A diameter formula for an undirected double-loop network *

Bao-Xing Chen[1,2] Ji-Xiang Meng[2] Wen-Jun Xiao[3]

[1]Dept. of Computer Science, Zhangzhou Teacher's College, Zhangzhou, P.R. China

[2]College of Mathematics & System Science, Xinjiang University, Wulumuqi, P.R. China

[3]Dept. of Computer Science, South China University of Technology, Guangzhou, P.R. China

Abstract

Let n, s_1 and s_2 be positive integers such that $1 \leq s_1 < n/2$, $1 \leq s_2 < n/2$, $s_1 \neq s_2$ and $\gcd(n, s_1, s_2)=1$. An undirected double-loop network $G(n; \pm s_1, \pm s_2)$ is a graph (V, E), where $V=\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$, and $E=\{i \to i+s_1 \pmod n, i \to i-s_1 \pmod n, i \to i+s_2 \pmod n, i \to i-s_2 \pmod n \mid i = 0, 1, 2, \cdots, n-1\}$. In this paper, a diameter formula is given for an undirected double-loop network $G(n; \pm s_1, \pm s_2)$. As its application, two new optimal families of undirected double-loop networks are presented.

1 Introduction

An undirected double-loop network is very useful in designs of local area networks, multimodule memory organization, data alignment in parallel memory systems and super-computer architecture. Many researchers are interested in the case of undirected networks [2, 4, 5, 7, 8, 10, 13, 15-18], while others are interested in the case of directed ones [1, 3, 6, 9-12, 14]. Their interests mainly focus on routing, diameters and optimal double-loop networks. For more details we refer readers to [3] and [12] and the references therein.

Now we give definitions of some notations used in the following. Let G be a finite group with e as its identity. Let $S \subset G$ be a generator set of G such that $e \notin S$ and $g^{-1} \in S$ if $g \in S$. Define Cayley graph $Cay(G, S) = (V, E)$, where $V=G$ and $E=\{(x, y) \mid y=xg \text{ for some } g \in S\}$. Then $Cay(G, S)$ is a regular, vertex-transitive graph of degree$=|S|$.

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Let n, s_1 and s_2 be positive integers such that $1 \le s_1 < n/2$, $1 \le s_2 < n/2$, $s_1 \ne s_2$. The undirected double-loop network $G(n; \pm s_1, \pm s_2)$ is a graph (V, E), where $V = \mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$, and $E = \{i \to i + s_1 \pmod n, i \to i - s_1 \pmod n, i \to i + s_2 \pmod n, i \to i - s_2 \pmod n \mid i = 0, 1, 2, \cdots, n-1\}$. Thus $G(n; \pm s_1, \pm s_2)$ is a Cayley graph $Cay(\mathbb{Z}_n, \{s_1, s_2, -s_1, -s_2\})$.

Let $d(i, j)$ be the length of a shortest path from i to j. The maximum length among all pairs of nodes, denoted by $d(n; \pm s_1, \pm s_2)$, is the diameter of $G(n; \pm s_1, \pm s_2)$. As $G(n; \pm s_1, \pm s_2)$ is vertex symmetric, $d(n; \pm s_1, \pm s_2) = \max\{d(i, 0) \mid 0 \le i < n\}$. Let $D(n) = \min\{d(n; \pm s_1, \pm s_2) \mid 1 \le s_1 < s_2 < n/2\}$. Wong and Coppersmith [16] gave a lower bound $(\sqrt{2n} - 3)/2$ for $D(n)$. Boesch and Wang [4] sharpened the bound by giving $lb(n) = \lceil \frac{\sqrt{2n-1}-1}{2} \rceil$, where $\lceil x \rceil$ denotes the minimum integer $\ge x$. For any n, taking $s_1 = lb(n)$ and $s_2 = lb(n) + 1$(see [4, 17]) yields a graph $G(n; \pm s_1, \pm s_2)$ of diameter $lb(n)$. Du et al. [7] gave an upper bound of $\max\{q+1, r-2, h-r-1\}$ for $d(n; \pm 1, \pm h)$, where $n = qh + r, 0 \le r < h$. Mukhopadhyaya and Sinha [13] proposed an $O(D)$ time optimal routing for an undirected double-loop network, where D is the diameter of the network. They also listed some open problems in [13], one of which is to derive an analytical formula for the diameter of $G(n; \pm 1, \pm h)$. In this paper, we will give a diameter formula for an undirected double-loop network $G(n; \pm s_1, \pm s_2)$ and therefore solve this problem. This paper is organized in such a way that Section 2 provides some preliminary facts, observations, and known results concerning undirected double-loop networks. In Section 3, a diameter formula for $d(n; \pm s_1, \pm s_2)$ is presented. In Section 4, two new optimal families of undirected double-loop networks are given.

