_n, v_1)$ and the loop at vertex v_1 are negative, and the other arcs are positive. We shall show $l_S(k) = 2$ for $1 \leq k \leq n-1$.

For any vertices $v_i, v_j \in V(S)$, walks $v_i \to v_2 \to v_j$ and $v_i \to v_1 \to v_j$ form a pair of $SSSD$ walks of length 2 from v_i to v_j. Since there exists a loop at each vertex, so there exists a pair of $SSSD$ walks of length l from v_i to v_j for each integer $l \geq 2$. On the other hand, $l_S(v_i) \geq 2$ for any $S \in PNSSD$. So $l_S(v_i) = 2$ for any $v_i \in V(S)$.

Consequently, $l_S(k) = 2$ for $1 \leq k \leq n-1$. □

Since $1 \notin L(k)$ for $n \geq 3$ and $1 \leq k \leq n-1$, combining Theorem 3.1 and Lemmas 3.3–3.6, we obtain the following theorem.

Theorem 3.7 *For* $n \geq 3$ *and* $1 \leq k \leq n-1$, $L(k) = \{2, 3, \ldots, 2n-1\}$.

4 The extremal signed symmetric digraphs

In this section, we characterize all primitive non-powerful signed symmetric digraphs of order n with at least one loop whose kth local bases attain $l(k)$.

Lemma 4.1 *Let $n \geq 3$, $S \in PNSSD$ with D as the underlying digraph and there exist at least one negative 2-cycle in S. Then*

(1) $l_S(k) \leq 2n - 2$ for $1 \leq k \leq n - 2$.

(2) $l_S(n - 1) = 2n - 1$ if and only if D is isomorphic to D_3 (as given in Figure 3).

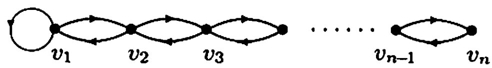

Fig. 3 Digraph D_3

Proof Let C_1 and C_2 be a loop and a negative 2-cycle of S, respectively, and $R = \{C_1, C_2\}$. Then C_1 and C_2 form a distinguished cycle pair. Clearly $d(D) \leq n - 1$. For any $v_i, v_j \in V(S)$, we consider the following cases.

Case 1. $d(D) \leq n - 2$. Then $d_R(v_i, v_j) \leq 2(n - 2)$ and $l_S(v_i, v_j) \leq 2(n-2) + 2 = 2n - 2$ by Lemma 2.2. So $l_S(v_i) \leq 2n - 2$ for any vertex v_i.

Case 2. $d(D) = n - 1$. Without loss of generality, let $d(v_1, v_n) = n - 1$, and the shortest path in D from v_1 to v_n be $v_1 \to v_2 \to \cdots \to v_n$. If either there exists a loop at vertex v_k, where $2 \leq k \leq n - 1$, or there exist loops at both vertices v_1 and v_n, then there exists a walk of length $2(n-2)$ from v_i to v_j such that it meets both a loop and a negative 2-cycle. So $l_S(v_i, v_j) \leq 2(n - 2) + 2 = 2n - 2$ by Lemma 2.2 and $l_S(v_i) \leq 2n - 2$ for any vertex v_i. If there exists a loop at exactly vertex v_1 or v_n, then D is isomorphic to D_3. Now, we calculate the base of each vertex.

For vertex v_n, since there exists a walk of length $2(n - 1)$ from v_n to v_j such that it meets both the loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_n, v_j) \leq 2(n-1)+2 = 2n$ by Lemma 2.2 and $l_S(v_n) \leq 2n$. On the other hand, there exists the unique walk of length $2n - 1$ from v_n to v_n. Thus there is no pair of $SSSD$ walks of length $2n - 1$ from v_n to v_n and $l_S(v_n) = 2n$.

For vertex v_{n-1}, since there exists a walk of length $2(n - 2) + 1$ from v_{n-1} to v_j such that it meets both the loop and a negative 2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_{n-1}, v_j) \leq 2(n-2)+1+2 = 2n-1$ by Lemma 2.2 and $l_S(v_{n-1}) \leq 2n - 1$. On the other hand, there exists the unique walk of length $2n - 2$ from v_{n-1} to v_n. Thus there is no pair of $SSSD$ walks of length $2n - 2$ from v_{n-1} to v_n and $l_S(v_{n-1}) = 2n - 1$.

For vertex v_i, where $1 \leq i \leq n - 2$, since there exists a walk of length $2(n - 2)$ from v_i to v_j such that it meets both the loop and a negative

2-cycle for any vertex $v_j \in V(S)$, so $l_S(v_i, v_j) \leq 2(n-2)+2 = 2n-2$ by Lemma 2.2 and $l_S(v_i) \leq 2n-2$.

