Combinatorial graph games

Eric Duchêne[◦,*] Sylvain Gravier[◦,†] Mehdi Mhalla[*,‡]

[◦]ERTé "Maths à Modeler", GéoD Research group
Leibniz Laboratory
46, avenue Félix Viallet 38000 Grenoble, France

[*]Dept. of Comp. Sci. University of Calgary
2500, University Drive N.W. Calgary, A.B. T2N 1N4

Abstract

In this paper, we consider the class of impartial combinatorial games for which the set of possible moves strictly decreases. Each game of this class can be considered as a domination game on a certain graph, called the move-graph. We analyze this equivalence for several families of combinatorial games, and introduce an interesting graph operation called *twin and match* that preserves the Grundy value. We then study another game on graphs related to the *dots and boxes* game, and we propose a way to solve it.

1 Introduction

In a graph $G = (V, E)$, we denote by $N_G(x)$ the neighborhood of $x \in V$, i.e. the set $\{y \in V/ \ y \textit{ is adjacent to } x\}$. When G is a directed graph, $N_G(x) = \{y \in V/ \textit{ there is an edge from } x \textit{ to } y\}$. The closed neighborhood of x is defined by $N_G[x] = N_G(x) \cup \{x\}$.
The *domination game* on a (directed) graph $G = (V, E)$ is the two-player game where each player chooses a vertex and removes its closed neighborhood from G. The first player unable to play loses.
We consider the class $\mathbb{I}$ of impartial games where the set of possible moves is finite, and strictly decreases after each player's turn. All the moves are available at the beginning and there is no new move that appears during the game.

*email: eric.duchene@imag.fr
†email: sylvain.gravier@imag.fr
‡email: mhalla@cpsc.ucalgary.ca

To any game of this class we associate a move-graph $G_m = (V, E)$, where V is the set of all possible moves of the graph. There is an edge from v_i to v_j if playing according to the move v_i forbids to play according to v_j in the continuation of the game.
We instantly deduce from these definitions that playing a game of $\mathbb{I}$ is equivalent to playing the domination game on its move-graph.

We now give some examples of this equivalence:
Given a partially-ordered set (poset) P, we define a poset game as a two-player game where each player alternately removes an element x from P and all the elements greater or equal to x. The player who removes the last element from P is the winner. By definition of a poset game, all of them are equivalent to a domination game on their move-graph.
The set of poset games includes lots of classical games : the game of Nim (see [4]), green Hackenbush (see [2]), the superset game (see [8]), or Chomp (see [7] or [5]) as examples. The latter is often played on a rectangular chocolate bar, where two players alternately select a square, remove (or eat) it and all the squares to the right and below it. The player eating the last square (the upper left one, supposed poisoned) loses the game.
Here is the move-graph of Chomp: we have a directed move-graph where each vertex corresponds to the selection of a square (see Figure 1).

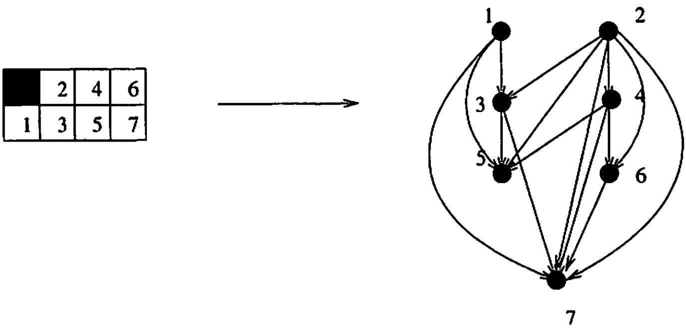

Figure 1: Move-graph of Chomp played on a 2×4 grid

Most removal games, such as the game of Nim and its variants, can be played on their move-graphs. Consider Wythoff's game (studied for the first time in [10]): two players alternately move from a given configuration, made up of two heaps of tokens. There are two different types of moves : removing any number of tokens from a single heap (the Nim rule), or removing the same number of tokens from both heaps. The winning player is the one taking the last token, the other loses as he is unable to move again. A game configuration is denoted by (a, b), where a and b are the number of tokens in each heap. As depicted below, the move-graph can be constructed with $(a + b + min(a, b))$ vertices.

