Bound on the exponents of a class of two-colored digraphs*

Yubin Gao[†] Yanling Shao
Department of Mathematics, North University of China
Taiyuan, Shanxi 030051, P.R. China

Abstract

A two-colored digraph D is primitive if there exist nonnegative integers h and k with $h+k>0$ such that for each pair (i,j) of vertices there exists an (h,k)-walk in D from i to j. The exponent of the primitive two-colored digraph D is the minimum value of $h+k$ taken over all such h and k. In this paper, we consider the exponents of families of two-colored digraphs of order n obtained by coloring the digraph that has the exponent $(n-1)^2$. We give the tight upper bound on the exponents, and the characterization of the extremal two-colored digraph.

AMS classification: 15A18, 05C15
Keywords: Exponent; Primitive digraph; Two-colored digraph

1 Introduction

A digraph D is *primitive* if there exists a nonnegative integer l such that for each pair (i,j) of vertices there exists a walk in D from i to j with length l. The *exponent* of D is defined to be the minimum value of l.

A *two-colored digraph* is a digraph whose arcs are colored red or blue. We allow loops and both a red arc and blue arc from i to j for all pairs (i,j) of vertices. The two-colored digraph D is *strongly connected* provided for each pair (i,j) of vertices there is a walk in D from i to j.

Given a walk w in D, $r(w)$ (respectively, $b(w)$) is the number of red arcs (respectively, blue arcs), and the *composition* of w is the vector $(r(w),b(w))$

*Research supported by NNSF of China (No. 10571163) and NSF of Shanxi (No. 20041010).

†Corresponding author. E-mail addresses: ybgao@nuc.edu.cn (Y.B. Gao), ylshao@nuc.edu.cn (Y.L. Shao).

or

$$\left[\begin{array}{c} r(w) \\ b(w) \end{array}\right].$$

If the composition of w is (h,k), we also say that w is an (h,k)-*walk*.

A two-colored digraph D is *primitive* if there exist nonnegative integers h and k with $h+k>0$ such that for each pair (i,j) of vertices there exists an (h,k)-walk in D from i to j. The *exponent* of the primitive two-colored digraph D is defined to be the minimum value of $h+k$ taken over all such h and k.

Let $C=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ be the set of cycles of D. Set M to be the $2\times l$ matrix whose ith column is the composition of γ_i. We call M the *cycle matrix* of D. The *content* of M, denoted content(M), is defined to be 0 if the rank of M is less than 2 and the greatest common divisor (i.e., g.c.d) of all 2×2 minors of M, otherwise.

Lemma 1.1 *([1]) Let D be a two-coloring digraph with cycle matrix M. Then D is primitive if and only if D is strongly connected and* content$(M)=1$.

There is a natural correspondence between two-colored digraphs and nonnegative matrix pairs (see [1]). The concept of the exponent of two-colored digraph arises in the study of finite Markov chains (see [1, 2]), and some results have already obtained ([1, 3, 4, 5, 6]). The paper [1] gives the exponents of families of primitive two-colored digraphs of order n obtained by coloring the digraph (Wielandt digraph) that has the largest exponent $(n-1)^2+1$. In this paper, we consider the class of two-colored digraphs of order n, denoted by $\mathcal{D}_n$, obtained by coloring the digraph as in Fig.1.

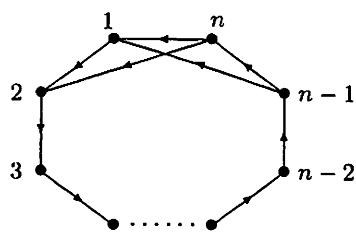

Fig. 1 The digraph

Clearly, for any $D\in\mathcal{D}_n$, D has one n-cycle and two $(n-1)$-cycles. Note that the path $(n-1)\to n\to 1\to 2$ has at least two arcs having the same color. Without loss of generality, we assume that the path $(n-1)\to n\to 1\to 2$ has at least two red arcs. Thus the two-colored digraphs in $\mathcal{D}_n$ have ten cases as in Table 1.

