Restricted total domination in graphs with minimum degree two

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Abstract

The k-restricted total domination number of a graph G is the smallest integer t_k such that given any subset U of k vertices of G, there exists a total dominating set of G of cardinality at most t_k containing U. Hence, the k-restricted total domination number of a graph G measures how many vertices are necessary to totally dominate a graph if an arbitrary set of k vertices are specified to be in the set. When $k = 0$, the k-restricted total domination number is the total domination number. For $1 \leq k \leq n$, we show that $t_k \leq 4(n + k)/7$ for all connected graphs of order n and minimum degree at least two and we characterize the graphs achieving equality. These results extend earlier results of the author (*J. Graph Theory* **35** (2000), 21–45). Using these results we show that if G is a connected graph of order n with the sum of the degrees of any two adjacent vertices at least four, then $\gamma_t(G) \leq 4n/7$ unless $G \in \{C_3, C_5, C_6, C_{10}\}$.

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1 Introduction

In this paper we continue the study of restricted dominating sets started by Sanchis [10]: the restricted version of a parameter considers the case when certain vertices are specified to be in the set. We establish a sharp bound for the case of total domination.

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [4] and is now well studied in graph theory (see, for example, [1] and [9]). A *total dominating set* (TDS) of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S (other than itself). The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set.

In this paper we study *restricted total domination* in graphs where we restrict the total dominating sets to contain any given subset of vertices. Let U be a subset of vertices of a graph G. The *restricted total domination number* $r(G, U, \gamma_t)$ *of* U is the minimum cardinality of a TDS of G containing U. A smallest possible TDS of G containing all the vertices in U, we call a $\gamma_t(G, U)$-set. The *k-restricted total domination number* of G is the smallest integer $r_k(G, \gamma_t)$ such that $r(G, U, \gamma_t) \leq r_k(G, \gamma_t)$ for all subsets U of G of cardinality k. Note that when $k = 0$, the k-restricted total domination number is the total domination number.

The concept of restricted domination in graphs, where we restrict the dominating sets to contain any given subset of vertices, was introduced by Sanchis in [10] and studied further in [7, 11]. Restricted total domination in graphs was introduced and studied in [8]. For more on domination, see the book [5].

For notation and graph theory terminology we in general follow [3]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E of size q, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the *open neighborhood* of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. For sets $S, T \subseteq V$, the set S *totally dominates* T if $T \subseteq N(S)$. In particular, if $T = V$, then S is a TDS of G. The subgraph of G induced by the vertices in S is denoted by $G[S]$. The minimum degree among the vertices of G is denoted by $\delta(G)$. The distance $d(v, S)$ of a vertex v from a set S of vertices is the minimum distance from v to a vertex of S. The *girth* of G is the length of a shortest cycle in G.

A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. For $m \geq 3$ and $n \geq 1$, we denote by $L_{m,n}$ the graph of order $m+n$ obtained by joining with an edge a vertex in C_m to an end-vertex of P_n. The graph $L_{m,n}$ is called a *key*. For integers $m, n \geq 3$, we define a *dumb-bell* $D_b(m,n)$ to be the graph of order $m+n$ obtained from the cycles C_m and C_n by joining a vertex of C_m to a vertex of C_n, and we define a *daisy* $D(m,n)$ to be the graph of order $m+n-1$ obtained from the cycles C_m and C_n by identifying a set of two vertices, one from each cycle, into one vertex. In this paper, we define

$$F(n,k) = \frac{4(n+k)}{7} \qquad \text{and} \qquad H(n,k) = \frac{n+k+1}{2}.$$

We let $x \equiv_\ell y$ mean $x \equiv y \pmod{\ell}$.

2 Main Results

A bound on the total domination number of a connected graph with minimum degree at least two is established in [6].

Theorem 1 ([6]) *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$.*

An upper bound on the restricted total domination number of a connected graph with minimum degree at least two in terms of its size is established in [8].

Theorem 2 ([8]) *For $1 \leq k \leq n$, if G is a connected graph of order n and size q with $\delta(G) \geq 2$, then*

$$r_k(G, \gamma_t) \leq \frac{q+k+1}{2}.$$

In this paper we have three immediate aims: First to extend the bound for the total domination number obtained in Theorem 1 to the restricted total domination number. Secondly to improve on the bound in Theorem 2 for dense graphs (or rather for graphs that are not very sparse). Thirdly to establish a sharp upper bound on the total domination number of a connected graph with the sum of the degrees of any two of its adjacent vertices at least four, in terms of its order.