Consequently, we have $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-2$, and $l_S(n-1) = 2n-1$ if and only if D is isomorphic to D_3. □

Lemma 4.2 *Let $n \geq 3$, and $S \in PNSSD$ with D as the underlying digraph. If each 2-cycle of S is positive, and there exist both a negative loop and a positive loop in S. Then $l_S(k) = 2n-1$ for $1 \leq k \leq n-1$ if and only if D is isomorphic to D_1.*

Proof Sufficiency is immediate from the proof of Theorem 3.1. Now we consider necessity. Let C_1 and C_2 be the negative loop and positive loop, respectively. Then C_1 and C_2 form a distinguished cycle pair of S. Denote $R = \{C_1, C_2\}$. For any $v_i, v_j \in V(S)$, if $d(D) \leq n-2$, then $d_R(v_i, v_j) \leq 2(n-2)$ and $l_S(v_i, v_j) \leq 2(n-2)+1 = 2n-3$ by Lemma 2.2. So $l_S(v_i) \leq 2n-3$ for any vertex v_i contradicting $l_S(k) = 2n-1$ for $1 \leq k \leq n-1$. So $d(D) = n-1$. Without loss of generality, let $d(v_1, v_n) = n-1$, the shortest path in D from v_1 to v_n be $v_1 \to v_2 \to \cdots \to v_n$. Let C_1 and C_2 be at vertices v_t and v_s, respectively, and $t > s$. If $v_s \neq v_1$ $(v_t \neq v_n)$, then for $v_i \neq v_1$ $(v_i \neq v_n)$, $d_R(v_i, v_j) \leq 2(n-2)+1$ for any $v_j \in V(S)$ and $l_S(v_i, v_j) \leq 2(n-2)+1+1 = 2n-2$ by Lemma 2.2. So $l_S(v_i) \leq 2n-2$ for any vertex $v_i \neq v_1$ $(v_i \neq v_n)$ contradicting $l_S(k) = 2n-1$ for $1 \leq k \leq n-1$. The lemma now follows. □

Lemma 4.3 *Let $n \geq 3$, and $S \in PNSSD$ with D as the underlying digraph. If each 2-cycle of S is positive, and the signs of all loops of S are the same, then $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-1$.*

Proof Let C_1 be a loop of S. Since S is primitive non-powerful, by Lemma 2.1, there is a cycle C in S such that C_1 and C form a distinguished cycle pair. Denote C_2 to be the cycle of length m satisfying: (1) C_1 and C_2 form a distinguished cycle pair; (2) there is no cycle C such that C_1 and C form a distinguished cycle pair and the length of C is less than m. Then C_2 must be a simple cycle and we call C_1 and C_2 the shortest distinguished cycle pair. Denote $R = \{C_1, C_2\}$. For any $v_i, v_j \in V(S)$, we consider the following cases.

Case 1. m is odd. Then $d(D) \leq n - \frac{m+1}{2}$ and $\text{sgn}C_1 = -\text{sgn}C_2$. Without loss of generality, let $\text{sgn}C_1 = +$. If $d(D) \leq n - \frac{m+1}{2} - 1$, then $d_R(v_i, v_j) \leq 2(n - \frac{m+1}{2} - 1)$ and $l_S(v_i, v_j) \leq 2(n - \frac{m+1}{2} - 1) + m = 2n-3$ by Lemma 2.2. So $l_S(v_i) \leq 2n-3$ for any vertex v_i. If $d(D) = n - \frac{m+1}{2}$, without loss of generality, let $d(v_1, v_{n-\frac{m+1}{2}+1}) = n - \frac{m+1}{2}$, the shortest path in D from v_1 to $v_{n-\frac{m+1}{2}+1}$ be $v_1 \to v_2 \to \cdots \to v_{n-\frac{m+1}{2}+1}$. Since C_1 and C_2 are the shortest distinguished cycle pair and $|V(C_2)\cap\{v_1, v_2, \dots, v_{n-\frac{m+1}{2}+1}\}| \geq \frac{m+1}{2}$, without loss of generality, we can let $C_2 = v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m-1}{2}} \to v_n \to v_{n-1} \to \cdots \to v_{n-\frac{m+1}{2}+2} \to v_t$, where $1 \leq t \leq n-m+1$.

Subcase 1.1. $2 \leq t \leq n-m$.