2 Preliminary Observations

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and nonnegative integers respectively. Let $\lfloor x \rfloor$ denote the maximum integer $\le x$. Given $G(n; \pm s_1, \pm s_2)$, an edge from i to $(i \pm s) \pmod n$ is called a $[\pm s]$ edge, where $s \in \{s_1, s_2\}$. It is known that $G(n; \pm s_1, \pm s_2)$ is connected if and only if $\gcd(n, s_1, s_2) = 1$. In the following we always assume that $1 \le s_1 < s_2 < n/2$ and $\gcd(n, s_1, s_2) = 1$.

Consider a path from i to j involving w, x, y, and z (all non-negative integers) number of $[+s_1]$, $[-s_1]$, $[+s_2]$, $[-s_2]$ edges respectively. Then $j \equiv (i + ws_1 - xs_1 + ys_2 - zs_2) \pmod n$. Since we are only interested in the length of the paths, we shall denote such a path by $w[+s_1] + x[-s_1] + y[+s_2] + z[-s_2]$. If a path $w[+s_1] + x[-s_1] + y[+s_2] + z[-s_2]$ from i to j is the shortest one, then at most one of w and x is nonzero and at most one of y and z is nonzero.

Given $G(n; \pm s_1, \pm s_2)$, we construct an infinite grid $G_{n, \pm s_1, \pm s_2}$ in $\mathbb{Z}^2$ by labelling each lattice point (i, j) by $is_1 + js_2 \pmod n$. We refer to a lattice point with label i as an i-point. If $is_1 + js_2 \equiv 0 \pmod n$, then we call (i, j) a 0-point. We define

$$dist((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

as the distance between lattice points (x_1, y_1) and (x_2, y_2).

Let $|\overrightarrow{\alpha}|$ denote the length of the vector $\overrightarrow{\alpha}$ and let $\overrightarrow{\alpha} \times \overrightarrow{\beta}$ be the vector product of $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$. Suppose that A, B, C, and D are $(0,0), (u,v), (u-a, v+b)$, and $(-a, b)$ respectively, where $u, v, a, b \in \mathbb{Z}^+$ and lattice points A, B, and D are not on the same line. Let $S_{\square ABCD}$ denote the area of the parallelogram $ABCD$. Notice that $S_{\square ABCD} = |\overrightarrow{AB} \times \overrightarrow{AD}|$, we have $S_{\square ABCD} = ub + va$.

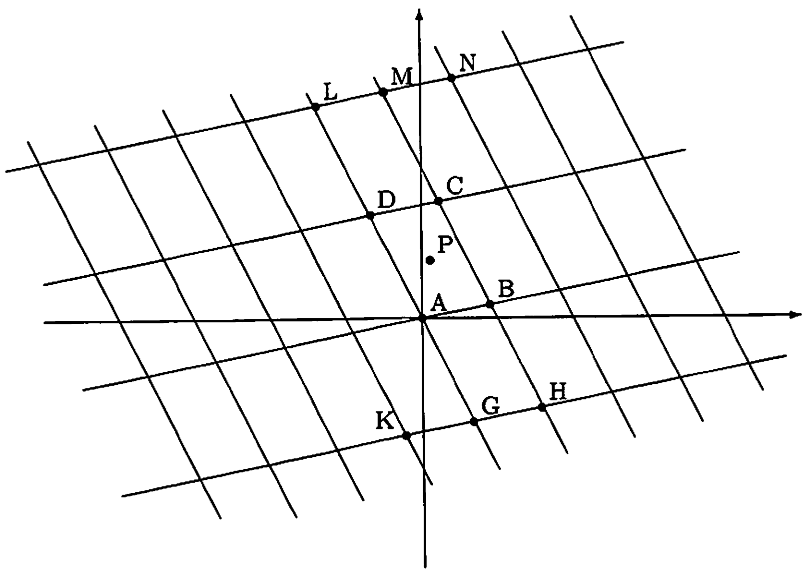

Fig. 1 $A, B, C, D, G, H, K, L, M$ and N are 0-points.

