# On a Conjecture Concerning the Friendly Index Sets of Trees

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**Abstract**

For a graph $G = (V, E)$ and a binary labeling $f : V(G) \to \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The labling $f$ is said to be friendly if $|v_f(1) - v_f(0)| \le 1$. Any vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = |f(x) - f(y)|$. Let $e_f(i) = |f^{*-1}(i)|$. The friendly index set of the graph G, denoted by $FI(G)$, is defined by

$$FI(G) = \{|e_f(1) - e_f(0)| : \ f \text{ is a friendly vertex labeling of } G\}.$$

In [15] Lee and Ng conjectured that the friendly index sets of trees will form an arithmetic progression. This conjecture has been mentioned in [17] and other manuscripts. In this paper we will first determine the friendly index sets of certain caterpillars of diameter four. Then we will disprove the conjecture by presenting an infinite number of trees whose friendly index sets do not form an arithmetic progression.

**Key Words:** Labeling; friendly labeling; friendly index set; caterpillars.

**AMS Subject Classification:** 05C78

## 1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [6]. Let $G = (V, E)$ be a graph and $f : V(G) \to \mathbb{Z}_2$ a vertex labeling (coloring) of $G$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring $f$ is said to be *friendly* if $|v_f(1) - v_f(0)| \le 1$. For example, consider the graph $H$, depicted in Figure 1, which consists of eight vertices. The provision $|v_f(1) - v_f(0)| \le 1$

stipulates that in order to have a friendly coloring, four vertices must be labeled 0, and the remaining four vertices must be labeled 1.
Any vertex labeling $f : V(G) \to \mathbb{Z}_2$ generates an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = |f(x) - f(y)|$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f^{*-1}(i)|$. The number $N(f) = |e_f(1) - e_f(0)|$ is called the *friendly index* of $f$. The *friendly index set* of the graph $G$, denoted by $FI(G)$, is defined by

$$FI(G) = \{N(f) : \ f \text{ is a friendly coloring of } G\}.$$

For example, $FI(H) = \{1,3,5,7\}$. Three other friendly colorings of $H$ which provide the indices $3, 5, 7$ are found in Figure 2.

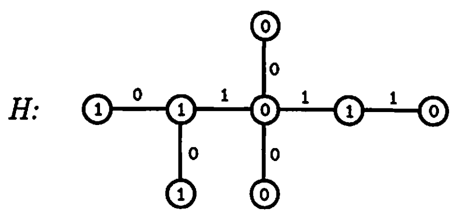


Figure 1: A typical friendly labeling of $H$ and its induced edge labeling.

For a graph $G$, the maximum element of $FI(G)$ is called the *maximum friendly index* of $G$ and the friendly coloring $f : V(G) \to \mathbb{Z}_2$ that provides this index is called the *maximum friendly coloring* of $G$. The following useful observation will be used on several occasions.

**Observation 1.1.** *If $f : V(G) \to \mathbb{Z}_2$ is a friendly coloring, then so is its complementary (inverse) coloring $g : V(G) \to \mathbb{Z}_2$ defined by $g(v) = 1 - f(v)$ $\forall v \in V(G)$. Furthermore, $N(g) = N(f)$.*

Readers interested in friendly colorings and friendly index sets of graphs are referred to a number of relevant literature that are mentioned in the bibliography section, including [15, 17].
In general, the elements of $FI(G)$ do not necessarily form an arithmetic progression. However, Lee-Ng [15] conjectured that the elements of the friendly index set of any tree will form an arithmetic progression. This has since been verified for several classes of trees [17] and is also supported by graph $H$. Nonetheless, in this paper we will determine the friendly index sets of a class of caterpillars of diameter 4 and will present an infinite number of trees whose friendly index sets do not form an arithmetic progression. First a few well known results [15, 17]:

**Theorem 1.2.** (Lee-Ng) *For any graph $G$ with $q$ edges, $FI(G) \subset \{q - 2i : i = 0, 1, 2, \cdots, \lfloor q/2 \rfloor\}$.*

