Bases of primitive non-powerful signed symmetric digraphs with loops*

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Abstract

Let S be a primitive non-powerful signed digraph. The base $l(S)$ of S is the smallest positive integer l such that for all ordered pairs of vertices i and j (not necessarily distinct), there exists a pair of $SSSD$ walks of length t from i to j for each integer $t \geq l$. In this work, we use $PNSSD$ to denote the class of all primitive non-powerful signed symmetric digraphs of order n with at least one loop. Let $l(n)$ be the largest value of $l(S)$ for $S \in PNSSD$, and $L(n) = \{l(S) \mid S \in PNSSD\}$. For $n \geq 3$, we show $L(n) = \{2, 3, \ldots, 2n\}$. Further, we characterize all primitive non-powerful signed symmetric digraphs of order n with at least one loop whose bases attain $l(n)$.

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1 Introduction

Let D be a digraph (permits loops but no multiple arcs). Digraph D is called *primitive* if there is a positive integer k such that for all ordered pairs of vertices i and j (not necessarily distinct) in D, there exists a walk of length k from i to j([1]).

A *signed digraph* S is a digraph where each arc of S is assigned a sign 1 or -1. The *sign* of the walk W (in a signed digraph), denoted by $\text{sgn}(W)$, is defined to be the product of signs of all arcs in W. Two walks W_1 and

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W_2 in a signed digraph is called *a pair of SSSD walks*, if they have the same initial vertex, same terminal vertex, same length, but different signs. A signed digraph S is called *powerful* if S contains no pair of $SSSD$ walks.

Let S be a primitive non-powerful signed digraph. For any $i, j \in V(S)$, we define the *base from i to j*, denoted by $l_S(i,j)$, to be the smallest positive integer p such that for each integer $t \geq p$, there exists a pair of $SSSD$ walks of length t from i to j. The *base* of S, denoted by $l(S)$, is defined to be the smallest positive integer l such that for all ordered pairs of vertices i and j (not necessarily distinct), there exists a pair of $SSSD$ walks of length t from i to j for each integer $t \geq l$. Clearly, $l(S) = \max\{l_S(i,j) \mid i, j \in V(S)\}$.

A digraph D is *symmetric* if for any $i, j \in V(D)$, (i,j) is an arc if and only if (j,i) is an arc. A *signed symmetric digraph* S is a symmetric digraph where each arc of S is assigned a sign 1 or -1, and the sign of (i,j) may be different from the sign of (j,i).

In this work, we use $PNSSD$ to denote the class of all primitive non-powerful signed symmetric digraphs of order n with at least one loop. Let $l(n)$ be the largest value of $l(S)$ for $S \in PNSSD$, and $L(n) = \{l(S) \mid S \in PNSSD\}$. For $n \geq 3$, we show $L(n) = \{2, 3, \ldots, 2n\}$. Further, we characterize all primitive non-powerful signed symmetric digraphs of order n with at least one loop whose bases attain $l(n)$.

2 Some preliminaries

Lemma 2.1 ([2]) *Let S be a primitive signed digraph. Then S is non-powerful if and only if S contains a pair of cycles C_1 and C_2 (of lengths p_1 and p_2, respectively) satisfying one of the following two conditions:*

(1) *p_1 is odd, p_2 is even and* $\mathrm{sgn}C_2 = -1$;

(2) *Both p_1 and p_2 are odd and* $\mathrm{sgn}C_1 = -\mathrm{sgn}C_2$.

For convenience, we call a pair of cycles C_1 and C_2 satisfying (1) or (2) in Lemma 2.1 a *distinguished cycle pair.* If C_1 and C_2 form a distinguished cycle pair of lengths p_1 and p_2, respectively, then the closed walks $W_1 = p_2C_1$ (walk around C_1 p_2 times) and $W_2 = p_1C_2$ have the same length p_1p_2 but with different signs since $(\mathrm{sgn}C_1)^{p_2} = -(\mathrm{sgn}C_2)^{p_1}$.

