On perfect matchings of complements of line graphs

Xiaoping Liu, Xinhui An and Baoyindureng Wu*

School of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, P.R. China

baoyin@xju.edu.cn

Abstract

Let $G = (V(G), E(G))$ be a nonempty graph (may have parallel edges). The line graph $L(G)$ of G is the graph with $V(L(G)) = E(G)$, and in which two vertices e and e' are joined by an edge if and only if they have a common vertex in G. We call the complement of $L(G)$ as the jump graph. In this note, we give a simple sufficient and necessary condition for a jump graph to have a perfect matching.

Keywords: Line graph; Claw-free graph; Jump graph; Perfect matching

1 Introduction

We consider finite undirected graphs (may have parallel edges) without loops, and refer to [2] for undefined terminology and notations. For a graph, two edges are called parallel edges if they join the same pair of distinct vertices. A graph is simple if it has no loops and parallel edges. Let G be a graph with parallel edges, and let u and v be two vertices of G. $\mu(u, v)$ denotes the number of edges with their two end vertices as u and v. For

*Corresponding author. Supported by NSFC and the grant XJEDU2004G05.

every pair of adjacent vertices, by deleting from G all but one edge joining them, we obtain a simple spanning subgraph of G, called the underlying simple graph of G. We denote it by $\underline{G}$. Clearly, G is simple if and only if $G = \underline{G}$. Suppose that V' is a nonempty subset of $V(G)$. The subgraph $G[V']$ of G induced by V' is a graph with $V(G[V']) = V'$ and $uv \in E(G[V'])$ if and only if $uv \in E(G)$. As usual, $\varepsilon(G), \omega(G), \Delta(G)$, and $\delta(G)$ denote the number of edges, the number of components, the maximum degree, and the minimum degree of G, respectively. A subset M of E is called a matching of G if no two elements of M are adjacent in G. A matching M is called a perfect matching if every vertex of G is incident with an edge of M in G. A component of a graph is odd or even according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of G. Tutte [6] obtained a necessary and sufficient condition for a graph to have a perfect matching.

Theorem 1.1 (Tutte's Theorem). A graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for all proper subset S of $V(G)$.

For two graph G and H, Let $G = (V(G), E(G))$ and $H = (V(H), E(G))$ be two graphs. The *union* $G \cup H$ of G and H is the graph whose vertex set is $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. Particularly, we denote their union by $G + H$ if they are disjoint, i. e., $V(G) \cap V(H) = \phi$. The disjoint union of k copies of G is written as kG. C_n and K_n are the cycle and complete graph with n vertices respectively. K_4^- is the graph resulting from K_4 by deleting an edge. $K_{r,s}$ is the complete bipartite graph with two partite sets containing r and s vertices. In particular, if one of r and s is equal to 1, $K_{r,s}$ is called a star.

Let $G = (V(G), E(G))$ be a nonempty graph (i. e. G contains at least one edge). The line graph $L(G)$ of G is the graph with $V(L(G)) = E(G)$, and in which two vertices e and e' are joined by an edge if and only if they have a common vertex in G. For a graph G, we call the complement of $L(G)$ as the jump graph of G [3]. Clearly, both $L(G)$ and $J(G)$ are simple. It is well known that for a connected graph G, $L(G)$ has a perfect matching if and only if G has an even number of edges. So, it is natural to consider when the complement of a line graph, the jump graph, has a perfect matching. Wu and Wang [8] proved that for a simple graph $G \ncong K_3 + K_2$, $J(G)$ has a perfect matching if and only if $\varepsilon(G)$ is an even number not less than $2\Delta(G)$. In this note, we generalize the previous result to graphs with parallel edges. Before stating our main result, we need an additional notation. For a graph G, $\nabla(G) = max\{\varepsilon(H) \mid H$ is a subgraph of G with $\underline{H} \cong K_3\}$ if G contains a triangle, otherwise $\nabla(G) = 0$. The following is our main theorem.

Theorem 1.2. For a graph G, $J(G)$ has a perfect matching if and only if $\varepsilon(G)$ is an even number not less than $2max\{\Delta(G), \nabla(G)\}$.

2 Connectedness of jump graphs

Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph G, we assume the graph under consideration is nonempty and has no isolated vertices. It is easy to see that for a graph G, $L(G)$ is connected if and only if G is connected.