To achieve our aims, we will need the following terminology. For $k \geq 1$ an integer, we will refer to a graph G as a **$\frac{4}{7}$-k-minimal graph** if G is edge-minimal with respect to satisfying the following three conditions:

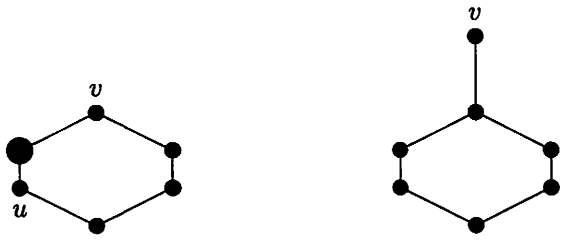
(i) $\delta(G) \geq 2$,
(ii) G is connected, and
(iii) $r_k(G, \gamma_t) \geq 4(n+k)/7$,

where n is the order of G.

Next we define a family of $\frac{4}{7}$-k-minimal graphs. For this purpose, we define a **unit** to be a graph that is isomorphic to a cycle C_6 or to a key $L_{6,1}$. There are two types of units and we call a unit **type (a)** or **type (b)** according to whether it is a cycle or a key, respectively.

In each unit, we define a link vertex (vertices) of the unit as follows. In a type (a) unit, we select two vertices at distance two apart in the unit and we call these two vertices the **link vertices** of the unit. In a type (b) unit we call the vertex of degree one the **link vertex** of the unit. In a type (a) unit, we call the vertex adjacent to the two link vertices the **pivot vertex** of the unit.

Figure 1 illustrated the two types of units. The link vertices in the type (a) unit in Figure 1 are labelled u and v with the pivot vertex indicated by the large darkened vertex, while the link vertex in the type (b) unit is labelled v.



(i) Type (a) unit (ii) Type (b) unit

Figure 1: The two types of units.

For nonnegative integers a and b with $a \geq 1$, let $\mathcal{G}_a$ denote the family of all *connected* graphs G that are obtained from the disjoint union of a units of type (a) and b units of type (b) (called the **units of G**) by adding $a+b-1$ edges in such a way that every added edge joins two link vertices and is a bridge of G (which we call a **link edge** of G). Let $\mathcal{G} = \{G \mid G \in \mathcal{G}_a$ for some integer $a \geq 1\}$. A graph in the family $\mathcal{G}$ with $a = 2$ and $b = 2$ is shown in Figure 2 (the pivot vertices are indicated in the figure by the two large darkened vertices, while the set of link-vertices is given by $\{u, v, w, x, y, z\}$).

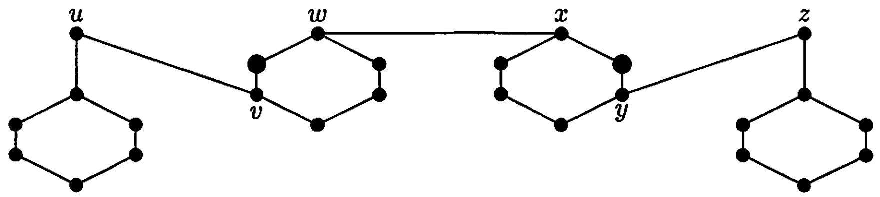


Figure 2: A graph in the family $\mathcal{G}$.

The following result, a proof of which is presented in Section 5, establishes properties about graphs in the family $\mathcal{G}$.

Theorem 3 *Let $G \in \mathcal{G}$ have order n. For $1 \le k \le n$, let U be a subset of k vertices of G. Then, $r(G, U, \gamma_t) \le F(n, k)$ with equality if and only if there is a selection of units and link vertices of G so that the resulting set of pivot vertices is precisely the set U. Furthermore, if $r(G, U, \gamma_t) = F(n, k)$, then for any vertex v of G there is a $\gamma_t(G, U)$-set containing v.*

The following result, a proof of which is presented in Section 6, characterizes $\frac{4}{7}$-k-minimal graphs.