**Theorem 1.3.** (Lee-Ng) *Let $1 \leq m \leq n$. For the complete bipartite graph $K_{m,n}$ we have*

$$FI(K_{m,n}) = \begin{cases} \{(m-2i)^2 : 0 \leq i \leq \lfloor m/2 \rfloor\} & \text{if } \ m+n \text{ is even;} \\ \{i(i+1) : 0 \leq i \leq m\} & \text{if } \ m+n \text{ is odd.} \end{cases}$$

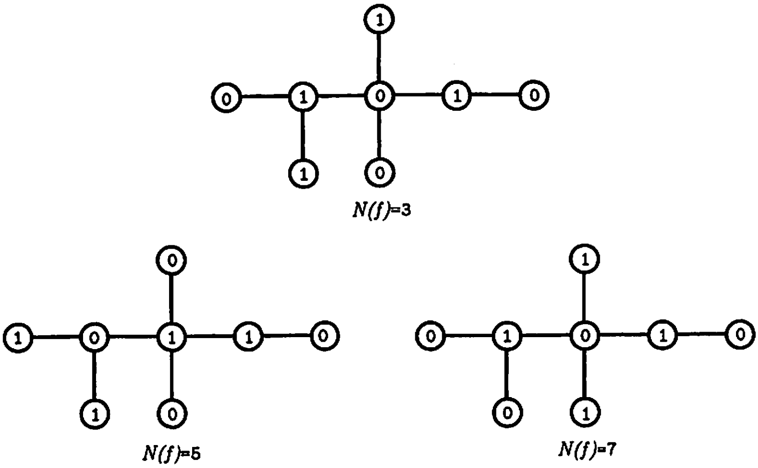


Figure 2: $FI(H) = \{1, 3, 5, 7\}$.

**Theorem 1.4.** (Lee-Ng) *The friendly index set of a full binary tree with depth $d > 1$ is*
$\{0, 2, 4, \cdots, 2^{d+1} - 4\}$.

**Theorem 1.5.** (Salehi-Lee) *If $T = (p, q)$ is a tree with perfect matching, then* $FI(T) = \{1, 3, 5, \cdots, q\}$.

**Theorem 1.6.** (Salehi-Lee) $FI(P_n) = \{n - 1 - 2i : 0 \le i \le \lfloor \frac{n-1}{2} \rfloor\}$, *where $P_n$ $(n \ge 2)$ is the path of order $n$.*

The Theorem 1.6 implies that for any natural number $n$ there is a connected graph $G$ such that $FI(G) = \{n - 2i : i = 1, 2, \cdots, \lfloor n/2 \rfloor\}$, which is an arithmetic progression with common difference being 2.

**Theorem 1.7.** (Salehi-Lee) *For $n \ge 3$, the friendly index set of the Fibonacci tree $FT_n$ is $\{ |E_n| - 2i : i = 0, 1, 2, \cdots, \lfloor q_n/2 \rfloor\}$.*

**Theorem 1.8.** (Salehi-Lee) *For $n \ge 3$, the friendly index set of the Lucas tree $LT_n$ is $\{0, 2, 4, \cdots, q\}$.*

## 2 Stars

For any $n \ge 1$, the complete bipartite graph $K(1, n)$ is called a *star* and is denoted by $ST(n)$.
The friendly index sets of stars can be obtained from Theorem 1.2. However, for the sake of completeness of this manuscript, we present a direct proof.

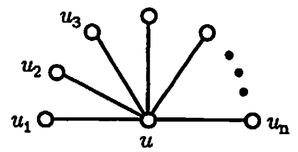


Figure 3: Star $ST(n) = K(1,n)$.