Subcase 1.1.1. C_1 is at vertex v, where $v \neq v_1$ and $v \neq v_{n-\frac{m+1}{2}+1}$. Then $d_R(v_i, v_j) \leq 2(n - \frac{m+1}{2} - 1)$ and $l_S(v_i, v_j) \leq 2(n - \frac{m+1}{2} - 1) + m = 2n-3$ by Lemma 2.2. So $l_S(v_i) \leq 2n-3 < 2n-2$ for any vertex v_i.

Subcase 1.1.2. C_1 is at vertex v_1 or $v_{n-\frac{m+1}{2}+1}$. Without loss of generality, let C_1 be at vertex v_1. Now, we calculate the base of each vertex.

Firstly, we consider vertex $v_{n-\frac{m+1}{2}+1}$. Since each 2-cycle of S is positive and C_2 is negative, the signs of $v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m-1}{2}}$ and $v_t \to v_{n-\frac{m+1}{2}+2} \to v_{n-\frac{m+1}{2}+3} \to \cdots \to v_n \to v_{t+\frac{m-1}{2}}$ is different. If $v_j \neq v_{n-\frac{m+1}{2}+1}$, then $d_R(v_{n-\frac{m+1}{2}+1}, v_j) \leq 2(n-\frac{m+1}{2})-1$ and $l_S(v_{n-\frac{m+1}{2}+1}, v_j) \leq 2(n - \frac{m+1}{2}) - 1 + m = 2n-2$ by Lemma 2.2. If $v_j = v_{n-\frac{m+1}{2}+1}$, since for any integer $l \geq 2(n - \frac{m+1}{2}) + 1$,

$$W_1 = (v_{n-\frac{m+1}{2}+1} \to v_{n-\frac{m+1}{2}} \to \cdots \to v_1) + (l - 2(n - \frac{m+1}{2}))C_1$$
$$+(v_1 \to \cdots \to v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m-1}{2}} \to \cdots \to v_{n-\frac{m+1}{2}+1})$$

and

$$W_2 = (v_{n-\frac{m+1}{2}+1} \to v_{n-\frac{m+1}{2}} \to \cdots \to v_1) + (l - 2(n - \frac{m+1}{2}) - 1)C_1$$
$$+(v_1 \to \cdots \to v_t \to v_{n-\frac{m+1}{2}+2} \to v_{n-\frac{m+1}{2}+3} \to \cdots \to v_n$$
$$\to v_{t+\frac{m-1}{2}} \to \cdots \to v_{n-\frac{m+1}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{n-\frac{m+1}{2}+1}$ to $v_{n-\frac{m+1}{2}+1}$, thus $l_S(v_{n-\frac{m+1}{2}+1}, v_{n-\frac{m+1}{2}+1}) \leq 2(n - \frac{m+1}{2}) + 1 = 2n-m$ and consequently $l_S(v_{n-\frac{m+1}{2}+1}) \leq 2n-2$.

Secondly, we consider vertex $v_i \neq v_{n-\frac{m+1}{2}+1}$. Since $d_R(v_i, v_j) \leq 2(n - \frac{m+1}{2} - 1)+1$ for any $v_j \in V(S)$, so $l_S(v_i, v_j) \leq 2(n-\frac{m+1}{2}-1)+1+m = 2n-2$ by Lemma 2.2 and $l_S(v_i) \leq 2n-2$ for any vertex $v_i \neq v_{n-\frac{m+1}{2}+1}$.

Consequently, $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-1$.

Subcase 1.2. $t = n-m+1$ or $t = 1$, and $m \leq n-1$. Without loss of generality, let $t = n-m+1$.

Subcase 1.2.1. C_1 is at vertex v, where $v \neq v_1$, $v \neq v_n$, and $v \neq v_{n-\frac{m+1}{2}+1}$. Since $m \leq n-1$, then $d_R(v_i, v_j) \leq 2(n - \frac{m+1}{2} - 1)$ and $l_S(v_i, v_j) \leq 2(n-\frac{m+1}{2}-1)+m = 2n-3$ by Lemma 2.2. Thus $l_S(v_i) \leq 2n-3$ for any vertex v_i.

Subcase 1.2.2. C_1 is at vertex v_1, or $v_{n-\frac{m+1}{2}+1}$, or v_n. Without loss of generality, let C_1 be at vertex v_1. Now, we calculate the base of each vertex.