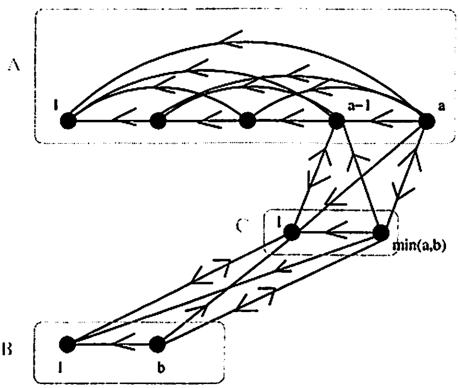

Figure 2: Move-graph of Wythoff's game from position (5,2)

Moves associated with the sets of vertices A and B define moves according to the Nim-rule (i.e. when a player removes tokens in a single heap), and are constructed as tournaments digraphs. A tournament digraph with n vertices is the digraph $T = (V, E)$, where $V = \{v_1, \dots, v_n\}$, and $(v_i, v_j) \in E$ iff $i > j$. Indeed, choosing the vertex labeled $(a - i)$ (resp. $(b - i)$) in the domination game amounts to leaving i tokens in the first (resp. second) heap. The set of vertices C defines moves according to both heaps. Its size is naturally equal to the minimum size of both heaps, and C is also build as a tournament digraph.
Edges from A/B to C : Moves leaving more than $min(a, b)$ tokens in a heap allow all moves according to both heaps, that is why there is no edge from such vertices to C. Moves leaving strictly less than $min(a, b)$ tokens in a heap forbid certain moves belonging to C : a move leaving $(min(a, b) - i)$ tokens in a heap has exactly i edges to the i smallest (according to their label) vertices of C.
Edges from C to A/B : Moves that remove i tokens in both heaps have i edges going to the greatest vertices of A and i other edges going to the greatest vertices of B (starting respectively at vertices $(a - min(a, b) + 1)$ and $(b - min(a, b) + 1)$). We use the fact that for all vertex u, $A \setminus u$ and $B \setminus u$ remain tournament graphs.

The move-graph is however not necessarily directed. This is the case when considering some octal games (see [2] and the description further), or the domino game (introduced by Conway), where two players alternately remove two adjacent squares (a domino) in a $m \times n$ grid. Its move-graph is depicted by figure 3, where vertices on the first and the third lines correspond to horizontal dominos and the second line to the vertical ones.

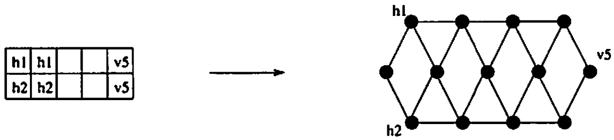

Figure 3: Move-graph of the domino game played on a 2×5 grid

The domination game is close to Fraenkel's game on hypergraphs (see [6]) (consider the move-graph as a hypergraph where a hyperedge represents an closed neighborhood). The difference is that removing vertices is not allowed.
As announced previously, we will refer in this paper to the set of octal games, introduced by Guy and Smith.

An octal game is a removal game played on heaps of tokens. At the beginning of the game, there is only one heap. Each octal game is encoded by an octal system, and can be written $.d_1d_2d_3\ldots$, with $d_i \in \{0\ldots 7\}$. The value of each d_i tells whether and how it is allowed to remove i adjacent tokens in a heap. Consider the binary coding of each d_i. It contains the two-power 2^k if and only if it is allowed to remove i adjacent tokens in a heap by splitting it into k non-empty heaps.

Consider for example the game .137. We have :

- $d_1 = 1$. Allowed to remove one token in a heap iff this token is the entire heap.
- $d_2 = 3 = 2 + 1$. Allowed to remove two adjacent tokens in a heap provided the heap is not splitted into two or more new heaps.
- $d_3 = 7 = 4 + 2 + 1$. Allowed to remove three adjacent tokens in a heap.
- $d_4 = d_5 = \ldots = 0$. By default. Not allowed to remove four or more tokens.

Note that these rules exactly define the domination game on a chain. The domino game played on a single row is an octal game encoded by .07.

We now present several classical definitions in the theory of combinatorial games: For any set S of nonnegative integers, we define $Mex(S)$ as the "Minimum excluded value" of S, i.e. the least nonnegative integer not in S.
For any game configuration C, the *options* of C are the set of possible resulting positions reachable from C. The *Grundy function* g associates to any game configuration C a positive integer value. It is generally recursively defined as $g(C) = Mex(g(F(C)))$, where $F(C)$ refers to the options of C. Zeros of the Grundy function correspond to second player win configurations (see [2]).

In the second section, we define an operation that preserves the Grundy value of a game belonging to $\mathbb{I}$. This leads to the construction of sets of equivalent games. In section 3, we study the particular case of powers of cycles. Section 4 is dedicated to a variant of the *dots and boxes* game.

2 Properties of the domination game

Given a graph $G = (V, E)$, an automorphism f of G is a bijection from V to V such that $(u, v) \in E$ iff $(f(u), f(v)) \in E$. f is a symmetric automorphism if $f = f^{-1}$.