Theorem 4 *For $k \ge 1$, a graph G is a $\frac{4}{7}$-k-minimal graph if and only if $G \in \mathcal{G}_k$.*

Since the restricted total domination number of a graph cannot decrease if edges are removed, we can use Theorems 3 and 4 to prove our main result which establishes an upper bound on the restricted total domination number of a connected graph with minimum degree at least two in terms of its order. For integers $k \ge 1$, let $\mathcal{G}_k^*$ denote the family of all graphs that can be obtained from a graph $G \in \mathcal{G}_k$ by adding any number of new edges, including the possibility of none, joining link vertices of G. We shall prove (see Section 7):

Theorem 5 *For $1 \le k \le n$, if G is a connected graph of order n with $\delta(G) \ge 2$, then $r_k(G, \gamma_t) \le F(n, k)$ with equality if and only if $G \in \mathcal{G}_k^*$.*

Remark 1. Since the vertex set of a (nontrivial) connected graph G of order n is a TDS of G, the restricted total domination number of G is at most n, i.e., $r_k(G, \gamma_t) \leq n$. Hence the result in Theorem 5 is only meaningful if $k \leq 3n/4$.

Remark 2. By Theorem 1, the upper bound of Theorem 5 does not necessarily hold if G is a disconnected graph, unless we insist that no component is a cycle of length 3, 5, 6 or 10 or that the subset U of k vertices of G contains at least one vertex from each component.

Remark 3. The result for the restricted total domination number obtained in Theorem 5 extends the result for the total domination number obtained in Theorem 1. Furthermore, the bound in Theorem 5 improves on the bound in Theorem 2 for dense graphs, namely those connected graphs of size q and order n satisfying $q > (8n + k - 7)/7$.

Using Theorems 1 and 5, we can readily achieve our third aim. (For a proof of Theorem 6 see Section 8.)

Theorem 6 *If G is a connected graph of order n such that* $\deg u + \deg v \geq 4$ *for every two adjacent vertices u and v of G, then $\gamma_t(G) \leq 4n/7$ unless $G \in \{C_3, C_5, C_6, C_{10}\}$.*

3 Total Domination in Graphs

The total domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices is easy to compute.

Theorem 7 [6] *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. Furthermore, if $n \equiv_4 2$ or if $n \equiv_4 3$, then there is a $\gamma_t(P_n)$-set that contains one of its end-vertices. If $n \equiv_4 2$, then there is a $\gamma_t(C_n)$-set that contains any two specified vertices of the cycle.*

The total domination number of a key $L_{m,n}$ of order (and size) $m+n$ was determined in [6]. As a consequence of this result, we have the following upper bound on $\gamma_t(L_{m,n})$.

Theorem 8 ([6]) *For $m \geq 3$ and $n \geq 1$, $\gamma_t(L_{m,n}) \leq (m + n + 2)/2$ with equality if and only if $m \equiv_4 2$ and $n \equiv_4 0$.*

The decision problem to determine the total domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph. Cockayne et al. [4] obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 9 ([4]) *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

A large family of graphs attaining the bound in Theorem 9 can be established using the following transformation of a graph. The *2-corona* of a graph H is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint. The 2-corona of a connected graph has total domination number two-thirds its order. Brigham, Carrington, and Vitray [2] obtained the following characterization of connected graphs of order at least 3 with total domination number exactly two-thirds their order.

Theorem 10 ([2]) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or the 2-corona of some connected graph.*

If we restrict the minimum degree to be at least two, then the upper bound in Theorem 9 can be improved. We will refer to a graph G as an **$\frac{4}{7}$-minimal graph** if G is edge-minimal with respect to satisfying the following three conditions: (i) $\delta(G) \geq 2$, (ii) G is connected, and (iii) $\gamma_t(G) \geq 4n/7$, where n is the order of G.

Let $\mathcal{H}$ be the family of graphs that can be obtained from a tree T of order at least 2 by adding for each vertex v of T, a 6-cycle and joining v to one vertex of this cycle. We refer to the tree T as the *underlying tree* of the resulting graph. It is shown in [6] that $\mathcal{H}$ is a family of $\frac{4}{7}$-minimal graphs. A graph in the family $\mathcal{H}$ with underlying tree $T \cong P_4$ is shown in Figure 3.