For a simple graph G, an edge e is called a *dominating* edge if it is adjacent to every other edge of G. Observe that if G has a dominating edge e, then e is an isolated vertex of $J(G)$, and thus $J(G)$ is not connected. So, if $J(G)$ is connected, then G contains no dominating edges. Chartrand et. al [3] proved that this necessary condition is almost sufficient for every simple graph to have its jump graph connected.

Lemma 2.1([3]). For a simple graph G with at least 5 vertices, $J(G)$ is connected if and only if it contains no dominating edges.

It is trivial to check that among the simple graphs with no more than 4 vertices, C_4 and K_4 are the only two graphs with the properties that they contain no dominating edge and their jump graphs are not connected. So, we have

Corollary 2.2. For a simple graph G with at least 2 edges, $J(G)$ is not connected if and only if either G contains a dominating edge or $G \in \{C_4, K_4\}$ up to isomorphism.

For a simple graph G, let $\xi(G) = max\{d(u) + d(v) |\ u$ and v are taken over any pair of adjacent vertices in $G\}$. Note that for a graph G, $\varepsilon(G) \geq \xi(G) - 1$, and the equality holds if and only if G contains a dominating edge. Thus Corollary 2.2 is equivalent to the following.

Corollary 2.3. For a simple graph G of size $q \geq 2$, $J(G)$ is not connected if and only if either $q = \xi(G) - 1$ or $G \in \{C_4, K_4\}$ up to isomorphism.

Let G be a graph, and $u, v \in V(G)$. We call u and v are twins if they have the same neighborhood in G. Obviously, if u and v are twins, they are not adjacent in G. The proof of Lemma 2.4 and Corollary 2.5 below are trivial, so it is omitted.

Lemma 2.4. Let G be graph, and u and v be twins. Then we have

(i). G is connected if and only if $G - u$ is connected.

(ii). G and $G - u$ have the same number of nontrivial components.

Corollary 2.5. For a graph G, the following statements hold:

(i). $J(G)$ is connected if and only if $J(\underline{G})$ is connected.

(ii). $J(G)$ and $J(\underline{G})$ have the same number of nontrivial components.

Theorem 2.6. Let G be a graph (may have parallel edges). Then

(i). $J(G)$ has at most three nontrivial components,

(ii). $J(G)$ has three nontrivial components if and only if $\underline{G} \cong K_4$,

(iii). $J(G)$ has exactly two nontrivial components if and only if $\underline{G} \cong K_4^-$ or C_4,

(iv). $J(G)$ has no nontrivial components if and only if $\underline{G} \cong K_3$ or a star,

(v). $J(G)$ is not connected and has just one nontrivial component if and only if G has a dominating edge, and $\underline{G}$ is not isomorphic to a star, or K_3, or K_4^-.

Proof. By Corollary 2.5 (*ii*), $J(G)$ and $J(\underline{G})$ have the same number of nontrivial components. So, to prove (*i*), it suffices to prove the result for $J(\underline{G})$. By contradiction, suppose H_1, H_2, H_3, and H_4 are four nontrivial components of $J(\underline{G})$. We take a vertex e_i from H_i for each $i = 1, 2, 3$, and 4. By the definition of jump graph, these e_is are pairwise adjacent in G, namely, they must have a common end vertex. Let e'_1 be a neighbor of e_1 in H_1. Then e_1 and at least one element of $\{e_2, e_3, e_4\}$, say e_2, are not adjacent to e'_1 in G. So, $e_1 e'_1 e_2$ is a path in $J(\underline{G})$, and e_1 and e_2 should be in the same component of $J(\underline{G})$. A contradiction.

The sufficiency of (*ii*) is obvious. To prove the necessity, we take two adjacent vertices e_i and e'_i from H_i for each $i = 1, 2, 3$. Let u_i and v_i be the two end vertices of e_i in G; u'_i and v'_i those of e'_i in G. By the definition of jump graph, $\{u_i, v_i\} \cap \{u'_i, v'_i\} = \phi$ for $i = 1, 2, 3$, and both e_i and e'_i are adjacent to each of e_j and e'_j in G. It follows that $\{u_i, v_i, u'_i, v'_i\} = \{u_j, v_j, u'_j, v'_j\}$ for any pair of i and j with $i \neq j$. Set $S = \{u_1, v_1, u'_1, v'_1\}$. Then $\underline{G}[S] \cong K_4$. Clearly, if there is an edge of $\underline{G}$ whose one end vertex is not in S, then $\underline{G}$ contains no dominating edge and $\underline{G} \ncong C_4$ or K_4. By Corollary 2.2 and Corollary 2.5 (*i*), $J(\underline{G})$ is connected, and so $J(G)$ is connected. A contradiction. So, $V(G) = V(\underline{G}) = S$ and since $\underline{G}[S] \cong K_4$, we have $\underline{G} \cong K_4$.