Table 1

	$(n-1) \to n$	$n \to 1$	$1 \to 2$	$(n-1) \to 1$	$n \to 2$
Case 1	red	red	red	red(blue)	blue(red)
Case 2	red	red	red	red	red
Case 3	red	red	red	blue	blue
Case 4	red	blue	red	red(blue)	blue(red)
Case 5	red	blue	red	red	red
Case 6	red	blue	red	blue	blue
Case 7	red(blue)	red	blue(red)	red	red
Case 8	red(blue)	red	blue(red)	red(blue)	blue(red)
Case 9	red(blue)	red	blue(red)	blue(red)	red(blue)
Case 10	red(blue)	red	blue(red)	blue	blue

Throughout the remainder of the paper, for any $D \in \mathcal{D}_n$, we let M be the cycle matrix of D, γ_1, γ_2 and γ_3 be three cycles of D, and the composition of γ_i be the ith column of M for $i = 1, 2, 3$.

2 The primitivity of a two-colored digraph in $\mathcal{D}_n$

Let $D \in \mathcal{D}_n$. Note that D is strongly connected. We assume that the path $2 \to 3 \to \cdots \to (n-2) \to (n-1)$ have a red arcs and $(n-a-3)$ blue arcs. Clearly, $0 \le a \le n-3$.

For Case 1, the cycle matrix of D is

$$M = \begin{bmatrix} a+3 & a+2 & a+1 \\ n-a-3 & n-a-3 & n-a-2 \end{bmatrix}. \quad (1)$$

Then content(M) = g.c.d$\{n-a-3, 2n-a-3, n-1\} = 1$, and so D is primitive.

For Case 2, the cycle matrix of D is

$$M = \begin{bmatrix} a+3 & a+2 & a+2 \\ n-a-3 & n-a-3 & n-a-3 \end{bmatrix}. \quad (2)$$

Then content$(M) = n-a-3$, and so D is primitive if and only if $a = n-4$.

For Case 3, the cycle matrix of D is

$$M = \begin{bmatrix} a+3 & a+1 & a+1 \\ n-a-3 & n-a-2 & n-a-2 \end{bmatrix}. \quad (3)$$

Then content$(M) = 2n-a-3 \neq 1$, and so D is not primitive.

For Case 4, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+2 & a+1 \\ n-a-2 & n-a-3 & n-a-2 \end{bmatrix}. \quad (4)$$

Then content(M) = g.c.d$\{-(a+2), n-a-2, n-1\} = 1$, and so D is primitive.

For Case 5, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+2 & a+2 \\ n-a-2 & n-a-3 & n-a-3 \end{bmatrix}. \tag{5}$$

Then content$(M) = -(a+2) \neq 1$, and so D is not primitive.

For Case 6, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+1 & a+1 \\ n-a-2 & n-a-2 & n-a-2 \end{bmatrix}. \tag{6}$$

Then content$(M) = n-a-2$, and so D is primitive if and only if $a = n-3$.

For Case 7, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+2 & a+1 \\ n-a-2 & n-a-3 & n-a-2 \end{bmatrix}. \tag{7}$$

Then content(M) = g.c.d$\{n-a-2, -(a+2), -(n-1)\} = 1$, and so D is primitive.

For Case 8, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+1 & a+1 \\ n-a-2 & n-a-2 & n-a-2 \end{bmatrix}. \tag{8}$$

Then content$(M) = n-a-2$, and so D is primitive if and only if $a = n-3$.

For Case 9, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+2 & a \\ n-a-2 & n-a-3 & n-a-1 \end{bmatrix}. \tag{9}$$

Then content(M) = g.c.d$\{2n-a-2, -(a+2), 2(n-1)\}$, and so D is primitive if and only if a is odd.

For Case 10, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+1 & a \\ n-a-2 & n-a-2 & n-a-1 \end{bmatrix}. \tag{10}$$

Then content(M) = g.c.d$\{2n-a-2, n-a-2, n-1\} = 1$, and so D is primitive.

To combine above discussions, we have the following result.

