Potentially K_{r+1}^{-p}-graphic sequences *

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Abstract. Let $0 \le p \le [\frac{r+1}{2}]$ and $\sigma(K_{r+1}^{-p}, n)$ be the smallest even integer such that each n-term graphic sequence with term sum at least $\sigma(K_{r+1}^{-p}, n)$ has a realization containing K_{r+1}^{-p} as a subgraph, where K_{r+1}^{-p} is a graph obtained from a complete graph K_{r+1} on $r+1$ vertices by deleting p edges which form a matching. In this paper, we determine $\sigma(K_{r+1}^{-p}, n)$ for $r \ge 2$, $1 \le p \le [\frac{r+1}{2}]$ and $n \ge 3r+3$. As a corollary, we also determine $\sigma(K_{1^s,2^t}, n)$ for $t \ge 1$ and $n \ge 3s+6t$, where $K_{1^s,2^t}$ is an $r_1 \times r_2 \times \cdots \times r_{s+t}$ complete $(s+t)$-partite graph with $r_1 = r_2 = \cdots = r_s = 1$ and $r_{s+1} = r_{s+2} = \cdots = r_{s+t} = 2$ and $\sigma(K_{1^s,2^t}, n)$ is the smallest even integer such that each n-term graphic sequence with term sum at least $\sigma(K_{1^s,2^t}, n)$ has a realization containing $K_{1^s,2^t}$ as a subgraph.
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1. Introduction

The set of all non-increasing nonnegative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. For a nonnegative integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For a given graph H, a sequence $\pi \in GS_n$ is said to be *potentially H-graphic* if there is a realization of π containing H as a subgraph. Gould et al. [4] considered the following variation of the classical

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Turán-type extremal problems: determine the smallest even integer $\sigma(H,n)$ such that every sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H,n)$ is potentially H-graphic. If $H = K_{r+1}$, a complete graph on $r+1$ vertices, this problem was considered by Erdős et al. [3] where they showed that $\sigma(K_3,n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_{r+1},n) = (r-1)(2n-r)+2$ for sufficiently large n. Gould et al. [4] and Li and Song [6] independently proved it for $r = 3$. Recently, Li et al. [7,8] proved that the conjecture is true for $r = 4$ and $n \geq 10$ and for $r \geq 5$ and $n \geq \binom{r}{2} + 3$. Li and Yin [9] further determined $\sigma(K_{r+1},n)$ for $r \geq 6$ and $n \geq 2r+3$. The problem about determining $\sigma(K_{r+1},n)$ was completely solved. For $H = K_{r,s}$, an $r \times s$ complete bipartite graph, Gould et al. [4] determined $\sigma(K_{2,2},n)$ for $n \geq 4$ and Yin and Li [10] determined $\sigma(K_{3,3},n)$ for $n \geq 6$ and $\sigma(K_{4,4},n)$ for $n \geq 8$. Recently, Yin, Li and Chen [11,12,13] further determined $\sigma(K_{r,s},n)$ for sufficiently large n. If $H = K_{r+1}^{-p}$, a graph obtained from K_{r+1} by deleting p edges which form a matching, Lai [5] determined $\sigma(K_4^{-1},n)$ for $n \geq 4$, Gould et al. [4] determined $\sigma(K_4^{-2},n)$ for $n \geq 4$, Yin et al. [15] determined $\sigma(K_5^{-1},n)$ for $n \geq 5$, Chen et al. [1] determined $\sigma(K_5^{-2},n)$ for $n \geq 11$. Recently, Yin and Li [14] further determined $\sigma(K_{r+1}^{-1},n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$. The purpose of this paper is to determine $\sigma(K_{r+1}^{-p},n)$ for $r \geq 2$, $1 \leq p \leq [\frac{r+1}{2}]$ and $n \geq 3r+3$. As a corollary, the values of $\sigma(K_{1^s,2^t},n)$ for $t \geq 1$ and $n \geq 3s+6t$ are determined, where $K_{1^s,2^t}$ is an $r_1 \times r_2 \times \cdots \times r_{s+t}$ complete $(s+t)$-partite graph with $r_1 = r_2 = \cdots = r_s = 1$ and $r_{s+1} = r_{s+2} = \cdots = r_{s+t} = 2$.

2. Main Results

In order to prove our main results, we need the following known theorems.

Theorem 2.1 [2] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any t, $1 \leq t \leq n-1$,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_j\}.$$

Theorem 2.2 [14] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \ldots, r-1$, then π is potentially K_{r+1}^{-1}-graphic.

Theorem 2.3 [14] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{r-1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially K_{r+1}^{-1}-graphic.

We first prove the lower bound of $\sigma(K_{r+1}^{-p},n)$.