The sufficiency of (*iii*) is also obvious. Now we show its necessity. Let H_1 and H_2 be the two nontrivial components of $J(G)$. We take two adjacent vertices e_i and e'_i from H_i, $i = 1, 2$. Let u_i and v_i be the two end vertices of e_i, and u'_i and v'_i those of e'_i for $i = 1, 2$ in G. By the similar arguments as in proof of (*ii*), it follows that $\{u_1, v_1, u'_1, v'_1\} = \{u_2, v_2, u'_2, v'_2\}$. Let $S = \{u_1, v_1, u'_1, v'_1\}$. Then $\underline{G}[S]$ contains C_4, and thus combining with the result of (*ii*), we have $\underline{G}[S] \cong C_4$ or K_4^-. Note that if there is an edge of $\underline{G}$ whose one end vertex is not in S, then $J(\underline{G})$ contains at most one nontrivial component. This contradicts the assumption. Hence $V(G) = V(\underline{G}) = S$, and moreover, if $\omega(J(G)) = 2$, then $\underline{G} \cong C_4$, and if $\omega(J(G)) \geq 3$, then $\underline{G} \cong K_4^-$.

The result of (iv) is obvious.

(v) follows from Corollary 2.2 and the results (i) – (iv).

3 Proof of Theorem 1.2

We first prove the necessity. Let G be a graph whose jump graph has a perfect matching. Then clearly $\varepsilon(G)$ is even. Let E' be a maximum independent set of $J(G)$. Then any two elements of E' are adjacent in G, and $|E'| = max\{\Delta(G), \nabla(G)\}$. Set $S = E(G)\backslash E'$. Since $J(G)$ has a perfect matching, by Tutte's Theorem, we have $o(J(G) - S) = \omega(J(G) - S) = |E'| \leq |S|$, and $\varepsilon(G) = |E'| + |S| \geq 2|E'| = 2max\{\Delta(G), \nabla(G)\}$.

Next we show the sufficiency. Suppose $J(G)$ is not connected. Then $\underline{G} \cong K_4$ or C_4 by $\varepsilon(G) \geq 2max\{\Delta(G), \nabla(G)\} \geq \xi(G)$ and Corollary 2.3. Let $V(G) = \{v_1, v_2, v_3, v_4\}$. First assume that $\underline{G} \cong C_4$, and v_i and v_{i+1} are adjacent in $\underline{G}$ for $i = 1, 2, 3, 4$, where the subscript is taken modulo 4. The fact that $J(G)$ is not connected and $\varepsilon(G) \geq 2max\{\Delta(G), \nabla(G)\}$ implies $\varepsilon(G) = 2\Delta(G)$, $\mu(v_1, v_2) = \mu(v_3, v_4)$ and $\mu(v_2, v_3) = \mu(v_1, v_4)$. Let $a = \mu(v_1, v_2)$, and $b = \mu(v_2, v_3)$. Thus $J(G) \cong K_{a,a} + K_{b,b}$, and $J(G)$ has a perfect matching. If $\underline{G} \cong K_4$, then similarly we have $\varepsilon(G) = 2\Delta(G)$, and $\mu(v_1, v_2) = \mu(v_3, v_4)$, $\mu(v_1, v_3) = \mu(v_2, v_4)$, and $\mu(v_1, v_4) = \mu(v_2, v_3)$. Let $\mu(v_1, v_2) = a$, $\mu(v_1, v_3) = b$, and $\mu(v_1, v_4) = c$. Then $J(G) \cong K_{a,a} + K_{b,b} + K_{c,c}$, and $J(G)$ has a perfect matching.

Now suppose G is a graph with properties that $\varepsilon(G)$ is an even number not less than $2max\{\Delta(G), \nabla(G)\}$, and $J(G)$ has no perfect matching. Then $J(G)$ is connected and by Tutte's theorem, there exists a nonempty subset $S \subseteq V(J(G))$ with $o(J(G) - S) \geq |S| + 2$. Therefore, $o(J(G) - S) \geq 3$, and $J(G) - S$ has at most three nontrivial components by (i) of Theorem 2.6. Clearly, $J(G) - S = J(G - S)$. Let $n = |V(G)|$ and $q = \varepsilon(G)$. We consider the following cases.