Theorem 2.1 *Let $D \in \mathcal{D}_n$. Then D is primitive if and only if D is one of the eight types in Table 2.*

Table 2

	$(n-1) \to n$	$n \to 1$	$1 \to 2$	$(n-1) \to 1$	$n \to 2$	a
Type 1	red	red	red	red(blue)	blue(red)	
Type 2	red	red	red	red	red	$a = n-4$
Type 3	red	blue	red	red(blue)	blue(red)	
Type 4	red	blue	red	blue	blue	$a = n-3$
Type 5	red(blue)	red	blue(red)	red	red	
Type 6	red(blue)	red	blue(red)	red(blue)	blue(red)	$a = n-3$
Type 7	red(blue)	red	blue(red)	blue(red)	red(blue)	a is odd
Type 8	red(blue)	red	blue(red)	blue	blue	

3 The tight bound on the exponents

In this section, we give the tight upper bound on the exponents of primitive two-colored digraphs in $\mathcal{D}_n$, and the characterization of the extremal two-colored digraph. The main result is Theorem 3.9.

Lemma 3.1 *Let $D \in \mathcal{D}_n$ be primitive. If D is Type 1 in Table 2, then* $\exp(D) \le 2n^2 - 4n + 2$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} a+3 & a+2 & a+1 \\ n-a-3 & n-a-3 & n-a-2 \end{bmatrix},$$

where $0 \le a \le n-3$.

For any pair (i, j) of vertices of D, we prove that there is a $(2na + 4n - a^2 - 4a - 5, 2n^2 - 3na - 8n + a^2 + 5a + 7)$-walk in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \le b \le n-a-2$ and $0 \le r + b \le n-1$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n-1-r-b)$ times around γ_1, $(r+2b)$ times around γ_2, and $(n-a-2-b)$ times around γ_3. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (n-1-r-b)\begin{bmatrix} a+3 \\ n-a-3 \end{bmatrix} + (r+2b)\begin{bmatrix} a+2 \\ n-a-3 \end{bmatrix}$$

$$+(n-a-2-b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix} = \begin{bmatrix} 2na+4n-a^2-4a-5 \\ 2n^2-3na-8n+a^2+5a+7 \end{bmatrix}.$$

Hence $\exp(D) \le 2n^2 - na - 4n + a + 2 \le 2n^2 - 4n + 2$. □

Lemma 3.2 *Let $D \in \mathcal{D}_n$ be primitive. If D is Type 2 in Table 2, then* $\exp(D) \le 2n^2 - 4n + 1$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} n-1 & n-2 & n-2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Clearly, D has only one blue arc, and the blue arc is in the path $2 \to 3 \to \cdots \to (n-1)$.

For any pair (i,j) of vertices of D, we prove that there is a $(2n^2-6n+4, 2n-3)$-walk in D. Let p_{ij} be the shortest walk from i to j containing the blue arc. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $b = 1$ and $0 \le r \le 2n-4$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(2n-4-r)$ times around γ_1, and r times around γ_2. Such a walk has composition

$$\begin{bmatrix} r \\ 1 \end{bmatrix} + (2n-4-r)\begin{bmatrix} n-1 \\ 1 \end{bmatrix} + r\begin{bmatrix} n-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2n^2-6n+4 \\ 2n-3 \end{bmatrix}.$$

Hence $\exp(D) \le 2n^2 - 4n + 1$. □

Lemma 3.3 *Let $D \in \mathcal{D}_n$ be primitive. If D is Type 3 in Table 2, then* $\exp(D) \le n^2 - n$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+2 & a+1 \\ n-a-2 & n-a-3 & n-a-2 \end{bmatrix},$$

where $0 \le a \le n-3$.

For any pair (i,j) of vertices of D, we prove that there is a $(na+2n-a-2, n^2-na-3n+a+2)$-walk in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \le r+b \le n-1$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n-1-r-b)$ times around γ_1, b times around γ_2, and r times around γ_3. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (n-1-r-b)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + b\begin{bmatrix} a+2 \\ n-a-3 \end{bmatrix} + r\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix}$$
$$= \begin{bmatrix} na+2n-a-2 \\ n^2-na-3n+a+2 \end{bmatrix}.$$

Hence $\exp(D) \le n^2 - n$. □

Lemma 3.4 *Let $D \in \mathcal{D}_n$ be primitive. If D is Type 4 in Table 2, then* $\exp(D) = 2n^2 - 3n + 1$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} n-1 & n-2 & n-2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Clearly, D has only three blue arcs, and they are $n \to 1$, $(n-1) \to 1$ and $n \to 2$.

First, we prove that $\exp(D) \le 2n^2 - 3n + 1$.