**Theorem 2.1.** *If* $n \geq 1$, *then* $FI(ST(n)) = \begin{cases} \{0,2\} & \text{if } n \text{ is even;} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$

*Proof.* Let $G = ST(n)$ and let $f : V(G) \to \mathbb{Z}_2$ be the labeling defined by

$$f(x) = \begin{cases} 0 & \text{if } x = u, u_1, u_2, \cdots, u_i; \\ 1 & \text{if } x = u_{n-i}, \cdots, u_{n-1}, u_n. \end{cases}$$

Then, $v_f(0) = i + 1$ and $v_f(1) = n - i$. Consequently, by the definition of the induced edge labeling, $i$ edges will be labeled 0, and $n - i$ edges will be labeled 1. Then the friendly index $N(f) = |e_f(1) - e_f(0)| = |(n-i) - i| = |n - 2i|$. We consider two cases:
**Case 1:** $n$ is even.
In order for $f$ to be a friendly labeling, either $i+1 = n-i+1$, or $i+1 = n-i-1$. With the former, $i = \frac{n}{2}$ and $N(f) = |n - 2(\frac{n}{2})| = 0$. With the latter, $i = \frac{n-2}{2}$ and $N(f) = |n - 2(\frac{n-2}{2})| = 2$.
**Case 2:** $n$ is odd.
In order for $f$ to be a friendly labeling, $i + 1 = n - i$. Thus, $i = \frac{n-1}{2}$ and $N(f) = |n - 2(\frac{n-1}{2})| = 1$. □

A *double star* is a tree of diameter 3. Double stars have two central vertices $u$ and $v$ and are denoted by $DS(a,b)$, where $\deg u = a$ and $\deg v = b$, as illustrated in Figure 4.

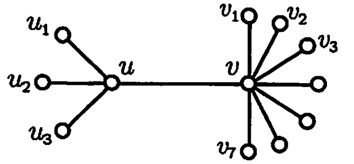


Figure 4: Double Star, $DS(4,8)$, with central vertices $u$ and $v$.

**Theorem 2.2.** *Let* $a \leq b$. *Then*

$$FI(DS(a,b)) = \begin{cases} \{1, 3, \cdots, 2a-1\} & \text{if } a+b \text{ is even;} \\ \{0, 2, \cdots, 2a\} & \text{if } a+b \text{ is odd.} \end{cases}$$

*Proof.* Consider $G = DS(a,b)$ with $a \leq b$, which has $a+b$ vertices. Let $f : V(G) \to \mathbb{Z}_2$ be a friendly labeling defined by $f(u) = 0$ and $f(v) = 1$, with the remaining vertices also labeled 1, excluding $f(u_1) = f(u_2) = \cdots = f(u_i) = 0$ and $f(v_1) = f(v_2) = \cdots = f(v_j) = 0$. In this way, $i$ of the vertices adjacent to vertex $a$ are labeled 0 and the remaining $a-i-1$ vertices adjacent to vertex $a$ (excluding vertex $b$) are labeled 1. Consequently, by the induced edge labeling, $i$ edges will be labeled 0 and $a-i$ edges (including edge $ab$) will be labeled 1. Likewise, $j$ of the vertices adjacent to vertex $b$ must be labeled 0 (excluding vertex $a$) and the remaining $b-j-1$ vertices adjacent to vertex $b$ must be labeled 1. So, $j$ edges will be labeled 1 and $b-j-1$ edges will be labeled 0. Therefore, $e_f(0) = b-j-1+i$ and $e_f(1) = a-i+j$ so that $N(f) = |e_f(0)-e_f(1)| = |(b-j-1+i)-(a-i+j)| = |b-a-1-2(j-i)|$.

**Case 1:** $a+b$ is even.
In order for $f$ to be a friendly labeling, we require that $i+j+1 = \frac{a+b}{2}$. Therefore, $N(f) = |e(1)-e(0)| = |2a-4i-1|$, where $0 \leq i \leq a-1$.

**Case 2:** $a+b$ is odd.
In order for $f$ to be a friendly labeling, $i+j+1 = a-i-1+(b-j-1)+1+1$, or $i+j+1 = a-i-1+(b-j-1)+1-1$. With the former, $j-i = \frac{a+b-4i-1}{2}$ and $N(f) = |b-a-1-2(\frac{a+b-4i-1}{2})| = |2a-4i|$. With the latter, $j-i = \frac{a+b-4i-3}{2}$ and $N(f) = |b-a-1-2(\frac{a+b-4i-3}{2})| = |2a-4i-2|$. When $0 \leq i \leq a-1$, the indices $0, 2, \cdots, 2a$ will be produced.