Case 1. $J(G) - S$ has three nontrivial components.

By (*ii*) of Theorem 2.6, $\omega(J(G-S)) = \omega(J(G)-S) = 3$ and $\underline{G-S} \cong K_4 + (n-4)K_1$. Together with $\omega(J(G)-S) \geq o(J(G)-S) \geq |S|+2$, it follows that $|S| = 1$, each component of $J(G)-S$ is odd, and $\underline{G-S} \cong K_4 + K_1$. Let $\{v_1, v_2, v_3, v_4\}$ be the set of vertices in the nontrivial component of $G-S$. Then $\underline{G}[\{v_1, v_2, v_3, v_4\}] \cong K_4$. Let $a = max\{\mu(v_1, v_2), \mu(v_3, v_4)\}$, $b = max\{\mu(v_1, v_3), \mu(v_2, v_4)\}$, and $c = max\{\mu(v_1, v_4), \mu(v_2, v_3)\}$. Then

$$\begin{aligned} q-1 &\leq a+(a-1)+b+(b-1)+c+(c-1) \\ &= 2(a+b+c)-3 \\ &\leq 2max\{\Delta(G), \nabla(G)\}-3, \end{aligned}$$

equivalently, $q \leq 2max\{\Delta(G), \nabla(G)\} - 2$. Thus it contradicts with fact that $q \geq 2max\{\Delta(G), \nabla(G)\}$.

Case 2. $J(G)-S$ has exactly two nontrivial components.

By (*iii*) of Theorem 2 , $\underline{G-S}$ is isomorphic to $K_4^- + (n-4)K_1$ or $C_4 + (n-4)K_1$. Since $\omega(J(C_4 + (n-4)K_1)) = 2$ and $\omega(J(G)-S) \geq 3$, we have $\underline{G-S} \cong K_4^- + (n-4)K_1$. Let $\{v_1, v_2, v_3, v_4\}$ be the set of vertices in the nontrivial component of $G-S$. Hence $\underline{G}[\{v_1, v_2, v_3, v_4\}] \cong K_4^-$, where we assume that v_2 and v_4 are not adjacent in $G-S$. Let $a = max\{\mu(v_1, v_2), \mu(v_3, v_4)\}$, $b = \mu(v_1, v_3)$, $c = max\{\mu(v_1, v_4), \mu(v_2, v_3)\}$. We consider three subcases below.

Subcase 2.1. The two nontrivial components of $J(G)-S$ are both odd. Then $2+b = o(J(G)-S) \geq |S|+2$, and $|S| \leq b$. So

$$\begin{aligned} q &= |E(G) \setminus S| + |S| \\ &\leq a+(a-1)+c+(c-1)+b+b \\ &= 2(a+b+c)-2 \\ &\leq 2max\{\Delta(G), \nabla(G)\}-2. \end{aligned}$$

But $q \geq 2max\{\Delta(G), \nabla(G)\}$, a contradiction.

Subcase 2.2. The two nontrivial components of $J(G)-S$ are both even.

Then $b = o(J(G) - S) \geq |S| + 2$, and thus $q = |E(G) \setminus S| + |S| \leq 2a + 2c + b + b - 2 = 2(a + b + c) - 2$, a contradiction.

Subcase 2.3. One of the two nontrivial components is even, and the other is odd.

Then $b + 1 = o(J(G) - S) \geq |S| + 2$, and $b \geq |S| + 1$. Therefore $q = |E(G) \setminus S| + |S| \leq 2a + 2c - 1 + b + b - 1 = 2(a + b + c) - 2 \leq 2max\{\Delta(G), \nabla(G)\} - 2$, a contradiction.

Case 3. There is no nontrivial components in $J(G) - S$.

Then $o(J(G) - S) = \omega(J(G) - S) = q - |S|$. Since $o(J(G) - S) \geq |S| + 2$, we have $q \geq 2|S| + 2$. On the other hand, as $G - S$ is isomorphic to K_3 or a star by (iv) of Theorem 2.6, we have $q - |S| = max\{\Delta(G - S), \nabla(G - S)\} \leq max\{\Delta(G), \nabla(G)\} \leq \frac{q}{2}$, i. e., $q \leq 2|S|$, a contradiction.

Case 4. $J(G) - S$ has only one nontrivial component.