Since the following three lemmas can be proved just like Lemma 1, Lemma 2 and Lemma 3 in [7], their proofs are omitted here.

Lemma 1: Suppose that 0-points A, B, C, D have coordinates $(0,0), (u,v)$, $(u-a, v+b)$, and $(-a, b)$, respectively, with u, v, a, b are all nonnegative integers. If the area of the region Σ covered by the parallelogram $ABCD$, excluding the two edges BC and CD(and by implication, the lattice points B, C and D), is n, then Σ contains exactly n lattice points whose labels are $0, 1, 2, \cdots, n-1$.

Corollary 1: The region Σ and lattice points A, B, C, D are defined as in Lemma 1. Suppose that the area of Σ is n. If (p, q) is a 0-point, then there exist two integers t_1, t_2 such that $(p,q) = t_1(u,v) + t_2(-a,b)$.

Proof. As A, B and D are not on the same line, there exist two real numbers t_1, t_2 such that $(p,q) = t_1(u,v) + t_2(-a,b)$. If t_1 and t_2 are not both integers, as B and D are 0-points, then $T = (t_1 - \lfloor t_1 \rfloor)(u,v) + (t_2 - \lfloor t_2 \rfloor)(-a,b)$ is a 0-point. As T is in Σ, we know that there are two 0-points A and T in Σ. This contradicts the conclusion of Lemma 1. □

Lemma 2: Suppose that the region Σ and the four lattice points A, B, C, D are defined as in Lemma 1 and that $u \geq v, a < u, a \leq b$ and $v < b$. Consider the points P and Q with coordinates $(\lfloor (u-a)/2 \rfloor, \lceil (v+b)/2 \rceil)$ and $(\lceil (u -$

$a)/2\rceil, \lceil(v+b)/2\rceil)$, respectively. If Σ includes n lattice points, then no 0-point is closer to P than the nearest of the points A, B, C, D. Thus, the shortest distance from node 0 to node $\lfloor(u-a)/2\rfloor s_1 + \lceil(v+b)/2\rceil s_2$ in $G(n; \pm s_1, \pm s_2)$ is $\min\{dist(A,P), dist(B,P), dist(C,P), dist(D,P)\}$. The shortest distance from node 0 to node $\lceil(u-a)/2\rceil s_1 + \lceil(v+b)/2\rceil s_2$ is similarly related to point Q.

Lemma 3: Suppose that the region Σ and the four lattice points A, B, C, D are defined as in Lemma 1 and that $u < v, b < a, u < a$ and $b \le v$. Consider the points P' and Q' with coordinates $(-\lfloor(a-u)/2\rfloor, \lceil(v+b)/2\rceil)$ and $(-\lceil(a-u)/2\rceil, \lceil(v+b)/2\rceil)$, respectively. If Σ includes n lattice points, then no 0-point is closer to P' than the nearest of the points A, B, C, D. Thus, the shortest distance from node 0 to node $-\lfloor(a-u)/2\rfloor s_1 + \lceil(v+b)/2\rceil s_2$ in $G(n; \pm s_1, \pm s_2)$ is $\min\{dist(A,P'), dist(B,P'), dist(C,P'), dist(D,P')\}$. The shortest distance from node 0 to node $-\lceil(u-a)/2\rceil s_1 + \lceil(v+b)/2\rceil s_2$ is similarly related to point Q'.

By Lemma 2 and Lemma 3, we know that

$$d(n; \pm s_1, \pm s_2) \ge \min\{dist(P,A), dist(P,B), dist(P,C), \; dist(P,D)\},$$

where P is a lattice point near or in the center of the parallelogram $ABCD$. This inequality is helpful in studying diameters of undirected double-loop networks in the next section.

3 A diameter formula for an undirected double-loop network

In this section, we will give a diameter formula for for an undirected double-loop network.