Note that if one assigns the same labels to the central vertices $u$, $v$; for example, $f(u) = f(v) = 1$, then $e(1) = i+j$ and $e(0) = a+b-i-j-1$. Hence $e(0)-e(1) = a+b-2(i+j)-1$. Since $f$ is friendly, then either $2(i+j) = a+b$ or $2(i+j) = a+b\pm1$. Therefore, $N(f)$ does not produce an additional index. □

**Observation 2.3.** If $\deg u = 1$, then $DS(a,b)$ would become $ST(b)$. Therefore, we can assume that $a \geq 2$. Nevertheless, we can see that Theorem 2.2 generates the same index set as Theorem 2.1. To illustrate, let $a = 1$. Then $i = 0$. We consider the two cases:
Case 1: $a+b$ is even. Since $a = 1$, $b$ must be odd. $N(f) = |2a-4i-1| = |2(1)-4(0)-1| = 1$.
Case 2: $a+b$ is odd. Since $a = 1$, $b$ must be even. $N(f) = |2a-4i| = |2(1)-4(0)| = 2$ and $N(f) = |2a-4i-2| = |2(1)-4(0)-2| = 0$.

**Examples 2.4.**

(a) $FI(DS(a,a)) = \{1, 3, \cdots, 2a-1\}$, where the maximum friendly index is equal to the number of edges in $DS(a,a)$.

(b) $FI(DS(4,4)) = \{1,3,5,7\}$, where $a = 4$ and $a+b = 4+4$ is even. The number of edges in $DS(4,4)$ is 7, which is also its maximum friendly index.

(c) $FI(DS(a,a+1)) = \{0, 2, \cdots, 2a\}$, where the maximum friendly index is also equal to the size of $DS(a,a+1)$.

(d) $FI(DS(5,6)) = \{0,2,4,6,8,10\}$, where $a = 5$ and $a+b = 5+6$ is odd. The number of edges in $DS(5,6)$ is 10, which is also its maximum friendly index.

(e) For $k \geq 2$, the friendly index sets of all double stars of the form $DS(a, a+k)$ with $2a+k$ even, are identical. Likewise, double stars of the form $DS(a, a+k)$ with $2a+k$ odd, have identical friendly index sets.

(f) $FI(DS(3,5)) = \{1,3,5\}$, where $a = 3$ and $a+b = 3+5$ is even.

(g) $FI(DS(3,11)) = \{1,3,5\}$, where $a = 3$ and $a+b = 3+11$ is even.

(h) $FI(DS(4,9)) = \{0,2,4,6,8\}$, where $a = 4$ and $a+b = 4+9$ is odd.

(i) $FI(DS(4,15)) = \{0,2,4,6,8\}$, where $a = 4$ and $a+b = 4+15$ is odd.

# 3 Caterpillars

A *caterpillar* is a tree having the property that the removal of its end-vertices results in a path (the spine). We use $CR(a_1, a_2, \cdots, a_n)$ to denote the caterpillar with a $P_n$-spine, where the $i$th vertex of $P_n$ has degree $a_i$. Since $CR(1, a_1, \cdots, a_n, 1) = CR(a_1, \cdots, a_n)$ and $a_i \neq 1$ $(2 \leq i \leq n-1)$, we will assume that $a_i \geq 2$.

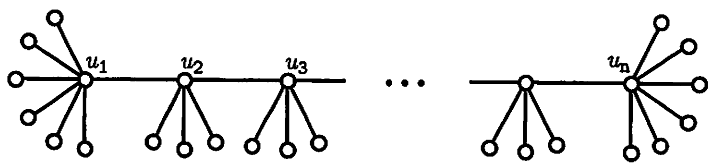


Figure 5: A Caterpillar of diameter $n+1$ ($P_n$-spine).

In this paper we will concentrate on caterpillars whose spines are $P_3$ and will use the notation $G = CR(a, b, c)$, where $\deg u = a$, $\deg v = b$, and $\deg w = c$, as illustrated in Figure 6. This caterpillar has $a+b+c-1$ vertices and $a+b+c-2$ edges.