By (v) of Theorem 2.6, $G - S$ has a dominating edge, say e, and let u and v be the two end vertices of e. If there does not exist other dominating edge that is not parallel to e in $G - S$, then $o(J(G) - S) = \mu_{G-S}(u, v)$ or $\mu_{G-S}(u, v) + 1$. Since $o(J(G) - S) \geq |S| + 2$, we have

$$\begin{aligned} q &= |E(G) \setminus S| + |S| \\ &= d_{G-S}(u) + d_{G-S}(v) - \mu_{G-S}(u, v) + |S| \\ &\leq d_{G-S}(u) + d_{G-S}(v) - 1 \\ &\leq 2\Delta(G) - 1, \end{aligned}$$

a contradiction.

Now suppose there exist a dominating edge e' that is not parallel to e in $G - S$. Then e and e' have one common vertex, say u, and let w be the other end vertex of e'. Since both e and e' are dominating edges of $G - S$, and $J(G) - S$ has exactly one nontrivial component, $G - S$ is isomorphic to the graph obtained from a star with at least 4 vertices by joining its two vertices of degrees one. Let $a = \mu_{G-S}(u, v)$, $b = \mu_{G-S}(u, w)$, $d = \mu_{G-S}(v, w)$, and $c = d_{G-S}(u) - a - b$. Then we have $q - |S| = a + b + c + d$. Observe that

$o(J(G)-S) \leq w(J(G)-S) = a+b+1$ and by $o(J(G)-S) \geq |S|+2$, it follows that $|S| \leq a+b-1$. So, $q \leq (a+b+c+d)+(a+b-1) = 2(a+b)+c+d-1$. On the other hand, $q \geq 2max\{\Delta(G), \nabla(G)\} \geq 2max\{\Delta(G-S), \nabla(G-S)\} \geq 2max\{a+b+c, a+b+d\} \geq a+b+c+a+b+d = 2(a+b)+c+d$, a contradiction.

For all cases, we obtain a contradiction. So for any proper subset S of $V(J(G))$, we have $o(J(G)-S) \leq |S|$. By Tutte's Theorem, $J(G)$ has a perfect matching. The proof is complete.

4 Concluding remarks

In this note, we give a simple necessary and sufficient condition for a jump graph $J(G)$ (G may not be a simple graph) to have a perfect matching. Wu and Meng [7] showed that for a simple graph G with $\varepsilon(G) \geq 11$, $J(G)$ is hamiltonian if and only if $\varepsilon(G) > 2\Delta(G)$, or $\varepsilon(G) = 2\Delta(G)$ and G has no edge uv with $d(u) = d(v) = \Delta(G)$. So, the condition for a jump graph having a hamiltonian cycle is slightly stronger than that for it having a perfect matching. It is interesting to give a necessary and sufficient condition for a jump graph $J(G)$ (G is not a simple) to be hamiltonian.

There is a natural superclass of line graphs, called claw-free graphs. A graph is said to be claw-free if it contains no induced subgraph isomorphic to $K_{1,3}$. It is clear that line graphs are claw-free by the forbidden subgraph characterization of line graphs by Beineke [1]. Sumner [5], independently Las Vergnas [4], proved that if G is a connected claw-free graph of even number of vertices, then G has a perfect matching. Motivated from our results, one may consider the corresponding problems on complements of claw-free graphs. However, it is certainly a difficult task to characterize those with perfect matchings or with hamiltonian cycles, since triangle-free graphs are a special class of complements of claw-free graphs, and there is no efficient way to determine if a triangle-free graph has a perfect matching.

References

[1] L. Beineke, "Derived Graphs and Digraphs", Beiträge zur Graphentheorie, Teubner, Leipzig, 1968.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[3] G. Chartrand, H. Hevia, E.B. Jarrett, M. Schultz, Subgraph distances in graphs defined by edge transfers, Discrete Math. **170**(1997) 63-79.

[4] M. Las Vergas, A note on matchings in graphs, Cahiers Centre Etudes Rech. Opér. 17(1975) 257-260.

[5] D.P. Sumner, Graphs with 1-factors, Proc. Amer. Math. Soc. 42(1974) 8-12.

[6] W. T. Tutte, The factorization of linear graphs, J. London Math. Soc. 22(1947), 107-111.

[7] B. Wu and J. Meng, Hamiltonian jump graphs, Discrete Math. **289**(2004) 95-106.

[8] B. Wu and W. Wang, Jump graphs with perfect matchings, Graph Theory Notes of New York XXXIX (2000) 23-25.