Let $R = \{C_1, \ldots, C_r\}$ be the set of some distinct cycles of signed digraph S. For any $x, y \in V(S)$, $d_R(x,y)$ denotes the length of the shortest walk from x to y which meets at least one vertex of C_i for each $i = 1, \ldots, r$. The following is clear.

Lemma 2.2 *Let S be a primitive non-powerful signed digraph with at least one loop, and C_1 and C_2 be a distinguished cycle pair of lengths p_1 and p_2, respectively. Denote $R = \{C_1, C_2\}$. If* $\min\{p_1, p_2\} = 1$, *then $l_S(i,j) \leq d_R(i,j) + p_1p_2$ for any $i, j \in V(S)$.*

3 Main results

Theorem 3.1 *Let $n \geq 3$ and $S \in PNSSD$. Then $l(S) \leq 2n$, and the equality can occur.*

Proof Let C_1 be a loop of S. Since S is primitive non-powerful, by Lemma 2.1, there is a cycle C_2 of length m (m-cycle, for short) in S such that C_1 and C_2 form a distinguished cycle pair. Denote $R = \{C_1, C_2\}$. For any $i, j \in V(S)$, we consider the following three cases.

Case 1. $m = 1$. Then $d_R(i,j) \leq 2(n-1)$ and $l_S(i,j) \leq 2(n-1)+1 = 2n-1$ by Lemma 2.2.

Case 2. $m = 2$. Then $d_R(i,j) \leq 2(n-1)$ and $l_S(i,j) \leq 2(n-1)+2 = 2n$ by Lemma 2.2.

Case 3. $m \geq 3$. If m is odd, then $d_R(i,j) \leq 2(n - \frac{m+1}{2})$ and $l_S(i,j) \leq 2(n - \frac{m+1}{2}) + m = 2n-1$ by Lemma 2.2. If m is even, then $d_R(i,j) \leq 2(n - \frac{m}{2})$ and $l_S(i,j) \leq 2(n - \frac{m}{2}) + m = 2n$ by Lemma 2.2.

Combining the above cases, we have $l(S) \leq 2n$.

On the other hand, take $S_1 \in PNSSD$ with D_1 (as given in Figure 1) as the underlying digraph and contains at least one negative 2-cycle.

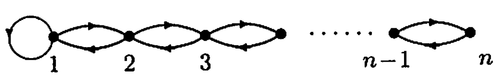

Fig. 1 Digraph D_1

Since there exists unique walk in D_1 of length $2n-1$ from n to n, so there is no pair of $SSSD$ walks in S_1 of length $2n-1$ from n to n and $l(S_1) = 2n$. □

Corollary 3.2 *For $n \geq 3$, $l(n) = 2n$.*

Lemma 3.3 *For $n \geq 3$, and $1 \leq k \leq n-1$, $2k+2 \in L(n)$.*

Proof Let $1 \leq k \leq n-1$. Take $S \in PNSSD$ with D_2 (as given in Figure 2) as the underlying digraph, the arc (k, n) of S is negative, and the other arcs of S are positive. We shall show $l(S) = 2k+2$.

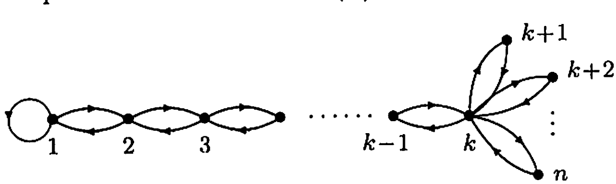

Fig. 2 Digraph D_2

The loop at vertex 1, denoted by C_1, and the negative 2-cycle $k \to n \to k$, denoted by C_2, form a distinguished cycle pair of S. Denote $R = \{C_1, C_2\}$. For any $i, j \in V(S)$, $d_R(i,j) \le 2k$, and so $l_S(i,j) \le 2k+2$ by Lemma 2.2. Then $l(S) \le 2k+2$. On the other hand, since there exists unique walk in D_2 of length $2k+1$ from n to n, so there is no pair of $SSSD$ walks in S of length $2k+1$ from n to n and $l(S) = 2k+2$. □

Lemma 3.4 *For $n \ge 3$, and $1 \le k \le n-1$, $2k+1 \in L(n)$.*

Proof Let $1 \le k \le n-1$. Take $S \in PNSSD$ such that its underlying digraph is the digraph obtained from D_2 by adding loops at vertices $k+1, k+2, \ldots, n$, respectively, the loop at vertex 1 is negative, and the other arcs are positive. We shall show $l(S) = 2k+1$.