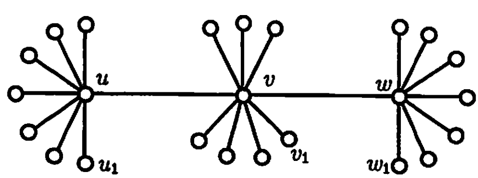


Figure 6: A Caterpillar of diameter 4, $CR(8,9,8)$.

**Theorem 3.1.** *Let $a, b, c \geq 2$ and $a+b+c$ be odd. Then $FI(CR(a,b,c)) = A \cup B \cup C$, where*

$$\begin{aligned} A &= \{\mid 2a - 4i - 1 \mid : m_A \leq i \leq M_A\}; \\ B &= \{\mid 2b - 4j - 1 \mid : m_B \leq j \leq M_B\}; \\ C &= \{\mid 2c - 4k - 1 \mid : m_C \leq k \leq M_C\}; \end{aligned}$$

*and*

$$m_A = \max\{0, \tfrac{a-b-c+3}{2}\}; \quad M_A = \min\{a-1, \tfrac{a+b+c-3}{2}\};$$

$$m_B = \max\{0, \tfrac{-a+b-c+1}{2}\}; \quad M_B = \min\{b-2, \tfrac{a+b+c-3}{2}\};$$

$$m_C = \max\{0, \tfrac{-a-b+c+3}{2}\}; \quad M_C = \min\{c-1, \tfrac{a+b+c-3}{2}\};$$

*Proof.* Assume $a, c > 1, b \geq 2$ and let $f : V(G) \to \{0,1\}$ be a friendly labeling. We will consider the following cases:
**Case 1.** Let $f(u) = 0$, and $f(v) = f(w) = 1$ be the labeling of the central vertices and all other labels be 1 except

$$\begin{array}{llll} f(u_1) & = f(u_2) & = \cdots & = f(u_i) = 0; \\ f(v_1) & = f(v_2) & = \cdots & = f(v_j) = 0; \\ f(w_1) & = f(w_2) & = \cdots & = f(w_k) = 0. \end{array} \tag{3.1}$$

Then $v_f(0) = i + j + k + 1$ and $v_f(1) = a + b + c - i - j - k - 2$. For this labeling to be friendly we need

$$i + j + k + 1 = \frac{a + b + c - 1}{2}, \tag{3.2}$$

which implies $|e(1) - e(0)| = |2a - 4i - 1|$. Moreover, $i + 1 \leq (a + b + c - 1)/2$ and $a - i + 1 \leq (a + b + c - 1)/2$, which provide the inequalities

$$(a - b - c + 3)/2 \leq i \leq (a + b + c - 3)/2.$$

Therefore, the possible friendly indices obtained in this case would be

$$A = \{|2a - 4i - 1| : m_A \leq i \leq M_A\}, \tag{3.3}$$

where $m_A = \max\{0, (a - b - c + 3)/2\}$ and $M_A = \min\{a - 1, (a + b + c - 3)/2\}$. The label assignments $f(u) = 1$, and $f(v) = f(w) = 0$, will result in $|e(1) - e(0)| = |2a - 4i - 3|$, which is the complementary labeling and will provide the same friendly indices.
**Case 2.** We only change the labels of the central vertices by $f(u) = f(w) = 1$, and $f(v) = 0$. Then with a similar argument as presented in the case 1, we may get the friendly indices

$$B = \{|2b - 4j - 1| : m_B \leq j \leq M_B\}, \tag{3.4}$$

where $m_B = \max\{0, (-a + b - c + 1)/2\}$ and $M_B = \min\{b - 2, (a + b + c - 3)/2\}$. The label assignments $f(u) = f(w) = 0$ and $f(v) = 1$, will result in $|e(1) - e(0)| = |2b - 4j - 7|$, which is the complementary labeling and will provide the same friendly indices.
**Case 3.** We only change the labels of the central vertices by $f(u) = f(v) = 1$, and $f(w) = 0$. Then with a similar argument as presented in the case 1, the possible friendly indices would be

$$C = \{|2c - 4k - 1| : m_C \leq j \leq M_C\}, \tag{3.5}$$

where $m_C = \max\{0, (-a-b+c+3)/2\}$ and $M_C = \min\{c-1, (a+b+c-3)/2\}$. The label assignments $f(u) = f(v) = 0$ and $f(w) = 1$, will result in $|e(1) - e(0)| = |2c-4k-3|$, which is the complementary labeling and will provide the same friendly indices. Thus, $FI(G)$ is a subset of the union of the three sets (3.3), (3.4) and (3.5).