Definition 1: (a_1, a_2) is said to be a non-negative solution of the congruence equation

$$xs_1 + ys_2 \equiv 0 \pmod n \qquad (1)$$

if $a_1 s_1 + a_2 s_2 \equiv 0 \pmod n$, $a_1, a_2 \in \mathbb{Z}^+$ and $(a_1, a_2) \ne (0,0)$. (u, v) is said to be the smallest non-negative solution of the congruence equation (1) if (u, v) is a non-negative solution of the equation (1) and the following conditions hold:
(1) if (a_1, a_2) is a non-negative solution of the equation (1), then $u + v \le a_1 + a_2$.
(2) if (a_1, a_2) is a non-negative solution of the equation (1) with $(a_1, a_2) \ne (u, v)$ and $u + v = a_1 + a_2$, then $u > a_1$.

For example, it is easy to see that $(4,1), (2,3), (0,5), (8,2), (4,6), \cdots$ are non-negative solutions of the equation $x + 6y \equiv 0 \pmod{10}$. Thus $(4,1)$ is the smallest non-negative solution of the equation $x + 6y \equiv 0 \pmod{10}$.

Definition 2: Let (u, v) be the smallest non-negative solution of the congruence equation (1). $(-a_1, a_2)$ is said to be a cross solution of the congruence equation (1) if $-a_1 s_1 + a_2 s_2 \equiv 0 \pmod n$, $a_1, a_2 \in \mathbb{Z}^+$, $(-a_1, a_2) \ne (0,0)$, and $(-a_1, a_2)$, $(0,0)$, (u, v) are not on the same line. $(-a, b)$ is said to be the smallest cross solution of the congruence equation (1) if $(-a, b)$ is a cross solution of the equation (1) and the following conditions hold:
(1) if $(-a_1, a_2)$ is a cross solution of the equation (1), then $a + b \le a_1 + a_2$.

(2) if $(-a_1, a_2)$ is a cross solution of the equation (1) with $(-a_1, a_2) \neq (-a, b)$ and $a + b = a_1 + a_2$, then $b > a_2$.

For example, it is easy to see that $(2, 2)$ is the smallest non-negative solution of the equation $4x+5y \equiv 0 \pmod{18}$, and $(-9,0), (-7,2), (-5,4), (-3,6), (-1,8), (-18,0), (-14,4), (-16,2), \cdots$ are cross solutions of the congruence equation $4x+5y \equiv 0 \pmod{18}$. Thus $(-1, 8)$ is the smallest cross solution of the congruence equation $4x + 5y \equiv 0 \pmod{18}$.

Lemma 4: Let (u, v) be the smallest non-negative solution of the congruence equation (1) and $(-a, b)$ be the smallest cross solution of the congruence equation (1). If $u < v$, then $a > u, a > b, b < v$.

Proof. Firstly, we claim that $b < v$. In fact, if $b \geq v$, then $(-a - u, b - v)$ is a cross solution of the equation (1) and $a + u + b - v < a + b$. This contradicts the hypothesis that $(-a, b)$ is the smallest cross solution of the equation (1).

Secondly, we prove $a > b$. If $a \leq b$, since $b < v$, $(u + a, v - b)$ must be a non-negative solution of the equation (1) and $u + a + v - b \leq u + v$. This contradicts the hypothesis that (u, v) is the smallest non-negative solution of the equation (1).

Finally, we prove $a > u$. If $a \leq u$, since $a > b$, $(u - a, v + b)$ must be a non-negative solution of the equation (1) and $u - a + b + v < u + v$. This contradicts the hypothesis that (u, v) is the smallest non-negative solution of the equation (1). □

By similar arguments, we can show the following Lemma 5.

Lemma 5: Let (u, v) be the smallest non-negative solution of the equation (1), and $(-a, b)$ be the smallest cross solution of the equation (1). If $u \geq v$, then $a < u, a \leq b, v < b$.

Lemma 6: Let (u, v) be the smallest non-negative solution of the congruence equation (1) and $(-a, b)$ be the smallest cross solution of the congruence equation (1). Then $ub + va = n$.

Proof. We consider two cases.

Case 1: $u \geq v$, by Lemma 5 we know that $a < u, a \leq b, v < b$.

As any one of two cases: (1) $u + v > a + b$, (2) $u + v \leq a + b$ may happen, for convenience, in the following we just consider the first case: $u + v > a + b$. The other case can be similarly proved.