Firstly, we consider vertex $v_{n-\frac{m+1}{2}+1}$ and v_n. If $v_j \neq v_{n-\frac{m+1}{2}+1}$ and $v_j \neq v_n$, then $d_R\ (v_{n-\frac{m+1}{2}+1}, v_j) \leq 2(n-\frac{m+1}{2})-1$ and $l_S(v_{n-\frac{m+1}{2}+1}, v_j) \leq 2(n-\frac{m+1}{2})-1+m = 2n-2$ by Lemma 2.2. For vertices $v_{n-\frac{m+1}{2}+1}$ and v_n, since for any integer $l \geq 2(n-\frac{m+1}{2})+1$,

$$W_1 = (v_{n-\frac{m+1}{2}+1} \to v_{n-\frac{m+1}{2}} \to \cdots \to v_1) + (l-2(n-\frac{m+1}{2}))C_1$$

$$+(v_1 \to \cdots \to v_{n-m+1} \to v_{n-m+2} \to \cdots \to v_{n-\frac{m+1}{2}+1})$$

and

$$W_2 = (v_{n-\frac{m+1}{2}+1} \to v_{n-\frac{m+1}{2}} \to \cdots \to v_1) + (l-2(n-\frac{m+1}{2})-1)C_1 +$$

$$(v_1 \to \cdots \to v_{n-m+1} \to v_{n-\frac{m+1}{2}+2} \to v_{n-\frac{m+1}{2}+3} \to \cdots \to v_n \to v_{n-\frac{m+1}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{n-\frac{m+1}{2}+1}$ to $v_{n-\frac{m+1}{2}+1}$, and $W_1 + (v_{n-\frac{m+1}{2}+1} \to v_n)$ and $W_2 + (v_{n-\frac{m+1}{2}+1} \to v_n)$ form a pair of $SSSD$ walks of length $l+1$ from $v_{n-\frac{m+1}{2}+1}$ to v_n, thus $l_S(v_{n-\frac{m+1}{2}+1}, v_{n-\frac{m+1}{2}+1}) \leq 2(n-\frac{m+1}{2})+1 = 2n-m$ and $l_S(v_{n-\frac{m+1}{2}+1}, v_n) \leq 2n-m+1$. So $l_S(v_{n-\frac{m+1}{2}+1}) \leq 2n-2$. Similarly, we can show $l_S(v_n) \leq 2n-2$.

Secondly, we consider vertex v_i, where $v_i \neq v_{n-\frac{m+1}{2}+1}$ and $v_i \neq v_n$. Since $d_R(v_i, v_j) \leq 2(n-\frac{m+1}{2}-1)+1$ for any vertex v_j of S, so $l_S(v_i, v_j) \leq 2(n-\frac{m+1}{2}-1)+1+m = 2n-2$ by Lemma 2.2 and $l_S(v_i) \leq 2n-2$.

Consequently, $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-1$.

Subcase 1.3. $m = n$. Without loss of generality, let $C_2 = v_1 \to v_2 \to \cdots \to v_{\frac{n+1}{2}} \to v_n \to v_{n-1} \to \cdots \to v_{\frac{n+1}{2}+1} \to v_1$, and C_1 be at vertex v_1. Now, we calculate the base of each vertex.

Firstly, we consider vertex $v_{\frac{n+1}{2}}$ and v_n. If $v_j \neq v_{\frac{n+1}{2}}$ and $v_j \neq v_n$, then $d_R(v_{\frac{n+1}{2}}, v_j) \leq n-2$ and $l_S(v_{\frac{n+1}{2}}, v_j) \leq n-2+n = 2n-2$ by Lemma 2.2. For vertices $v_{\frac{n+1}{2}}$ and v_n, since for any integer $l \geq n$,

$$W_1 = (v_{\frac{n+1}{2}} \to v_{\frac{n+1}{2}-1} \to \cdots \to v_1) + (l-n+1)C_1 + (v_1 \to v_2 \to \cdots \to v_{\frac{n+1}{2}})$$

and

$$W_2 = (v_{\frac{n+1}{2}} \to v_{\frac{n+1}{2}-1} \to \cdots \to v_1)$$

$$+(l-n)C_1 + (v_1 \to v_{\frac{n+1}{2}+1} \to v_{\frac{n+1}{2}+2} \to \cdots \to v_n \to v_{\frac{n+1}{2}})$$

form a pair of $SSSD$ walks of length l from $v_{\frac{n+1}{2}}$ to $v_{\frac{n+1}{2}}$, and $W_1+(v_{\frac{n+1}{2}} \to v_n)$ and $W_2 + (v_{\frac{n+1}{2}} \to v_n)$ form a pair of $SSSD$ walks of length $l+1$ from $v_{\frac{n+1}{2}}$ to v_n, thus $l_S(v_{\frac{n+1}{2}}, v_{\frac{n+1}{2}}) \le n < 2n-2$, and $l_S(v_{\frac{n+1}{2}}, v_n) \le n+1 \le 2n-2$. So $l_S(v_{\frac{n+1}{2}}) \le 2n-2$. Similarly, we can show $l_S(v_n) \le 2n-2$.

