Some Properties of Macula's matrix and its complement *

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Abstract

Anthony J. Macula constructed a $d-$disjunct matrix $\delta(n,d,k)$ in [1], and we now know it is determined by one type of pooling space. In this paper, we give some properties of $\delta(n,d,k)$ and its complement $\delta^c(n,d,k)$.

Key words: $d-$disjunct matrix pooling design

1 Introduction

Group testing has many applications such as screening blood samples for diseases, screening vaccines for contamination and DNA library screening. A group testing algorithm is non-adaptive if all tests must be specified without knowing the outcomes of other tests and a mathematical model of non-adaptive group testing design is a $d-$disjunct matrix. A group testing algorithm is error tolerant if it can detect or correct some e errors in test outcomes. We know if we view the $d-$disjunct matrices as $i-$disjunct matrices$(0 < i < d)$, then they can detect e errors. In this paper we count the number e for each i with $0 < i < d$ and we show if $\delta(n,d,k)$ is $d-$disjunct, then $\delta^c(n,d,k)$ is $m-$disjunct for some m with $0 \le m \le n$.

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2 Preliminary results

Let n be a positive integer and $[n]$ denote $\{1, 2, \cdots, n\}$. Let $(\binom{n}{j})$ denote the family of $j-$subsets of $[n]$. For $d < k < n$, we define the $\binom{n}{d} \times \binom{n}{k}$ $\{0,1\}$ matrix $\delta(n,d,k)([1])$ by letting the rows and the columns be, respectively, represented by the members of $(\binom{n}{d})$ and $(\binom{n}{k})$ in the following way: For a given $D \in (\binom{n}{d})$ and $K \in (\binom{n}{k})$, the matrix $\delta(n,d,k)$ has a 1 in its (D,K)th entry if and only if $D \subset K$.

Consider a $t \times n$ $\{0,1\}$ matrix μ. Let R_i and C_j denote row i and column j respectively. Abusing notation, we also let R_i (resp. C_j) denote the set of column (resp. row) indices corresponding to the 1 entries.

Definition 2.1. *([2]) A $t \times n$ matrix μ is said to be $d-$disjunct if the union of any d columns does not contain another column.*

Definition 2.2. *([2]) A $t \times n$ μ is said to be $(d,e)-$disjunct if for any $d+1$ columns $C_0, C_1, \cdots, C_d$ of μ there are at least $e+1$ elements in $C_0 - \bigcup_{i=0}^{d} C_i$.*

The definition also can described in this way: A $d-$disjunct matrix μ is called $(d,e)-$disjunct if and only if given any $d+1$ columns of μ with one designated, there are $e+1$ rows with a 1 in the designated column and a 0 in each of the other d columns. From a coding theory point of view, a $(d,e)-$disjunct matrix is equivalent to a superimposed distance code with strength d and distance $e+1$.

Proposition 2.3. *([3, 4]) A matrix μ is $d-$disjunct if and only if it is $(d,0)-$disjunct.*

Proposition 2.4. *([1]) $\delta(n,d,k)$ is a $\binom{n}{d} \times \binom{n}{k}$ $d-$disjunct matrix with column weight $\binom{k}{d}$ and row weight $\binom{n-d}{k-d}$.*

Proposition 2.5. *([5, 6]) $\delta(n,s,k)(d \le s < k)$ is a $(d, \binom{k-d}{s-d} - 1)-$disjunct matrix.*

From Proposition 2.5 we can easily have that Matrix $\delta(n,d,k)$ is

$$\begin{aligned}
(d,0) &- disjunct,\\
(d-1,\tbinom{k-(d-1)}{d-(d-1)}-1) &- disjunct,\\
(d-2,\tbinom{k-(d-2)}{d-(d-2)}-1) &- disjunct,\\
&\vdots\\
(d-i,\tbinom{k-(d-i)}{d-(d-i)}-1) &- disjunct,\\
&\vdots\\
(d-(d-1),\tbinom{k-1}{d-1}-1) &- disjunct.
\end{aligned}$$