Theorem 1 *If a graph G admits a symmetric automorphism s such that for every vertex u, $s(u) \notin N_G[u]$, then G is second player win for the domination game.*

proof:
Given any vertex $u \in V$, let $G' = G \setminus (N_G[u] \bigcup N_G[s(u)])$. Since $s^{-1} = s$, s remains an automorphism of G' such that for every vertex u, $s(u) \notin N_{G'}[u]$. If the first player chooses a vertex u, then a winning strategy for the second player consists in choosing the vertex $s(u)$. □

This theorem can be applied to particular cases of the domino game (which remains an open problem in the general case).

Corollary 1 *A configuration of the domino game is second player win if the length and the width of the grid are both even and first player win if they have a different parity.*

proof:
Consider the move-graph associated with the domino game on a $w \times l$ grid. Label the vertices of the graph with $h_{i,j}$ ($v_{i,j}$) for the move consisting in removing the horizontal (vertical) domino that starts in the square of index (i, j). The symmetric automorphism s that associates $h_{i,j}$ with $h_{w-i+1,l-j}$ and $v_{i,j}$ with $v_{w-i,l-j+1}$ satisfies the assumption of Theorem 1: on G if w and l are both even, on $G \setminus N_G[h_{(w+1)/2,l/2}]$ if w is odd and l even, and on $G \setminus N_G[v_{w/2,(l+1)/2}]$ if w is even and l odd. □

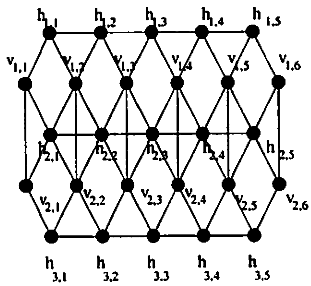

Figure 4: Move-graph of the domino game on a 3×6 grid

We now consider the case where the move-graph is a strong product of two other graphs.

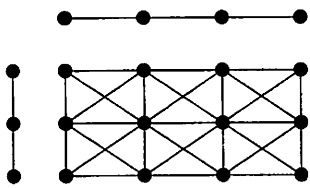

Figure 5: $P3 \otimes P4$

lemma 1 *If G_1 or G_2 is a second player win configuration, then $G = G_1 \otimes G_2$ is second player win too.*

proof:
If G_1 is second player win, then all the copies of G_1 are second player win too. When the first player chooses to move on a certain copy of G_1, his opponent keeps this property by applying its winning strategy on the same copy of G_1. □

We now introduce two graph operations preserving the Grundy value of a configuration of the domination game. Two vertices u and v are called *twins* if and only if $u \in N(v)$ and $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

lemma 2 *When playing the domination game on a graph G, the Grundy value of G is invariant by adding a twin v_0' to any vertex v_0.*

proof:
By induction, this is true for an isolated vertex (the empty graph is the unique option). Suppose now that the property is true for graphs with less than n vertices. Let $G = (V, E)$ be a configuration of the domination game with n vertices, and let v_0 be any vertex of G. Consider the graph $G_2 = twin_{v_0}(G)$ with $n+1$ vertices, and obtained from G by adding a twin v_0' to the vertex v_0 (v_0' is such that $N_{G_2}[v_0'] = N_G[v_0]$). For any vertex $u \in V$, removing $N_{G_2}[u]$ can lead to several options:

- if $u = v_0$ or $u = v_0'$ and since v_0 and v_0' are adjacent, the resulting graph is identical to the one obtained from G by choosing v_0.
- if u is a neighbor of v_0, then the resulting graph is the same as the one obtained from G by choosing u.
- if u is not in the neighborhood of v_0, then by induction hypothesis, the Grundy value of the resulting graph is the same as the one obtained from G by choosing u.

□

We call *twin and match* the operation that consists in twinning two non-adjacent vertices v_i and v_j, and adding a matching between the pairs of twins $\{v_i, v_i'\}$ and $\{v_j, v_j'\}$.