Theorem 2.4 Let $r \geq 2$, $1 \leq p \leq [\frac{r+1}{2}]$ and $n \geq r+1$. Then

$$\sigma(K_{r+1}^{-p}, n) \geq \begin{cases} (r-1)(2n-r)+2-(n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r)+1-(n-r) & \text{if } n-r \text{ is odd.} \end{cases}$$

Proof. Let

$$\pi = \begin{cases} ((n-1)^{r-2}, (r-1)^{n-r+2}) & \text{if } n-r \text{ is even,} \\ ((n-1)^{r-2}, (r-1)^{n-r+1}, r-2) & \text{if } n-r \text{ is odd,} \end{cases}$$

where the symbol x^y in a sequence stands for y consecutive terms, each equal to x. Then

$$G = \begin{cases} K_{r-2} + (\frac{n-r}{2}+1)K_2 & \text{if } n-r \text{ is even,} \\ K_{r-2} + (\frac{n-r+1}{2}K_2 \cup K_1) & \text{if } n-r \text{ is odd,} \end{cases}$$

is the unique realization of π, where $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 and mK_2 denotes the union of m complete graphs K_2. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \ldots, v_{r-2}\}$ and $d(v_i) = n-1$ for $1 \leq i \leq r-2$. Then, it is easy to see that any induced subgraph of $r+1$ vertices in G has at least three vertices coming from V_2, and hence contains no K_{r+1}^{-p} as a subgraph. Thus, π is not potentially K_{r+1}^{-p}-graphic, in other words,

$$\begin{aligned} \sigma(K_{r+1}^{-p}, n) &\geq \sigma(\pi)+2 \\ &= \begin{cases} (r-1)(2n-r)+2-(n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r)+1-(n-r) & \text{if } n-r \text{ is odd.} \end{cases} \end{aligned}$$

□

We now prove the following main result.

Theorem 2.5 Let $r \geq 2$, $1 \leq p \leq [\frac{r+1}{2}]$ and $n \geq 3r+3$. Then

$$\sigma(K_{r+1}^{-p}, n) = \begin{cases} (r-1)(2n-r)+2-(n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r)+1-(n-r) & \text{if } n-r \text{ is odd.} \end{cases}$$

Proof. By Theorem 2.4, it is enough to show that for $r \geq 2$, $1 \leq p \leq [\frac{r+1}{2}]$ and $n \geq 3r+3$,

$$\sigma(K_{r+1}^{-p}, n) \leq \begin{cases} (r-1)(2n-r)+2-(n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r)+1-(n-r) & \text{if } n-r \text{ is odd.} \end{cases}$$

We now prove that if $n \geq 3r+3$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (r-1)(2n-r)+2-(n-r)$, then π is potentially K_{r+1}^{-p}-graphic. If $d_{r-1} \leq r-1$, then $\sigma(\pi) \leq (n-1)(r-2)+(n-r+2)(r-1) = (r-1)(2n-

$r) - (n-r) < \sigma(\pi)$, a contradiction. Hence $d_{r-1} \geq r$. If $d_{r+1} \leq r-2$, then by Theorem 2.1,

$$\begin{aligned}
\sigma(\pi) &= \sum_{i=1}^{n} d_i = \sum_{i=1}^{r} d_i + \sum_{i=r+1}^{n} d_i \\
&\leq r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} + \sum_{i=r+1}^{n} d_i \\
&= r(r-1) + 2\sum_{i=r+1}^{n} d_i \\
&\leq r(r-1) + 2(n-r)(r-2) \\
&< (r-1)(2n-r) + 2 - (n-r) \leq \sigma(\pi), \quad \text{a contradiction.}
\end{aligned}$$

Hence $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $1 \leq i \leq r-1$ or $d_{2r+2} \geq r-1$, then by Theorem 2.2 or 2.3, π is potentially K_{r+1}^{-1}-graphic, and hence π is potentially K_{r+1}^{-p}-graphic. If $d_{2r+2} \leq r-2$ and there exists an integer $i, 1 \leq i \leq r-1$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned}
\sigma(\pi) &\leq (n-1)(i-1) + (2r-i-1)(2r+2-i) + (n-2r-1)(r-2) \\
&= i^2 + (n-4r-2)i - (n-1) + (2r-1)(2r+2) \\
&\quad +(n-2r-1)(r-2).
\end{aligned}$$

Since $n \geq 3r+3$, it is easy to see that $i^2 + (n-4r-2)i$, considered as a function of i, attains its maximum value when $i = r-1$. Hence,

$$\begin{aligned}
\sigma(\pi) &\leq (r-1)^2 + (n-4r-2)(r-1) - (n-1) + (2r-1)(2r+2) \\
&\quad +(n-2r-1)(r-2) \\
&= (r-1)(2n-r) + 2 - (n-r) - n + 3r + 2 \\
&< (r-1)(2n-r) + 2 - (n-r) \leq \sigma(\pi), \quad \text{a contradiction.}
\end{aligned}$$

Thus, $\sigma(K_{r+1}^{-p}, n) \leq (r-1)(2n-r) + 2 - (n-r)$ for $n \geq 3r+3$. Since $\sigma(K_{r+1}^{-p}, n)$ is even, we have

$$\sigma(K_{r+1}^{-p}, n) \leq \begin{cases} (r-1)(2n-r) + 2 - (n-r) & \text{if } n-r \text{ is even,} \\ (r-1)(2n-r) + 1 - (n-r) & \text{if } n-r \text{ is odd.} \end{cases}$$

□

Remark Theorem 2.5 is nice also in the sense that the value of $\sigma(K_{r+1}^{-p}, n)$ is independent of p.

By $K_{1^s,2^t} = K_{(s+2t-1)+1}^{-t}$, we have the following

Corollary 2.1 If $t \geq 1$ and $n \geq 3s+6t$, then

$$\sigma(K_{1^s,2^t}, n) = \begin{cases} (s+2t-2)(2n-s-2t+1) + 2 - (n-s-2t+1) \\ \text{if } n-s-2t \text{ is odd,} \\ (s+2t-2)(2n-s-2t+1) + 1 - (n-s-2t+1) \\ \text{if } n-s-2t \text{ is even.} \end{cases}$$

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