Secondly, we consider vertex v_i, where $v_i \neq v_{\frac{n+1}{2}}$ and $v_i \neq v_n$. Since $d_R(v_i, v_j) \le n-2$ for any vertex v_j of S, so $l_S(v_i, v_j) \le n-2+n = 2n-2$ by Lemma 2.2 and $l_S(v_i) \le 2n-2$.

Consequently, $l_S(k) \le 2n-2$ for $1 \le k \le n-1$.

Case 2. m is even. Clearly, $d(D) \le n - \frac{m}{2}$. If $d(D) \le n - \frac{m}{2} - 1$, then $d_R(v_i, v_j) \le 2(n - \frac{m}{2} - 1)$. So $l_S(v_i, v_j) \le 2(n - \frac{m}{2} - 1) + m = 2n-2$ by Lemma 2.2 and $l_S(v_i) \le 2n-2$ for any vertex v_i. If $d(D) = n - \frac{m}{2}$, without loss of generality, let $d(v_1, v_{n-\frac{m}{2}+1}) = n - \frac{m}{2}$, the shortest path from v_1 to $v_{n-\frac{m}{2}+1}$ be $v_1 \to v_2 \to \cdots \to v_{n-\frac{m}{2}+1}$. Since C_1 and C_2 are the shortest distinguished cycle pair and $|V(C_2) \cap \{v_1, v_2, \ldots, v_{n-\frac{m}{2}+1}\}| \ge \frac{m}{2}+1$, without loss of generality, we can let $C_2 = v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m}{2}} \to v_n \to v_{n-1} \to \cdots \to v_{n-\frac{m}{2}+2} \to v_t$, where $1 \le t \le n-m+1$.

Subcase 2.1. $2 \le t \le n-m$.

Subcase 2.1.1. C_1 is at vertex v, where $v \neq v_1$ and $v \neq v_{n-\frac{m}{2}+1}$. Then $d_R(v_i, v_j) \le 2(n - \frac{m}{2} - 1)$ and $l_S(v_i, v_j) \le 2(n - \frac{m}{2} - 1) + m = 2n-2$ by Lemma 2.2. Thus $l_S(v_i) \le 2n-2$ for any vertex v_i.

Subcase 2.1.2. C_1 is at vertex v_1 or $v_{n-\frac{m}{2}+1}$. Without loss of generality, let C_1 be at vertex v_1. Now, we calculate the base of each vertex.

Firstly, we consider vertex $v_{n-\frac{m}{2}+1}$. Since each 2-cycle of S is positive and C_2 is negative, the signs of $v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m}{2}}$ and $v_t \to v_{n-\frac{m}{2}+2} \to v_{n-\frac{m}{2}+3} \to \cdots \to v_n \to v_{t+\frac{m}{2}}$ is different. If $v_j \neq v_{n-\frac{m}{2}+1}$ and $v_j \neq v_{n-\frac{m}{2}}$, then $d_R(v_{n-\frac{m}{2}+1}, v_j) \le 2(n - \frac{m}{2} - 1)$ and $l_S(v_{n-\frac{m}{2}+1}, v_j) \le 2(n - \frac{m}{2} - 1) + m = 2n-2$ by Lemma 2.2. For vertices $v_{n-\frac{m}{2}+1}$ and $v_{n-\frac{m}{2}}$, since for any integer $l \ge 2n-m$,

$$W_1 = (v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}} \to \cdots \to v_1) + (l-(2n-m))C_1$$
$$+(v_1 \to \cdots \to v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m}{2}} \to \cdots \to v_{n-\frac{m}{2}+1})$$

and

$$W_2 = (v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}} \to \cdots \to v_1) + (l-(2n-m))C_1 + (v_1 \to \cdots$$
$$\to v_t \to v_{n-\frac{m}{2}+2} \to v_{n-\frac{m}{2}+3} \to \cdots \to v_n \to v_{t+\frac{m}{2}} \to \cdots \to v_{n-\frac{m}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{n-\frac{m}{2}+1}$ to $v_{n-\frac{m}{2}+1}$, and $W_1+(v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}})$ and $W_2+(v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}})$ form a pair of $SSSD$ walks of length $l+1$ from $v_{n-\frac{m}{2}+1}$ to $v_{n-\frac{m}{2}}$, thus $l_S(v_{n-\frac{m}{2}+1}, v_{n-\frac{m}{2}+1}) \le 2n-m$ and $l_S(v_{n-\frac{m+1}{2}+1}, v_{n-\frac{m}{2}}) \le 2n-m+1$. So $l_S(v_{n-\frac{m}{2}+1}) \le 2n-2$.