3 Main results

Theorem 3.1. *If the intersection of any m $k-$subsets in $(\binom{n}{k})$ has at least d elements, whereas the intersection of any $m+1$ $k-$subsets has at most $d-1$ elements, then the complement of $\delta(n,d,k)$, $\delta^c(n,d,k)$, is at most $m-$disjunct.*

Proof. Let $C_{j_0}, C_{j_1}, \cdots, C_{j_m}$ be $m+1$ columns of $\delta(n,d,k)$ with C_{j_0} being distinguished. We know there is a row with all 1 entries in columns $C_{j_1}, \cdots, C_{j_m}$ and there does not exist a row with all 1 entries in columns $C_{j_0}, C_{j_1}, \cdots, C_{j_m}$. So in matrix $\delta^c(n,d,k)$ there is a row with all 0 entries in columns $C_{j_1}, \cdots, C_{j_m}$ and there does not exist a row with all 0 entries in columns $C_{j_0}, C_{j_1}, \cdots, C_{j_m}$. Thus C_{j_0} does not contain in the union of $C_{j_1}, \cdots, C_{j_m}$. Therefore $\delta^c(n,d,k)$ is at most $m-$disjunct. □

For example, $\delta^c(5,2,3)$ is 1−disjunct, $\delta^c(6,2,4)$ is 1−disjunct, and $\delta^c(5,2,4)$ is 3−disjunct.

Corollary 3.2. *$\delta^c(n,d,n-1)$ is $(n-d)-$disjunct.*

Proof. It is easy to see that the intersection of any m columns of $\delta(n,d,n-1)$ has $n-m$ elements. so there are $\binom{n-m}{d}$ rows with all 1 entries in these columns in $\delta(n,d,n-1)$. Now we consider $\delta^c(n,d,n-1)$. There are exact

$\binom{n-m}{d}$ rows with all 0 entries in these columns. Observe that $\binom{n-(n-d)}{d} = 1$ and $\binom{n-(n-d+1)}{d} = 0$. Our assertion is proved. □

Corollary 3.3. *For $d < l < n-1$, $\delta^c(n,d,n-1)$ is $(n-l, \binom{l-1}{l-d} - 1)-$disjunct.*

4 Remarks

Tayuan Huang and Chih-wen Weng define a pooling space and show us how to construct $d-$disjunct matrices from a pooling space in [2]. $\delta(n,d,k)$ is a type of $d-$disjunct matrix constructed from a pooling space which is a ranked partially ordered set and its partial order relation is the inclusion relation between subsets. In fact, these $d-$disjunct matrices determined by the pooling spaces mentioned in [2] also have the similar properties above. For example, from the attenuated space $A_q(D,N)(D \leq N)$ we can constructed a type of $d-$disjunct matrix $\eta(D,d,k)$. $\eta(D,d,k)$ is

$$\begin{gathered}(d,0)-disjunct,\\ (d-1, {\textstyle\begin{bmatrix} k-(d-1) \\ d-(d-1)\end{bmatrix}_q} q^{(d-(d-1))(N-D)} - 1) - disjunct,\\ (d-2, {\textstyle\begin{bmatrix} k-(d-2) \\ d-(d-2)\end{bmatrix}_q} q^{(d-(d-2))(N-D)} - 1) - disjunct,\\ \vdots \\ (d-i, {\textstyle\begin{bmatrix} k-(d-i) \\ d-(d-i)\end{bmatrix}_q} q^{(d-(d-i))(N-D)} - 1) - disjunct,\\ \vdots \\ (d-(d-1), {\textstyle\begin{bmatrix} k-1 \\ d-1\end{bmatrix}_q} q^{(d-1)(N-D)} - 1) - disjunct,\end{gathered}$$

and its complement $\eta^c(D,d,D-1)$ is $(D-d)-$disjunct.

5 Acknowledgement

The authors are grateful to the referee for his valuable suggestions. The third author would like to thank Professor Tayuan Huang for introducing him $d-$disjunct matrices and sending him the papers[2-4]. This paper was supported by Natural Science Foundation of Hebei Province, China, (No. A2005000141).

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