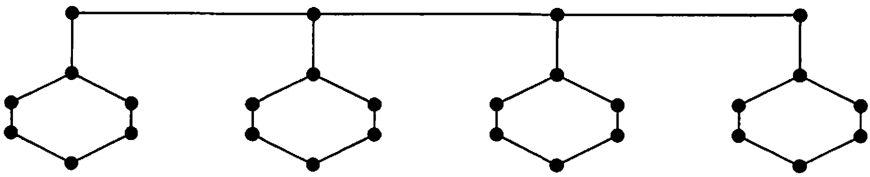


Figure 3: A graph in the family $\mathcal{H}$ of $\frac{4}{7}$-minimal graphs with underlying tree P_4.

Let H_1 be the graph obtained from a 6-cycle by adding a new vertex and joining this vertex to two vertices at distance 2 apart on the cycle. The graph H_1 is shown in Figure 4.

H_1 :

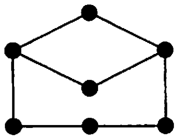


Figure 4: A graph H_1.

Observation 11 *If $G \in \mathcal{H} \cup \{H_1\}$, then G is a connected graph, $\delta(G) = 2$, $\gamma_t(G) = 4n/7$, and for any vertex v of G, there is a $\gamma_t(G)$-set containing v.*

The following result characterizes $\frac{4}{7}$-minimal graphs.

Theorem 12 ([6]) *A graph G is a $\frac{4}{7}$-minimal graph if and only if $G \in \{C_3, C_5, C_6, C_7, C_{10}, C_{14}, H_1\} \cup \mathcal{H}$.*

Theorem 1 is an immediate consequence of Theorem 12.

4 Known Results on Restricted Total Domination

For $n/2 \le k \le n$, it is shown in [8] that if G is a connected graph of order $n \ge 2$, then $r_k(G, \gamma_t) \le n$ and the graphs attaining this bound are characterized. For $1 \le k < n/2$, we have the following result which extends the bounds for the total domination number obtained in Theorem 9 and Theorem 10 to the restricted total domination number.

Theorem 13 ([8]) *For $1 \le k < n/2$, if $G = (V, E)$ is a connected graph of order $n \ge 3$, then*

$$r_k(G, \gamma_t) \le \frac{2(n+k)}{3}$$

with equality if and only if G is obtained from a connected graph F of order $k + \ell$, where $\ell \ge 1$, by attaching a path of length 1 to k vertices of F and by attaching a path of length 2 to each of the remaining ℓ vertices of F so that the resulting paths are vertex disjoint.

In order to extend the bound for the total domination number obtained in Theorem 1 to the restricted total domination number, we need to determine upper bounds for the restricted total domination numbers of simple structures such as cycles, paths and keys. For this purpose, suppose U is a subset of vertices of a cycle C. We say that two vertices x and y in U are **consecutive** on C if there is an x–y path on C that contains no vertices of U other than x and y.

Theorem 14 ([8]) *For $1 \le k \le n$, let U be a subset of k vertices of a cycle C_n. Then,*

$$r(C_n, U, \gamma_t) \le H(n, k)$$

with equality if and only if

(i) $k = 1$ *and* $n \equiv_4 2$, *or*

(ii) $k \ge 3$ *is odd and every two consecutive vertices x and y of U on the cycle have ℓ vertices between them (not including x and y) where $\ell \equiv_4 1$.*

Let U be a subset of vertices of a path P. We say that two vertices x and y in U are **consecutive** on P if the x–y path contains no vertices of U other than x and y. The following two results establish upper bounds on the restricted total domination number of a path.

Theorem 15 ([8]) *For $1 \le k \le n$, let U be a subset of k vertices of a u–v path P_n. Then,*

$$r(P_n, U, \gamma_t) \le H(n+1, k)$$

with strict inequality if $d(u, U) \equiv_4 1, 2$ or $d(v, U) \equiv_4 1, 2$.

Corollary 16 ([8]) *For $1 \le k \le n$, let U be a subset of k vertices of a u–v path P_n. If $v \notin U$ and $r(P_n, U, \gamma_t) = H(n+1, k)$, then $r(P_n - v, U, \gamma_t) \le H(n-1, k)$. If $v \in U$ and $r(P_n, U, \gamma_t) = H(n+1, k)$, then there exists a set S' containing U such that $|S'| \le H(n-1, k)$ and every vertex of P_n different from v is adjacent to some vertex of S'.*

The following result establishes an upper bound on the restricted total domination number of a key.