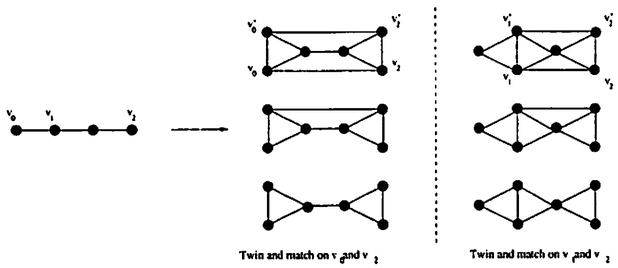

Figure 6: Examples of twin and match from a $P4$

Theorem 2 *The Grundy value of a configuration is invariant by the twin and match operation.*

proof:
If the matching contains no edges, then Lemma 2 concludes. Otherwise, the property is true for graphs with 2 vertices, as depicted below :

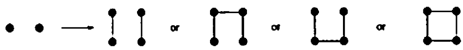

Figure 7: Twin and match operation applied on a stable of size two

Indeed, the stable of size 2 is second player win. When applying any twin and match operation on it, the resulting graph remains winning for the second player.
Suppose the theorem true for graphs with less than n vertices. Let $G = (V, E)$ be a configuration of the domination game with n vertices, and let v_0 and v_1 be any two non-adjacent vertices of G. Consider a graph G_2 with $n+2$ vertices, obtained from G by constructing matched twins $\{v_0, v_0'\}$ and $\{v_1, v_1'\}$. We consider several cases when removing the closed neighborhood of a vertex u:

- u is v_0 or v_0' and u is not a neighbor of v_1 or v_1'. Then $G_2 \setminus N_{G_2}[u] = twin_{v_1}(G \setminus N_G[v_0])$ and by Lemma 2 the Grundy value of the new graph is the same as the one obtained from G by choosing v_0.

- u is v_0 or v_0' and u is a neighbor of v_1 or v_1' (because of the matching). Then $G_2 \setminus N_{G_2}[u] = G \setminus N_G[v_0]$. The resulting graph is the same as the one obtained from G by choosing v_0.

- u belongs to both neighborhoods in G, i.e. $u \in N_G(v_0)$ and $u \in N_G(v_1)$. Then $G_2 \setminus N_{G_2}[u] = G \setminus N_G[u]$, and the resulting graph is identical to the one obtained from G by choosing u.

- u is in a neighborhood of only one of the twinned vertices. For example $u \in N_G[v_0]$ and $u \notin N_G[v_1]$. Hence we have $G_2 \setminus N[u] = twin_{v_1}(G \setminus N[u])$ and then by Lemma 2, the Grundy value of the resulting graph is the same as the one obtained from G by choosing u.

- u is outside both neighborhoods. Then the induction hypothesis ensures that the Grundy value of the resulting graph is identical to the one obtained from G by choosing u.

□

An example of equivalent games

In this section we use Theorem 2 to build an example of a new game equivalent to another known game.

From the octal game .07, we get a new domino game with an additional rule that allows to remove a trimino (three adjacent squares) starting at an index congruent to 2 or 3 mod 4.

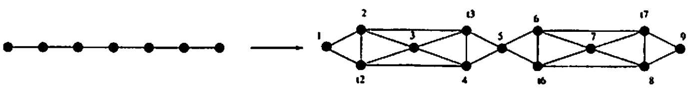

Figure 8: Equivalent graphs

Starting configuration: a row of n squares (squares of indices 2 or 3 mod 4 are white and the others black)

Rules: remove two adjacent squares or three adjacent squares starting with a white square

Analysis: the Grundy value has a pseudo-period 34 (see [2]) and we can apply the winning strategy of the domino game described in [9].

Figure 9: A .07 equivalent game

3 Domination game on powers of cycles

Denote by $C(n,k)$ the k^{th} power of a cycle of size n. Denote by $.0\ldots7_k$ the octal game where the only allowed move consists in removing k adjacent squares from a row. $g_{.0\ldots7_k}(n)$ defines the value of the Grundy function of this game when the starting configuration is a row of size n. $g(C(n,k))$ defines the Grundy value of the domination game when $G = C(n,k)$.

Theorem 3 $g(C(n,k)) = 1$ *iff* $g_{.0\ldots7_k}(n-k-1) = 0$
$g(C(n,k)) = 0$ *iff* $g_{.0\ldots7_k}(n-k-1) > 0$

proof:
From a $C(n,k)$ there exists a unique option, which is the k^{th} power of a chain of size $n-2k-1$ (denoted by $P(n-2k-1,k)$). Hence the Grundy value $g(C(n,k))$ is equal to 0 or 1 depending whether this power of chain is first or second player win.
Consider now the game $.0\ldots7_k$ played on a row of size $n-k-1$. Its move-graph is $P(n-2k-1,k)$, which concludes the proof. □

For example, $C(14,2)$ gives a $P(9,2)$ after one move, which is the move-graph of the octal game .007, called also the "trimino game" on a row with 11 squares.