**Case 4.** If we assign the same labels to the central vertices $u$, $v$, and $w$; for example, $f(u) = f(v) = f(w) = 1$, then $e(1) = i+j+k$ and $e(0) = a+b+c-i-j-k-2$. Hence $e(0) - e(1) = a+b+c-2(i+j+k)-2$. Since $f$ is friendly and there are an even number $a+b+c-1$ of vertices, then $2(i+j+k) = a+b+c-1$. Therefore, $N(f) = 1$ is a member of $FI(CR(a,b,c))$.

This shows that $FI(CR(a,b,c)) \subset A \cup B \cup C$. On the other hand, every element of $A \cup B \cup C$ is generated by a friendly coloring of $CR(a,b,c)$, which implies that $FI(CR(a,b,c)) = A \cup B \cup C$. □

**Observation 3.2.** *If $a = c = 1$, then $FI(CR(1,b,1)) = \{1\}$.*

*Proof.* Note that if $a = c = 1$, then $CR(1,b,1) = ST(b)$ and $m_A = M_A = m_C = M_C = 0$ or $A = C = \{1\}$. Also, $m_B = M_B = \dfrac{b-1}{2}$, which implies that $B = \{1\}$ or $FI(CR(1,b,1)) = \{1\}$. This result is consistent with Theorem 2.1. □

**Observation 3.3.** *If $a = 1$ and $b, c \geq 2$, then $FI(CR(1,b,c)) = \{1, 3, \ldots, 2x-1\}$, where $x = \min\{b,c\}$.*

*Proof.* Since $a = 1$, then $m_A = M_A = 0$ or $A = \{1\}$. Also, $CR(1,b,c) = DS(b,c)$ and $a+b$ is even, which implies that $a$ and $b$ have the same parity. Without loss of generality we may assume that $b \leq c$. We will consider the following three cases:

**Case 1.** If $b = c$, then $m_B = 0$, $M_B = b-2$, $m_C = 1$ and $M_C = b-1$. Therefore,

$$\begin{aligned} B &= \{|\, 2b - 4j - 1 \,| : 0 \leq j \leq b-2\}; \\ C &= \{|\, 2b - 4k - 1 \,| : 1 \leq k \leq b-1\}, \end{aligned}$$

consequently, $B \cup C = \{|\, 2b - 4k - 1 \,| : 0 \leq k \leq b-1\} = \{1, 3, \cdots, 2b-1\}$.

**Case 2.** If $c = b+2$, then $m_B = 0$, $M_B = b-2$, $m_C = 2$ and $M_C = b$. Therefore,

$$\begin{aligned} B &= \{|\, 2b - 4j - 1 \,| : 0 \leq j \leq b-2\}; \\ C &= \{|\, 2b + 4 - 4k - 1 \,| : 2 \leq k \leq b\} = \{|\, 2b - 4k - 1 \,| : 1 \leq k \leq b-1\}, \end{aligned}$$

consequently, $B \cup C = \{|\, 2b - 4k - 1 \,| : 0 \leq k \leq b-1\} = \{1, 3, \cdots, 2b-1\}$.