Let $M = \begin{pmatrix} u & -a \\ v & b \end{pmatrix}$. The set $M\mathbb{Z}^2$, whose elements are linear combinations (with integral coefficients) of the (column) vectors $m_1 = \binom{u}{v}$ and $m_2 = \binom{-a}{b}$, is said to be the lattice generated by M. Clearly, $M\mathbb{Z}^2$ with usual vector addition is a normal subgroup of $\mathbb{Z}^2$.

Let $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$. Now define a map φ: Cay$(\mathbb{Z}^2/M\mathbb{Z}^2, \{e_1, e_2, -e_1, -e_2\})$ $\rightarrow$ Cay$(\mathbb{Z}_n, \{s_1, s_2, -s_1, -s_2\})$ by $\varphi(\binom{x_1}{x_2}) = x_1 s_1 + x_2 s_2$.

For any $t \in \mathbb{Z}_n$, as $\gcd(n, s_1, s_2) = 1$, there exist two integers x_1, x_2 such that $x_1 s_1 + x_2 s_2 \equiv t \pmod{n}$. That is, φ is a surjective map.

In the following we will prove φ is injective.

Since $m_1 = \binom{u}{v}$ and $m_2 = \binom{-a}{b}$ are linear independent, for any two integers x_1, x_2, there exist two real numbers t_1, t_2 such that $\binom{x_1}{x_2} = t_1 m_1 + t_2 m_2$. If $x_1 s_1 + x_2 s_2 \equiv 0 \pmod{n}$, then we will prove that t_1, t_2 are both integers. That is, $\binom{x_1}{x_2} \in M\mathbb{Z}^2$.

If t_1, t_2 are not both integers, then $t_1 = \lfloor t_1 \rfloor + r_1$, $t_2 = \lfloor t_2 \rfloor + r_2$, where $0 \leq r_1 < 1, 0 \leq r_2 < 1$ and at least one of r_1, r_2 is not zero. Thus $(r_1 u - r_2 a)s_1 + (r_1 v + r_2 b)s_2 \equiv 0 \pmod{n}$.

If $r_1 u - r_2 a < 0$, since $-r_1 u + r_2 a + r_1 v + r_2 b < a + b$, it is contradictory to that $(-a, b)$ is the smallest cross solution of the congruence equation (1).

If $r_1 u - r_2 a \geq 0$, three subcases are considered.

Subcase 1.1: $r_1 u - r_2 a \leq u - a$. As $(r_1 u - r_2 a, r_1 v + r_2 b)$ and $(u - a - r_1 u + r_2 a, b + v - r_1 v - r_2 b)$ are two non-negative solutions of the equation (1) and $r_1 u - r_2 a + r_1 v + r_2 b + u - a - r_1 u + r_2 a + b + v - r_1 v - r_2 b = u - a + b + v \leq a + b + u + v < 2u + 2v$, we see that either $r_1 u - r_2 a + r_1 v + r_2 b < u + v$ or $u - a - r_1 u + r_2 a + b + v - r_1 v - r_2 b < u + v$ holds. This contradicts the hypothesis that (u, v) is the smallest non-negative solution of the equation (1).

Subcase 1.2: $r_1 u - r_2 a > u - a$ and $r_1 v + r_2 b < v$. As $(u - r_1 u + r_2 a, v - r_1 v - r_2 b)$ is a non-negative solution of the equation (1) and $u - r_1 u + r_2 a + v - r_1 v - r_2 b < u + v$, it contradicts the hypothesis that (u, v) is the smallest non-negative solution of the equation (1).

Subcase 1.3: $r_1 u - r_2 a > u - a$ and $r_1 v + r_2 b \geq v$. As $(u - a - r_1 u + r_2 a, b + v - r_1 v - r_2 b)$ is a cross solution of the equation (1) and $-(u - a - r_1 u + r_2 a) + b + v - r_1 v - r_2 b < a + b$, it contradicts the hypothesis that $(-a, b)$ is the smallest cross solution of the congruence equation (1).

From above, we see that φ is injective. It is easy to verify that φ is a homomorphism. Thus φ is an isomorphism between $\text{Cay}(\mathbb{Z}^2/M\mathbb{Z}^2, \{e_1, e_2, -e_1, -e_2\})$ and $\text{Cay}(\mathbb{Z}_n, \{s_1, s_2, -s_1, -s_2\})$. So $|\mathbb{Z}^2/M\mathbb{Z}^2| = |\mathbb{Z}_n| = n$. By Proposition 2.1 [10], we have $|detM| = |\mathbb{Z}^2/M\mathbb{Z}^2|$. Thus $n = |detM| = ub + va$.