Secondly, we consider vertex $v_{n-\frac{m}{2}}$. If $v_j \neq v_{n-\frac{m}{2}+1}$, then $d_R(v_{n-\frac{m}{2}}, v_j) \leq 2(n-\frac{m}{2}-1)$ and $l_S(v_{n-\frac{m}{2}}, v_j) \leq 2(n-\frac{m}{2}-1)+m = 2n-2$ by Lemma 2.2. For vertex $v_{n-\frac{m}{2}+1}$, since for any integer $l \geq 2n-m-1$,

$$W_1 = (v_{n-\frac{m}{2}} \to v_{n-\frac{m}{2}-1} \to \cdots \to v_1) + (l-(2n-m-1))C_1$$

$$+(v_1 \to \cdots \to v_t \to v_{t+1} \to \cdots \to v_{t+\frac{m}{2}} \to \cdots \to v_{n-\frac{m}{2}+1})$$

and

$$W_2 = (v_{n-\frac{m}{2}} \to v_{n-\frac{m}{2}-1} \to \cdots \to v_1) + (l-(2n-m-1))C_1 + (v_1 \to \cdots$$

$$\to v_t \to v_{n-\frac{m}{2}+2} \to v_{n-\frac{m}{2}+3} \to \cdots \to v_n \to v_{t+\frac{m}{2}} \to \cdots \to v_{n-\frac{m}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{n-\frac{m}{2}}$ to $v_{n-\frac{m}{2}+1}$, thus $l_S(v_{n-\frac{m}{2}}, v_{n-\frac{m}{2}+1}) \leq 2n-m-1$. So $l_S(v_{n-\frac{m}{2}}) \leq 2n-2$.

Lastly, we consider vertex v_i, where $v_i \neq v_{n-\frac{m}{2}+1}$ and $v_i \neq v_{n-\frac{m}{2}}$. Since $d_R(v_i, v_j) \leq 2(n-\frac{m}{2}-1)$ for any vertex v_j of S, so $l_S(v_i, v_j) \leq 2(n-\frac{m}{2}-1)+m = 2n-2$ by Lemma 2.2 and $l_S(v_i) \leq 2n-2$.

Consequently, $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-1$.

Subcase 2.2. $t = n-m+1$ or $t = 1$, and $m \leq n-1$. Without loss of generality, let $t = n-m+1$.

Subcase 2.2.1. C_1 is at vertex v, where $v \neq v_1$ and $v \neq v_{n-\frac{m}{2}+1}$. Since $m \leq n-1$, then $d_R(v_i, v_j) \leq 2(n-\frac{m}{2}-1)$ and $l_S(v_i, v_j) \leq 2(n-\frac{m}{2}-1)+m = 2n-2$ by Lemma 2.2. Thus $l_S(v_i) \leq 2n-2$ for any vertex v_i.

Subcase 2.2.2. C_1 is at vertex v_1 or $v_{n-\frac{m}{2}+1}$. Without loss of generality, let C_1 be at vertex v_1. Now, we calculate the base of each vertex.

Firstly, we consider vertex $v_{n-\frac{m}{2}+1}$. If $v_j \neq v_{n-\frac{m}{2}+1}$, $v_j \neq v_{n-\frac{m}{2}}$, and $v_j \neq v_n$, then $d_R(v_{n-\frac{m}{2}+1}, v_j) \leq 2(n-\frac{m}{2}-1)$ and $l_S(v_{n-\frac{m}{2}+1}, v_j) \leq 2(n-\frac{m}{2}-1)+m = 2n-2$ by Lemma 2.2. For vertices $v_{n-\frac{m}{2}+1}$, $v_{n-\frac{m}{2}}$, and v_n, since for any integer $l \geq 2n-m$,

$$W_1 = (v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}} \to \cdots \to v_1) + (l-(2n-m))C_1$$

$$+(v_1 \to \cdots \to v_{n-m+1} \to v_{n-m+2} \to \cdots \to v_{n-\frac{m}{2}+1})$$

and

$$W_2 = (v_{n-\frac{m}{2}+1} \to v_{n+\frac{m}{2}} \to \cdots \to v_1) + (l-(2n-m))C_1$$