Let (i, j) be any pair of vertices of D, and p_{ij} be the shortest walk from i to j containing one blue arc. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $b = 1$ and $0 \le r \le 2n - 3$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(2n - 3 - r)$ times around γ_1, and r times around γ_2. Such a walk has composition

$$\begin{bmatrix} r \\ 1 \end{bmatrix} + (2n-3-r)\begin{bmatrix} n-1 \\ 1 \end{bmatrix} + r\begin{bmatrix} n-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2n^2-5n+3 \\ 2n-2 \end{bmatrix}.$$

Hence $\exp(D) \le 2n^2 - 3n + 1$.

Next, we prove that $\exp(D) \ge 2n^2 - 3n + 1$.

Note that the compositions of cycles γ_2 and γ_3 are the same. Now we set

$$N = \begin{bmatrix} n-1 & n-2 \\ 1 & 1 \end{bmatrix}.$$

Suppose that (h, k) is a pair of nonnegative integers such that for all pairs (i, j) of vertices there is an (h, k)-walk from i to j. By considering $i = j = 2$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}.$$

Taking $i = 1$ and $j = n$, then there is a unique path from i to j, and this path has composition $(n - 1, 0)$. Hence

$$Nz = \begin{bmatrix} h-(n-1) \\ k \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = N^{-1}\begin{bmatrix} h-(n-1) \\ k \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - N^{-1}\begin{bmatrix} n-1 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} n-1 \\ 1-n \end{bmatrix} \ge 0.$$

So $u \ge n - 1$. Taking $i = n$ and $j = 1$, then the path from i to j has composition either $(0, 1)$ or $(n - 3, 2)$, so we have that

$$Nz = \begin{bmatrix} h \\ k-1 \end{bmatrix} \quad \text{or} \quad Nz = \begin{bmatrix} h-(n-3) \\ k-2 \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - N^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 2-n \\ n-1 \end{bmatrix} \geq 0,$$

or

$$z = \begin{bmatrix} u \\ v \end{bmatrix} - N^{-1}\begin{bmatrix} n-3 \\ 2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -n+1 \\ n+1 \end{bmatrix} \geq 0.$$

So $v \geq n-1$. Thus

$$\exp(D) \geq h+k = \begin{bmatrix} 1 & 1 \end{bmatrix} N \begin{bmatrix} u \\ v \end{bmatrix} \geq \begin{bmatrix} n & n-1 \end{bmatrix} \begin{bmatrix} n-1 \\ n-1 \end{bmatrix} = 2n^2-3n+1.$$

The lemma follows. □

Lemma 3.5 *Let* $D \in \mathcal{D}_n$ *be primitive. If* D *is Type 5 in Table 2, then* $\exp(D) \leq n^2 - n$.

Proof The proof is similar to the proof of Lemma 3.3. We omit it. □

Lemma 3.6 *Let* $D \in \mathcal{D}_n$ *be primitive. If* D *is Type 6, then* $\exp(D) \leq 2n^2 - 4n + 1$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} n-1 & n-2 & n-2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Clearly, D has only two blue arcs, and they are $n \to 2$ and $1 \to 2$ (or $(n-1) \to n$ and $(n-1) \to 1$).

For any pair (i,j) of vertices of D, we prove that there is a $(2n^2-6n+4, 2n-3)$-walk in D. Let p_{ij} be the shortest walk from i to j containing one blue arc. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $b = 1$ and $0 \leq r \leq 2n-4$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(2n-4-r)$ times around γ_1, and r times around γ_2. Such a walk has composition

$$\begin{bmatrix} r \\ 1 \end{bmatrix} + (2n-4-r)\begin{bmatrix} n-1 \\ 1 \end{bmatrix} + r\begin{bmatrix} n-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2n^2-6n+4 \\ 2n-3 \end{bmatrix}.$$

Hence $\exp(D) \leq 2n^2 - 4n + 1$. □

Lemma 3.7 *Let* $D \in \mathcal{D}_n$ *be primitive. If* D *is Type 7 in Table 2, then* $\exp(D) \leq 2n^2 - 3n$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+2 & a \\ n-a-2 & n-a-3 & n-a-1 \end{bmatrix},$$

where a is odd, and $1 \leq a \leq n-3$.

For any pair (i,j) of vertices of D, we prove that there is a $(2na + 4n - 3a - 6, 2n^2 - 2na - 7n + 3a + 6)$-walk in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \leq r + b \leq n - 1$. We consider two cases.