For any $i, j \in V(S)$, since there exists a walk in S of length $2k$ from i to j such that it meets both a negative loop and a positive loop, so $l_S(i,j) \le 2k+1$ by Lemma 2.2 and $l(S) \le 2k+1$. On the other hand, since each walk in S of length $2k$ from n to n is positive, so there is no pair of $SSSD$ walks in S of length $2k$ from n to n and $l(S) = 2k+1$. □

Lemma 3.5 *For $n \ge 3$, $2 \in L(n)$.*

Proof Take $S \in PNSSD$ such that its underlying digraph is the symmetric complete digraph with a loop at each vertex, the arcs $(2,1), (3,1), \ldots, (n,1)$ and the loop at vertex 1 are negative, and the other arcs are positive. For any $i, j \in V(S)$, we shall show that there exists a pair of $SSSD$ walks in S of length l from i to j for each integer $l \ge 2$.

Case 1. $i = j$. If $i \ne 1$, then $i \to i \to i$ and $i \to 1 \to i$ form a pair of $SSSD$ walks of length 2 from i to j. If $i = 1$, then $1 \to 1 \to 1$ and $1 \to 2 \to 1$ form a pair of $SSSD$ walks of length 2 from i to j.

Case 2. $i \ne j$ and $2 \le i, j \le n$. Then $i \to 1 \to j$ and $i \to j \to j$ form a pair of $SSSD$ walks of length 2 from i to j.

Case 3. $i = 1$ and $j \ge 2$ (or $j = 1$ and $i \ge 2$). Then $i \to i \to j$ and $i \to j \to j$ form a pair of $SSSD$ walks of length 2 from i to j.

Since there exists a loop at each vertex, there exists a pair of $SSSD$ walks in S of length l from i to j for each integer $l \ge 2$. Noticing that $l(S) \ge 2$ for any $S \in PNSSD$, so $l(S) = 2$. □

Note: $1 \notin L(n)$ for $n \ge 3$. Combining Theorem 3.1 and Lemmas 3.3–3.5, we obtain the following theorem.

Theorem 3.6 *For $n \ge 3$, $L(n) = \{2, 3, \ldots, 2n\}$.*

4 The extremal signed symmetric digraphs

In this section, we characterize all primitive non-powerful signed symmetric digraphs of order n with at least one loop whose bases attain $l(n)$.

For a digraph D and any $x, y \in V(D)$, we use $d(D)$ and $d(x, y)$ to denote the diameter of D and the distance from x to y in D, respectively.

Lemma 4.1 *Let $n \geq 3$, $S \in PNSSD$ with D as the underlying digraph and there exist at least one negative 2-cycle. Then $l(S) = 2n$ if and only if D is isomorphic to D_1.*

Proof Sufficiency is immediate from the proof of Theorem 3.1. We now consider the necessity. Let C_1 and C_2 be a loop and negative 2-cycle, respectively. Then C_1 and C_2 form a distinguished cycle pair of S. Denote $R = \{C_1, C_2\}$. For any $i, j \in V(S)$, if $d(D) \leq n-2$, then $d_R(i, j) \leq 2(n-2)$. By Lemma 2.2, $l_S(i, j) \leq 2(n-2)+2 = 2n-2$ contradicting $l(S) = 2n$. So $d(D) = n-1$. Without loss of generality, let $d(1, n) = n-1$, and the shortest path in D from 1 to n is $1 \to 2 \to \cdots \to n$. If either there exists a loop at vertex x, where $x \neq 1$ and $x \neq n$, or there exist loops at both vertices 1 and n, then $d_R(i, j) \leq 2(n-2)$. By Lemma 2.2, $l_S(i, j) \leq 2(n-2)+2 = 2n-2$ contradicting $l(S) = 2n$. Thus there exists a loop only at vertex 1 or n, and D is isomorphic to D_1. □