Theorem 17 ([8]) *For $1 \le k \le m+n$, let U be a set of k vertices of $L_{m,n}$. Let v denote the end-vertex of $L_{m,n}$. Then,*

$$r(L_{m,n}, U, \gamma_t) \le H(m+n+1, k).$$

Furthermore, if $r(L_{m,n}, U, \gamma_t) = H(m+n+1, k)$, then $r(L_{m,n} - v, U, \gamma_t) \le H(m+n-1, k)$ or there exists a set S^* containing $U \cup \{v\}$ such that $|S^*| \le H(m+n-1, k)$ and every vertex of $L_{m,n}$ different from v is adjacent to some vertex of S^*.

5 The family $\mathcal{G}$

Note that it is possible for a graph in the family $\mathcal{G}$ to have a link vertex that is incident with no link edge. For example, if $G \in \mathcal{G}$ is the graph of Figure 2, then in the graph $(G - \{uv, yz\}) \cup \{uw, xz\} \in \mathcal{G}$ (as before, $a = 2$ and $b = 2$), each unit of type (a) has a link vertex incident with no link edge.

If $G \in \mathcal{G}$ has k units of type (a) and b units of type (b), then G has order $n = 6k + 7b$ and minimum degree at least two. Furthermore, if U is the set of k pivot vertices of G, then $r(G, U, \gamma_t) = 4k + 4b = 4(n+k)/7 = F(n, k)$ and for any vertex v of G, there is a $\gamma_t(G, U)$-set containing v. In particular, G is a $\frac{4}{7}$-k-minimal graph.

As a consequence of Theorem 14, one can readily establish the following observation about the restricted total domination number of a 6-cycle.

Observation 18 *Let $G = C_6$. For $1 \le k \le 6$, $r_k(G, \gamma_t) \le F(6, k)$ with equality if and only if $k = 1$. Furthermore, if U is a subset of k vertices of G, and v is any vertex of G, then there exists a* TDS *S of G containing $U \cup \{v\}$ such that $|S| = F(6, k)$ if $k = 1$ and $|S| < F(6, k)$ if $2 \le k \le 6$ unless $k = 2$ and $U \cup \{v\}$ consists of an independent set of three vertices, in which case $|S| = 5$.*

The following result is straightforward to verify. We omit the proof.

Observation 19 *Let $G = L_{6,1}$. Then, $\gamma_t(G) = 4 = F(7, 0)$ and for any vertex v of G, there is a $\gamma_t(G)$-set that contains v. For $1 \le k \le 7$, $r_k(G, \gamma_t) < F(7, k)$.*

We are now in a position to prove Theorem 3.

5.1 Proof of Theorem 3

As observed earlier, if $G \in \mathcal{G}$ has k units of type (a) and if U is the set of k pivot vertices of G, then $r(G, U, \gamma_t) = F(n, k)$ and for any vertex v of G, there is a $\gamma_t(G, U)$-set containing v. This establishes the sufficiency.

To prove the necessity, we proceed by induction on the number $a + b \geq 1$ of units (with $a \geq 1$ units of type (a) and $b \geq 0$ units of type (b)). When $a + b = 1$, $G = C_6$ and the statement holds by Observation 18. This establishes our base case. Assume then that $a + b \geq 2$ and that the statement holds for all graphs in $\mathcal{G}$ with fewer than $a + b$ units. Let $G \in \mathcal{G}$ have $a + b$ units.

Let T be the tree of order $a + b$ whose vertices correspond to the units of G, and where two vertices of T are adjacent if and only if there is a (link) edge joining the corresponding units of G. For each vertex w of T, let G_w be the unit of G corresponding to the vertex w. Among all end-vertices of T, let v be one for which G_v contains as few vertices of U as possible, and let $G' = G - V(G_v)$. Further, let $n' = |V(G')|$, $U' = U \cap V(G')$ and let $k' = |U'|$. By our choice of the vertex v, $1 \leq k' \leq n'$.

Applying the inductive hypothesis to the graph G', which has $a + b - 1$ units, we have $r(G', U', \gamma_t) \leq F(n', k')$ with equality if and only if there is a selection of units and link vertices of G' so that the resulting set of pivot vertices is precisely the set U'. Furthermore, if $r(G', U', \gamma_t) = F(n', k')$, then for any vertex u of G' there is a $\gamma_t(G', U')$-set containing u.