Case 2: $u < v$. The equality $n = ub + va$ can be similarly proved. □

Theorem 1: Given $G(n; \pm s_1, \pm s_2)$, where n, s_1 and s_2 are positive integers such that $1 \leq s_1 < n/2$, $1 \leq s_2 < n/2$, $s_1 \neq s_2$ and $\gcd(n, s_1, s_2) = 1$. Let (u, v) be the smallest non-negative solution of the congruence equation (1) and $(-a, b)$ be the smallest cross solution of the congruence equation (1). Let $r_1 = \lfloor (u + v)/2 \rfloor$, $r_2 = \lfloor (a + b)/2 \rfloor$, $r_3 = \lfloor (|u - a| + v + b)/2 \rfloor$, $r_4 = \lfloor (u + a + |v - b|)/2 \rfloor$, and $d_1 = \max\{r_1, r_2, \min\{r_3, r_4\}\}$. Then $d(n; \pm s_1, \pm s_2)$ equals $r_3 - 1$ if $r_3 = r_4$ and $(u + a)(v - b) \equiv 1 \pmod{2}$; otherwise, it equals d_1.

Proof. We consider two cases.

Case 1: $u \geq v$. By Lemma 5 and Lemma 6 we know that $a < u, a \leq b, b > v$ and $ub + va = n$.

Let lattice points A, B, C and D be $(0,0), (u, v), (u - a, v + b)$ and $(-a, b)$ respectively(see Fig. 1), and Σ be region surrounded by the parallelogram $ABCD$, excluding the edges BC and CD. As the area of Σ is $ub + va = n$, by Lemma 1, we see that Σ includes exactly n lattice points whose labels are $0, 1, 2, \cdots, n - 1$.

Since $u \geq v, a < u, a \leq b, b > v$ and Σ includes n lattice points, we can use Lemma 2, 3 and follow the proof of Lemma 4 [7] to prove that the diameter formula for $G(n; \pm s_1, \pm s_2)$ is true.

Case 2: $u < v$. The diameter formula can be similarly proved by using Lemma 2, 3, 4, 6 and Lemma 5[7]. □

Example 1: computing the diameter of $G(38; \pm 2, \pm 5)$. It is easy to see that

$(4,6)$ is the smallest non-negative solution of the congruence equation $2x+5y \equiv 0 \pmod{38}$, and $(-5,2)$ is the smallest cross solution of the congruence equation $2x+5y \equiv 0 \pmod{38}$. Thus, by Theorem 1, we have r_1=5, r_2=3, r_3=4 and r_4=6. As $r_3 \neq r_4$, we have $d(38;2,5)$=$\max\{r_1, r_2, \min\{r_3, r_4\}\}$=5.

Example 2: computing the diameter of $G(39;\pm1,\pm17)$. It is easy to see that $(5,2)$ is the smallest non-negative solution of the congruence equation $x+17y \equiv 0 \pmod{39}$, and $(-2,7)$ is the smallest cross solution of the congruence equation $x+17y \equiv 0 \pmod{39}$. Thus, by Theorem 1, we have r_1=3, r_2=4, r_3=6 and r_4=6. As r_3=r_4 and $(u+a)(v-b)$=$(5+2)*(2-7) \equiv 1 \pmod{2}$, we then have $d(39;1,17)$= $r_3 - 1$ =5.

4 Applications

Many optimal families of undirected double-loop networks are given in [2, 7, 15]. Two new optimal families of undirected double-loop networks will be given in this section.

In the following, we shall use the following notations:

$$n_k = 2k^2 + 2k + 1, k \geq 0;$$

$$R[k] = \{n_{k-1}+1, n_{k-1}+2, \cdots, n_k\}, k \geq 1;$$

Let $n \in R[k]$ and D_n^*=$\min\{d(n;\pm1,\pm s) \mid 1 < s < n/2\}$. Then

$$D_n^* \geq lb(n) = \lceil \frac{\sqrt{2n-1}-1}{2} \rceil = k.$$

If there exists some h_n such that $D_n^* = d(n;\pm1,\pm h_n) = k$, then n and $G(n;\pm1,\pm h_n)$ will be called optimal.