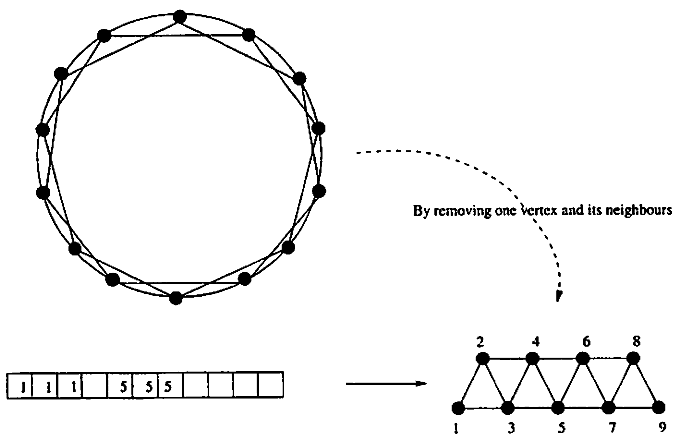

Figure 10: Move-graph for the trimino game on a chain with 11 squares

4 A forest removal game

In this section we study a game on graphs related to the *dots and boxes* game (see [1, 3]).
Given a graph G, we call *forest removal game* the two-player game where each player removes from G a set of edges constituting a forest. The first player unable to play loses.
In the *dots and boxes* game, a player removes an edge of a given graph G, and as long as each move disconnects a vertex of G, the same player plays again. The set of removed edges by a player in one turn is thus a forest. However, removing any forest does not correspond to an allowed move in the *dots and boxes* game.

We call a "non-adjacent cycles decomposition" of a graph G a covering of its edges by a pair (C, F), where C is a set of disjoint cycles (i.e. cycles having no common edge), and F is a forest with no common edge with C.

lemma 3 (non-adjacent cycles decomposition) *Any graph $G = (V, E)$ admits a non-adjacent cycles decomposition.*

proof:
By induction on the size of E. If there is no cycle, then G is a forest, otherwise remove the edges of a cycle from G and apply the induction hypothesis. □

lemma 4 *If a graph G has a non-adjacent cycles decomposition with an empty forest, then all its non-adjacent cycles decompositions have empty forests.*

proof:
By way of contradiction, suppose that a graph G has two non-adjacent cycles decomposition $(C_1, \emptyset)$ and (C_2, F). Let Δ be the symmetric difference. Then $C_1 \Delta C_2 = C_1 \setminus C_2 = F$ because C_1 contains all the edges of G. Since C_1 and C_2 are both sets of cycles, $F = C_1 \Delta C_2$ is also a set of cycles. □

The next theorem uses the following property, whose proof is immediate :
Graphs with a non-adjacent cycle decomposition of the form $(C, \emptyset)$ are graphs all of whose vertices have even degree.

Theorem 4 *Second player win configurations of this game are graphs all of whose vertices have an even degree.*

proof:
By induction on the number of edges.
It is true when the graph is a stable set.
Suppose that the graph admits a non-adjacent cycles decomposition with an empty forest. Remove a forest and the remaining graph has a non-adjacent cycles decomposition with a non-empty forest F. The other player chooses to remove F and lets a resulting graph, smaller than the previous one, and with a non-adjacent cycle decomposition containing only cycles. □

Acknowledgement
This research was partially supported by the Canadian Institute for Advanced Research (CIAR).

References

[1] E.R. Berlekamp. *The dot and boxes game.* A K Peters, 2000.

[2] E.R. Berlekamp, J.H. Conway and R.K. Guy. *Winning ways for your mathematical plays.* Academic Press, 1982.

[3] L. Blanc, E. Duchêne and S. Gravier *A deletion game on graphs: "le Pic'arete".* Cahiers du laboratoire Leibniz, Vol.107, 2004

[4] C.L. Bouton *Nim, a game with a complete mathematical theory.* Annals of Math. **3**, 35-39, 1905.

[5] S. Byrnes *Poset game periodicity.* Electr. J. Combin. Number Th. **3**, 2003.

[6] A.S. Fraenkel and E.R. Scheinerman *A deletion game on hypergraphs.* Discr. App.Math, Vol.30,155–162,1991.

[7] D. Gale *A curious nim-type game.* Amer. Math. Monthly **81**, 876-879, 1974.

[8] D. Gale and A. Neyman *Nim-types games.* Internat. J. Game Theory **11**, 17-20, 1982.

[9] S. Gravier, P. Jorrand, M. Mhalla and C. Payan *Let us play with qubits.* Cahiers du laboratoire Leibniz, Vol.98, 2003.

[10] W. Wythoff *A modification of the game of Nim.* Nieuw Arch. Wisk. **7**, 199-202, 1907.