$$+(v_1 \to \cdots \to v_{n-m+1} \to v_{n-\frac{m}{2}+2} \to v_{n-\frac{m}{2}+3} \to \cdots \to v_n \to v_{n-\frac{m}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{n-\frac{m}{2}+1}$ to $v_{n-\frac{m}{2}+1}$, $W_1 + (v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}})$ and $W_2 + (v_{n-\frac{m}{2}+1} \to v_{n-\frac{m}{2}})$ form a pair of $SSSD$ walks of length $l+1$ from $v_{n-\frac{m}{2}+1}$ to $v_{n-\frac{m}{2}}$, and $W_1 + (v_{n-\frac{m}{2}+1} \to v_n)$ and $W_2 + (v_{n-\frac{m}{2}+1} \to v_n)$ form a pair of $SSSD$ walks of length $l+1$ from $v_{n-\frac{m}{2}+1}$ to v_n, thus $l_S(v_{n-\frac{m}{2}+1}, v_{n-\frac{m}{2}+1}) \leq 2n-m$, $l_S(v_{n-\frac{m}{2}+1}, v_{n-\frac{m}{2}}) \leq 2n-m+1$, and $l_S(v_{n-\frac{m}{2}+1}, v_n) \leq 2n-m+1$. So $l_S(v_{n-\frac{m}{2}+1}) \leq 2n-2$.

Secondly, we consider vertex $v_{n-\frac{m}{2}}$ and v_n. If $v_j \neq v_{n-\frac{m}{2}+1}$, then $d_R(v_{n-\frac{m}{2}}, v_j) \leq 2(n-\frac{m}{2}-1)$ and $l_S(v_{n-\frac{m}{2}}, v_j) \leq 2(n-\frac{m}{2}-1)+m = 2n-2$ by Lemma 2.2. For vertex $v_{n-\frac{m}{2}+1}$, since for any integer $l \geq 2n-m-1$,

$$W_1 = (v_{n-\frac{m}{2}} \rightarrow v_{n-\frac{m}{2}-1} \rightarrow \cdots \rightarrow v_1) + (l-(2n-m-1))C_1$$

$$+(v_1 \rightarrow \cdots \rightarrow v_{n-m+1} \rightarrow v_{n-m+2} \rightarrow \cdots \rightarrow v_{n-\frac{m}{2}+1})$$

and

$$W_2 = (v_{n-\frac{m}{2}} \rightarrow v_{n+\frac{m}{2}-1} \rightarrow \cdots \rightarrow v_1) + (l-(2n-m-1))C_1$$

$$+(v_1 \rightarrow \cdots \rightarrow v_{n-m+1} \rightarrow v_{n-\frac{m}{2}+2} \rightarrow v_{n-\frac{m}{2}+3} \rightarrow \cdots \rightarrow v_n \rightarrow v_{n-\frac{m}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{n-\frac{m}{2}}$ to $v_{n-\frac{m}{2}+1}$, thus $l_S(v_{n-\frac{m}{2}}, v_{n-\frac{m}{2}+1}) \leq 2n-m-1$. So $l_S(v_{n-\frac{m}{2}}) \leq 2n-2$. Similarly we can show $l_S(v_n) \leq 2n-2$.

Lastly, we consider vertex v_i, where $v_i \neq v_{n-\frac{m}{2}+1}$, $v_i \neq v_n$, and $v_i \neq v_{n-\frac{m}{2}}$. Since $d_R(v_i, v_j) \leq 2(n-\frac{m}{2}-1)$ for any vertex v_j of S, so $l_S(v_i, v_j) \leq 2(n-\frac{m}{2}-1)+m = 2n-2$ by Lemma 2.2 and $l_S(v_i) \leq 2n-2$.

Consequently, $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-1$.

Subcase 2.3. $m = n$. Without loss of generality, let $C_2 = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{\frac{n}{2}+1} \rightarrow v_n \rightarrow v_{n-1} \rightarrow \cdots \rightarrow v_{\frac{n}{2}+2} \rightarrow v_1$ and C_1 be at vertex v_1. Now, we calculate the base of each vertex.