If there exists some h_n such that $D_n^* = d(n;\pm1,\pm h_n) = k+1$, then n and $G(n;\pm1,\pm h_n)$ will be called suboptimal.

A set Θ of natural numbers will be called an optimal (suboptimal) family if each $n \in \Theta$ is optimal (suboptimal).

Lemma 7[2]: Let $n \in R[k]$, then n is optimal in each of the cases:
(1) $\gcd(n,k) = 1$,
(2) $\gcd(n,k+1) = 1$,
(3) $\gcd(n,k-1) = 1$ and $n \leq 2k^2+1$.
and in each case the associated hop h_n is easily determined.

If k is odd, then $\gcd(2k^2-2,k) = 1$. By Lemma 7 it is easy to see that $2k^2-2$, where $k = 2e+3$, $e \in \mathbb{Z}^+$, is optimal. On the other hand, there exists k such that $2k^2-2$ is suboptimal. For example, when $k = 14$, $2k^2-2 = 390$ is suboptimal. One can refer to Appendix B in [15].

By using the algorithm given in [18] and computer search, we find that $\{2k^2-2 \mid k = 10e+10, e \in \{0,1,2,\cdots,20\}\}$ is an optimal family. Thus we conjecture that $\{2k^2-2 \mid k = 10e+10, e \in \mathbb{Z}^+\}$ is an optimal family. For $n = 2k^2-2, k = 10e+10$, we have $n = (10e+8)*(4e+2)+(10e+13)*(16e+14)$. Since $\gcd(10e+13, 4e+2) = 1$, there exist two integers α, β such that $\alpha(10e+13) -$

$\beta(4e+2) = 1$. Let $s \equiv \alpha(16e+14) + \beta(10e+8) \pmod{n}$. It is easy to see that s satisfies $10e+8+(10e+13)s \equiv 0 \pmod{n}$ and $(-16e-14)+(4e+2)s \equiv 0 \pmod{n}$. By Theorem 1, we can see that $d(n; \pm 1, \pm s) = k = lb(n)$.

For $n = 2k^2 - 2, k = 10e+18$, we can find s' as above such that $d(n; \pm 1, \pm s') = k$. In the following we will use diameter formulas in section 3 to prove the following theorem.

Theorem 2: (1) Let $\Theta = \{2k^2 - 2 \mid k = 10e+10, e \in \mathbb{Z}^+\}$. Then Θ is an optimal family.
(2) Let $\Phi = \{2k^2 - 2 \mid k = 10e+18, e \in \mathbb{Z}^+\}$. Then Φ is an optimal family.

Proof. (1) Let $n = 2k^2 - 2$, where $k = 10e+10$. Then $n = 200e^2 + 400e + 198$. Let $s \equiv 100e^5 + 50e^4 - 26e^3 - 86e^2 - 408e - 290 \pmod{n}$. It suffices to prove that $d(n; \pm 1, \pm s) = 10e + 10$.

Consider the congruence equation

$$x + ys \equiv 0 \pmod{n} \qquad (*)$$

Let $(u, v) = (10e+8, 10e+13)$ and $(-a, b) = (-16e-14, 4e+2)$. In the following we will prove that (u, v) is the smallest non-negative solution of the equation $(*)$ and $(-a, b)$ is the smallest cross solution of the equation $(*)$.

Since $10e+8+(10e+13)s = 1000e^6 + 1800e^5 + 390e^4 - 1198e^3 - 5198e^2 - 8194e - 3762 \equiv 0 \pmod{n}$, $(10e+8, 10e+13)$ is a non-negative solution of the equation $(*)$.

Since $-16e-14+(4e+2)s = 400e^6+400e^5-4e^4-396e^3-1804e^2-1992e-594 \equiv 0 \pmod{n}$, $(-16e-14, 4e+2)$ is a cross solution of the equation $(*)$.

Suppose that (p, q) is a non-negative solution of the equation $(*)$. As $ub + va = n$, by Corollary 1, we see that there exist two integers t_1, t_2 such that $(p, q) = t_1(u, v) + t_2(-a, b)$. Thus $(p, q) = (t_1(10e+8) + t_2(-16e-14), t_1(10e+13) + t_2(4e+2))$. As $p \geq 0$, we have $t_1 > t_2$.