**Case 3.** If $c > b+2$, then $m_B = 0$, $M_B = b-1$, $m_C = \dfrac{c-b}{2}$ and $M_C = \dfrac{b+c-2}{2}$. In this case, $B = C = \{1, 3, \cdots, 2b-1\}$, which completes the proof. □

The result of Observation 3.2 is consistent with the Theorem 2.1. Also, the result of Observation 3.3 is consistent with the Theorem 2.2. The combination of these two observations and Theorem 3.1 is summarized in the following theorem:

**Theorem 3.4.** *Let $a, c \geq 1$, $b \geq 2$ and $a+b+c$ be odd. Then $FI(CR(a,b,c)) = A \cup B \cup C$, where*

$$
\begin{aligned}
A &= \{\mid 2a-4i-1 \mid : m_A \le i \le M_A\};\\
B &= \{\mid 2b-4j-1 \mid : m_B \le j \le M_B\};\\
C &= \{\mid 2c-4k-1 \mid : m_C \le k \le M_C\};
\end{aligned}
$$

*and*

$$
\begin{aligned}
m_A &= \max\{0, \tfrac{a-b-c+3}{2}\}; & M_A &= \min\{a-1, \tfrac{a+b+c-3}{2}\};\\
m_B &= \max\{0, \tfrac{-a+b-c+1}{2}\}; & M_B &= \min\{b-2, \tfrac{a+b+c-3}{2}\};\\
m_C &= \max\{0, \tfrac{-a-b+c+3}{2}\}; & M_C &= \min\{c-1, \tfrac{a+b+c-3}{2}\};
\end{aligned}
$$

# 4 Counter Examples

Another useful observation of Theorem 3.4 is that if $a = c$, then the boundaries for $i$ and $k$ are the same, i.e. $A = C$. For example, when $G = CR(3,5,3)$, we have $A = C = \{1,3,5\}$ and $B = \{1,3,5,9\}$, i.e. $FI(CR(3,5,3)) = \{1,3,5,9\}$. We observe that the elements of the friendly index set of the caterpillar $CR(3,5,3)$ do *not* form an arithmetic progression. The next theorem will provide an infinite number of trees whose friendly index sets do not form an arithmetic progression.

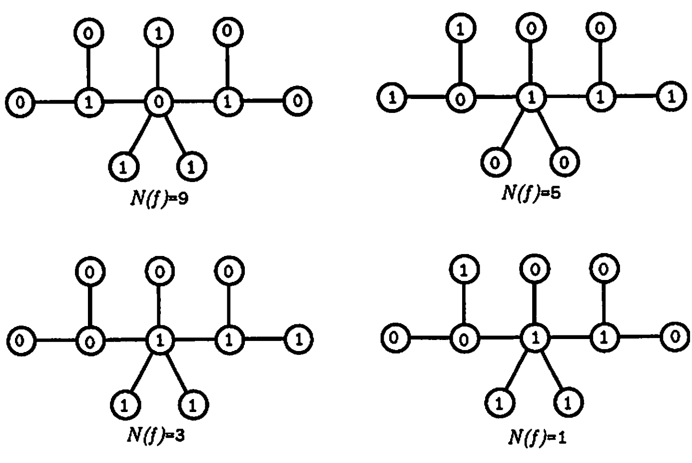


Figure 7: $FI(CR(3,5,3)) = \{1,3,5,9\}$.

**Theorem 4.1.** *If $n \ge 3$, then the elements of the friendly index set for the caterpillar of the form $CR(n, 2n-1, n)$, do not form an arithmetic progression.*

*Proof.* Let $n \ge 3$ and consider the graph $G = CR(n, 2n-1, n)$. Note that $a+b+c = 4n-1$ is odd. Therefore, by Theorem 3.4, $FI(G) = A \cup B \cup C$. Also, for this graph, $A = C = \{|2n-1-4i| : 0 \le i \le n-1\} = \{1, 3, 5, \cdots, 2n-1\}$ and $B = \{|4n-3-4j| : 0 \le j \le 2n-3\} = \{1, 3, \cdots, 4n-9, 4n-7, 4n-3\}$. Since

$A \subset B$, then $FI(G) = B$. We observe that $B$ has $2n - 2$ odd numbers with $\min B = 1$ and $\max B = 4n - 3$. Thus, one odd number is missing. In fact, the value $4n - 5$ cannot be obtained, thereby preventing the elements of the set from forming an arithmetic progression. □

Therefore, by Theorem 4.1 we see that caterpillars $CR(n, 2n - 1, n)$ provide infinitely many counterexamples to disprove the conjecture.

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