Case 1. r is even. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(2n - 3 - r - b)$ times around γ_1, $(b + \frac{r}{2})$ times around γ_2, and $\frac{r}{2}$ times around γ_3, has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (2n-3-r-b)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (b+\frac{r}{2})\begin{bmatrix} a+2 \\ n-a-3 \end{bmatrix}$$
$$+\frac{r}{2}\begin{bmatrix} a \\ n-a-1 \end{bmatrix} = \begin{bmatrix} 2na+4n-3a-6 \\ 2n^2-2na-7n+3a+6 \end{bmatrix}.$$

Case 2. r is odd. If $r + b = n - 1$, then $i = 1$ and $j = n$, and thus $r = a + 1$ is even. It is a contradiction. So $r + b \leq n - 2$. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n - 2 - r - b)$ times around γ_1, $(n + b - \frac{a+2-r}{2})$ times around γ_2, and $(\frac{r+a}{2} + 1)$ times around γ_3, has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (n-2-r-b)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n+b-\frac{a+2-r}{2})\begin{bmatrix} a+2 \\ n-a-3 \end{bmatrix}$$
$$+(\frac{r+a}{2}+1)\begin{bmatrix} a \\ n-a-1 \end{bmatrix} = \begin{bmatrix} 2na+4n-3a-6 \\ 2n^2-2na-7n+3a+6 \end{bmatrix}.$$

Hence $\exp(D) \leq 2n^2 - 3n$. □

Lemma 3.8 *Let $D \in \mathcal{D}_n$ be primitive. If D is Type 8 in Table 2, then* $\exp(D) \leq 2n^2 - 4n + 2$.

Proof The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & a+1 & a \\ n-a-2 & n-a-2 & n-a-1 \end{bmatrix},$$

where $0 \leq a \leq n-3$.

For any pair (i,j) of vertices of D, we prove that there is a $(2na + 2n - a^2 - 3a - 2, 2n^2 - 3na - 6n + a^2 + 4a + 4)$-walk in D. Let p_{ij} be the shortest path from i to j such that $0 \leq r(p_{ij}) + b(p_{ij}) \leq n - 1$ and

$0 \leq b(p_{ij}) \leq n-a-2$. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n-1-r-b)$ times around γ_1, $(r+2b)$ times around γ_2, and $(n-a-2-b)$ times around γ_3. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (n-1-r-b)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (r+2b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix}$$

$$+(n-a-2-b)\begin{bmatrix} a \\ n-a-1 \end{bmatrix} = \begin{bmatrix} 2na+2n-a^2-3a-2 \\ 2n^2-3na-6n+a^2+4a+4 \end{bmatrix}.$$

Hence $\exp(D) \leq 2n^2 - na - 4n + a + 2 \leq 2n^2 - 4n + 2$. □

From Lemmas 3.1–3.8, we obtain the tight upper bound on the exponents of primitive two-colored digraphs in $\mathcal{D}_n$, and the characterization of the extremal two-colored digraph.

Theorem 3.9 *Let $D \in \mathcal{D}_n$ be primitive. Then* $\exp(D) \leq 2n^2 - 3n + 1$, *and* $\exp(D) = 2n^2 - 3n + 1$ *if and only if D has only three blue arcs which are $n \to 1$, $(n-1) \to 1$ and $n \to 2$.*

References

[1] B.L. Shader, S. Suwilo, Exponents of nonnegative matrix pairs, *Linear Algebra Appl.*, 363(2003), 275–293.

[2] A. Berman, R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, Vol. 9, SIAM, Philadelphia, PA, 1994.

[3] D.D. Olesky, B.L. Shader, P. van den Driessche, Exponents of tuples of nonnegative matrices, *Linear Algebra Appl.*, 356(2002), 123–134.

[4] L.B. Beasley, S. Kirkland, A note on k-primitive directed graphs, *Linear Algebra Appl.*, 373(2003), 67–74.

[5] Yubin Gao, Yanling Shao, Exponents of two-colored digraphs with two cycles, *Linear Algebra Appl.*, 407(2005), 263–276.

[6] Yanling Shao, Yubin Gao, Liang Sun, Exponents of a class of two-colored digraphs, *Linear and Multilinear Algebra*, 53:3(2005), 175–188.