If $t_2 \leq -1$, as $q \geq 0$, we have $t_1 \geq 1$. Thus $p+q \geq p = t_1(10e+8)+t_2(-16e-14) \geq 10e+8+16e+14 > 10e+8+10e+13$.

If $t_2 = 0$, then $t_1 \geq 1$. Thus $p+q = t_1(10e+8)+t_1(10e+13) \geq 10e+8+10e+13$.

If $t_2 \geq 1$, then $t_1 \geq 2$. Thus $p+q \geq q = t_1(10e+13) + t_2(4e+2) \geq 2(10e+13) + 4e+2 > 10e+8+10e+13$.

From above we conclude that (1) $p+q \geq u+v$, (2) $p+q = u+v$ if and only if $t_1 = 1, t_2 = 0$. That is, $p+q = u+v$ if and only if $(p, q) = (u, v)$. Thus (u, v) is the smallest non-negative solution of the equation $(*)$. In the following we prove that $(-a, b)$ is the smallest cross solution of the equation $(*)$. Suppose that $(-p, q)$ is a cross solution of the equation $(*)$. By Corollary 1, we know that there exist two integers t_1, t_2 such that $(-p, q) = t_1(u, v) + t_2(-a, b)$. Thus $(-p, q) = (t_1(10e+8) + t_2(-16e-14), t_1(10e+13) + t_2(4e+2))$.

Now we prove that $t_2 > 0$. If $t_2 = 0$, as $p \geq 0, q \geq 0$, then $t_1 = 0$. Thus $(-p, q) = 0$, a contradiction. If $t_2 < 0$, as $p \geq 0$, we have $t_1 < 0$. On the other hand, since $q \geq 0$, we have $t_1 > 0$. A contradiction.

If $t_1 < 0$, since $q \geq 0$, we have $t_2 > -2t_1$. Thus $p+q \geq p = -t_1(10e+8) + t_2(16e+14) > 10e+8+2(16e+14) > 16e+14+4e+2$.

If $t_1 \geq 0$, then $p+q = -t_1(10e+8)+t_2(16e+14)+t_1(10e+13)+t_2(4e+2) = 20t_2e + 16t_2 + 5t_1 \geq 20t_2e + 16t_2 \geq 16e+14+4e+2$.

From above we conclude that (1) $p+q \geq a+b$, (2) $p+q=a+b$ if and only if $t_1=0, t_2=1$. That is, $p+q=a+b$ if and only if $(-p,q)=(-a,b)$. Thus $(-a,b)$ is the smallest cross solution of the equation $(*)$.

Thus, by Theorem 1, we have $r_1=\lfloor(10e+8+10e+13)/2\rfloor = 10e+10$, $r_2=\lfloor(16e+14+4e+2)/2\rfloor = 10e+8$, $r_3=\lfloor(|10e+8-16e-14|+10e+13+4e+2)/2\rfloor = 10e+10$ and $r_4=\lfloor(10e+8+16e+14+|10e+13-4e-2|)/2\rfloor = 16e+16$. Since $r_3 < r_4$, we then have $d(n;\pm1,\pm s) = 10e+10$.

(2) Let $n=2k^2-2$, where $k=10e+18$. Thus $n=200e^2+720e+646$. Let $s \equiv 100e^5+410e^4+478e^3+84e^2-42e+30 \pmod n$.

It suffices to prove that $d(n;\pm1,\pm s) = 10e+18$. Consider the congruence equation

$$x+ys \equiv 0 \pmod n \qquad (**)$$

It can be similarly proved that $(10e+16, 10e+21)$ is the smallest non-negative solution of the equation $(**)$ and $(-18e-30, 2e+1)$ is the smallest cross solution of the equation $(**)$.

Thus, by Theorem 1, we have $r_1=\lfloor(10e+16+10e+21)/2\rfloor = 10e+18$, $r_2=\lfloor(18e+30+2e+1)/2\rfloor = 10e+15$, $r_3=\lfloor(|10e+16-18e-30|+10e+21+2e+1)/2\rfloor = 10e+18$ and $r_4=\lfloor(10e+16+18e+30+|10e+21-2e-1|)/2\rfloor = 18e+33$. As $r_3 < r_4$, then we have $d(n;\pm1,\pm s) = 10e+18$. □

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