Firstly, we consider vertex $v_{\frac{n}{2}+1}$. If $v_j \neq v_{\frac{n}{2}+1}$, $v_j \neq v_{\frac{n}{2}}$, and $v_j \neq v_n$, then $d_R(v_{\frac{n}{2}+1}, v_j) \leq n-2$ and $l_S(v_{\frac{n}{2}+1}, v_j) \leq n-2+n = 2n-2$ by Lemma 2.2. For vertices $v_{\frac{n}{2}+1}$, $v_{\frac{n}{2}}$, and v_n, since for any integer $l \geq n$,

$$W_1 = (v_{\frac{n}{2}+1} \rightarrow v_{\frac{n}{2}} \rightarrow \cdots \rightarrow v_1) + (l-n)C_1 + (v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{\frac{n}{2}+1})$$

and

$$W_2 = (v_{\frac{n}{2}+1} \rightarrow v_{\frac{n}{2}} \rightarrow \cdots \rightarrow v_1) + (l-n)C_1$$

$$+(v_1 \rightarrow v_{\frac{n}{2}+2} \rightarrow v_{\frac{n}{2}+3} \rightarrow \cdots \rightarrow v_n \rightarrow v_{\frac{n}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{\frac{n}{2}+1}$ to $v_{\frac{n}{2}+1}$, $W_1 + (v_{\frac{n}{2}+1} \rightarrow v_{\frac{n}{2}})$ and $W_2 + (v_{\frac{n}{2}+1} \rightarrow v_{\frac{n}{2}})$ form a pair of $SSSD$ walks of length $l+1$ from $v_{\frac{n}{2}+1}$ to $v_{\frac{n}{2}}$, and $W_1 + (v_{\frac{n}{2}+1} \rightarrow v_n)$ and $W_2 + (v_{\frac{n}{2}+1} \rightarrow v_n)$ form a pair of $SSSD$ walks of length $l+1$ from $v_{\frac{n}{2}+1}$ to v_n, thus $l_S(v_{\frac{n}{2}+1}, v_{\frac{n}{2}+1}) \leq n < 2n-2$, $l_S(v_{\frac{n}{2}+1}, v_{\frac{n}{2}}) \leq n+1 < 2n-2$, and $l_S(v_{\frac{n}{2}+1}, v_n) \leq n+1 < 2n-2$. So $l_S(v_{\frac{n}{2}+1}) \leq 2n-2$.

Secondly, we consider vertex $v_{\frac{n}{2}}$ and v_n. If $v_j \neq v_{\frac{n}{2}+1}$, then $d_R(v_{\frac{n}{2}}, v_j) \leq n-2$ and $l_S(v_{\frac{n}{2}}, v_j) \leq n-2+n = 2n-2$ by Lemma 2.2. For vertex $v_{\frac{n}{2}+1}$, since for any integer $l \geq n-1$,

$$W_1 = (v_{\frac{n}{2}} \rightarrow v_{\frac{n}{2}-1} \rightarrow \cdots \rightarrow v_1) + (l-n+1)C_1$$

$$+(v_1 \to v_2 \to \cdots \to v_{\frac{n}{2}} \to v_{\frac{n}{2}+1})$$

and

$$W_2 = (v_{\frac{n}{2}} \to v_{\frac{n}{2}-1} \to \cdots \to v_1) + (l-n+1)C_1$$
$$+(v_1 \to v_{\frac{n}{2}+2} \to v_{\frac{n}{2}+3} \to \cdots \to v_n \to v_{\frac{n}{2}+1})$$

form a pair of $SSSD$ walks of length l from $v_{\frac{n}{2}}$ to $v_{\frac{n}{2}+1}$, thus $l_S(v_{\frac{n}{2}}, v_{\frac{n}{2}+1}) \leq n-1 < 2n-2$ and so $l_S(v_{\frac{n}{2}}) \leq 2n-2$. Similarly we can show $l_S(v_n) \leq 2n-2$.

Lastly, we consider vertex v_i, where $v_i \neq v_{\frac{n}{2}+1}$, $v_i \neq v_n$, and $v_i \neq v_{\frac{n}{2}}$. Since $d_R(v_i, v_j) \leq n-2$ for any vertex v_j of S, so $l_S(v_i, v_j) \leq n-2+n = 2n-2$ by Lemma 2.2 and $l_S(v_i) \leq 2n-2$.

Consequently, $l_S(k) \leq 2n-2$ for $1 \leq k \leq n-1$.

Combining the above cases, the lemma holds □

By Lemmas 4.1–4.3, we have the following result.

Theorem 4.4 *Let $n \geq 3$, $S \in PNSSD$ with D as the underlying digraph.*

(1) *For $1 \leq k \leq n-2$, $l_S(k) = 2n-1$ if and only if D is isomorphic to D_1, each 2-cycle of S is positive, and there exist both a negative loop and a positive loop in S.*

(2) *$l_S(n-1) = 2n-1$ if and only if either D is isomorphic to D_1, each 2-cycle of S is positive, and there exist both a negative loop and a positive loop in S, or D is isomorphic to D_3, and there exists at least one negative 2-cycle in S.*

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