Lemma 4.2 *Let $n \geq 3$ and $S \in PNSSD$. If each 2-cycle of S is positive, then $l(S) \leq 2n - 1$.*

Proof Let C_1 be a loop of S. Since S is primitive non-powerful, by Lemma 2.1, there is a m-cycle C_2 ($m \neq 2$) in S such that C_1 and C_2 form a distinguished cycle pair. If m is odd, then $l(S) \leq 2n - 1$ by the proof of Theorem 3.1. If m is even, then $m \geq 4$ and $d(D) \leq n - \frac{m}{2}$. Denote $R = \{C_1, C_2\}$. For any $i, j \in V(S)$, if $d(D) \leq n - \frac{m}{2} - 1$, then $d_R(i, j) \leq 2(n - \frac{m}{2} - 1)$, and $l_S(i, j) \leq 2(n - \frac{m}{2} - 1) + m = 2n - 2$ by Lemma 2.2. If $d(D) = n - \frac{m}{2}$, without loss of generality, let $d(1, n - \frac{m}{2} + 1) = n - \frac{m}{2}$, the shortest path in D from 1 to $n - \frac{m}{2} + 1$ be $1 \to 2 \to \cdots \to n - \frac{m}{2} + 1$, and $C_m = k \to k+1 \to \cdots \to k + \frac{m}{2} \to n \to n-1 \to n - \frac{m}{2} + 2 \to k$, where $1 \leq k \leq n - m + 1$. Consider the following cases.

Case 1. Either there exists a loop at vertex x, where $x \neq 1$ and $x \neq n - \frac{m}{2} + 1$, or there exist loops at both vertices 1 and $n - \frac{m}{2} + 1$. Then $d_R(i, j) \leq 2(n - \frac{m}{2} - 1)$, and $l_S(i, j) \leq 2(n - \frac{m}{2} - 1) + m = 2n - 2$ by Lemma 2.2.

Case 2. There exists a loop only at vertex 1 or $n - \frac{m}{2} + 1$. Without loss of generality, let there exist a loop at vertex 1. Since each 2-cycle of S is positive and C_2 is a negative even cycle, then $k \to k+1 \to \cdots \to k + \frac{m}{2}$

and $k \to n-\frac{m}{2}+2 \to n-\frac{m}{2}+3 \to \cdots \to n \to k+\frac{m}{2}$ form a pair of $SSSD$ walks of length $\frac{m}{2}$ from k to $k+\frac{m}{2}$. If either $i \neq n-\frac{m}{2}+1$ or $j \neq n-\frac{m}{2}+1$, then $d_R(i,j) \leq 2(n-\frac{m}{2}-1)+1$, and $l_S(i,j) \leq 2(n-\frac{m}{2}-1)+1+m = 2n-1$ by Lemma 2.2. If $i = n-\frac{m}{2}+1$ and $j = n-\frac{m}{2}+1$, then for $l \geq 2n-m$,

$$W_1 = (n-\frac{m}{2}+1 \to n-\frac{m}{2} \to \cdots \to 1)+(l-2n+m)C_1$$

$$+(1 \to \cdots \to k \to k+1 \to \cdots \to k+\frac{m}{2} \to \cdots \to n-\frac{m}{2}+1)$$

and

$$W_2 = (n-\frac{m}{2}+1 \to n+\frac{m}{2} \to \cdots \to 1)+(l-2n+m)C_1+(1 \to \cdots$$

$$\to k \to n-\frac{m}{2}+2 \to n-\frac{m}{2}+3 \to \cdots \to n \to k+\frac{m}{2} \to \cdots \to n-\frac{m}{2}+1)$$

form a pair of $SSSD$ walks of length l from i to j and $l_S(i,j) \leq 2n-m < 2n-1$.

Combining the above cases, we have $l(S) \leq 2n-1$. □

By Lemmas 4.1 and 4.2, we have the following result.

Theorem 4.3 *Let $n \geq 3$, $S \in PNSSD$ with D as the underlying digraph. Then $l(S) = 2n$ if and only if there exists at least one negative 2-cycle in S, and D is isomorphic to D_1.*

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