In what follows we shall adopt the following notation. If $r(G', U', \gamma_t) < F(n', k')$, we select the units, link vertices and link edges of G' as defined in G (that is, the units of G different from G_v are the units of G', the link vertices of G not in G_v are the link vertices of G', and the link edges of G different from the link edge that is incident with a link vertex of G_v are the link edges of G'). If $r(G', U', \gamma_t) = F(n', k')$, we select the units and link vertices of G' so that the resulting set of pivot vertices is precisely the set U' (such a selection of units and link vertices is possible by our induction hypothesis). In both cases, we let G_u be the unit of G' that is joined with an edge to G_v, and we let S' be a TDS of G' containing U' with $|S'| = r(G', U', \gamma_t)$ and we let $S_u = S' \cap V(G_u)$. We let $U_v = U \cap V(G_v)$, and so $|U_v| = k - k'$ and $U = U_v \cup U'$. Further, we let $U_u = U \cap V(G_u)$.

Suppose $r(G', U', \gamma_t) = F(n', k')$. Then, S' contains four vertices from each type (a) and type (b) unit of G', and each type (a) unit contains one vertex of U' (namely the pivot vertex of the unit), while each type (b) unit contains no vertex of U'. Thus we may assume that S' is chosen so that the restriction of S' to any unit of G' totally dominates that unit. More precisely, in each type (a) unit of G', let S' contain the pivot vertex of the unit, its two neighbors (i.e., the two link vertices), and one vertex in the unit at distance 2 from the pivot vertex. In each type (b) unit of G', let S' contain the link vertex of the unit, its neighbor in the unit, one vertex in the unit at distance 3 from the link vertex, and the vertex in the unit at distance 4 from the link vertex. If $G_u = L_{6,1}$ (and still $r(G', U', \gamma_t) = F(n', k')$), let G_u be obtained from the 6-cycle c, d, e, f, g, h, c by adding the pendant edge cc' to the vertex c (and so, c' is the link vertex of G_u and $U_u = \emptyset$). Renaming vertices if necessary, we may assume $e \in S_u$, and so $S_u = \{c, c', e, f\}$. If $G_u = C_6$, let G_u be given by c, d, e, f, g, h, c. For our selection of link vertices of G', let d be the pivot vertex of G_u (and so, c and e are the link vertices of G_u and $U_u = \{d\}$). Renaming vertices if necessary, we may assume that $f \in S_u$, and so $S_u = \{c, d, e, f\}$. (The names of the vertices in G_u are illustrated in Figure 5, where the pivot vertex is indicated by the large darkened vertex.)

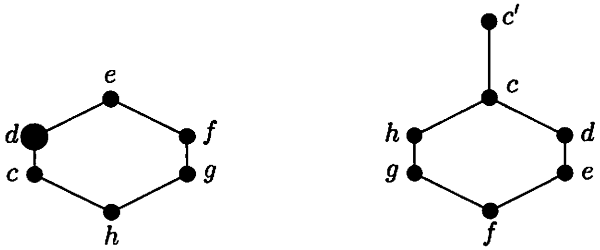


Figure 5: The unit G_u of G'.

We consider two possibilities, depending on whether G_v is a type (a) unit or a type (b) unit.

Case 1. $G_v = L_{6,1}$.

Then, $n' = n - 7$. By Observation 19, $r(G_v, U_v, \gamma_t) \le F(7, k - k')$. If $r(G_v, U_v, \gamma_t) < F(7, k - k')$ or $r(G', U', \gamma_t) < F(n', k')$, then $r(G, U, \gamma_t) \le r(G_v, U_v, \gamma_t) + r(G', U', \gamma_t) < F(7, k - k') + F(n', k') = F(n, k)$. Hence we may assume $r(G_v, U_v, \gamma_t) = F(7, k - k')$ and $r(G', U', \gamma_t) = F(n', k')$. Thus by Observation 19, $U_v = \emptyset$, i.e., $k = k'$. Let S_v be a $\gamma_t(G_v)$-set of G_v (of cardinality 4) containing the link vertex, z say, of G_v. Thus, $|S_v| + |S'| = F(n, k)$.

Suppose $G_u = L_{6,1}$. Since $G \in \mathcal{G}$, the only possible vertices in G_u adjacent to z are c, c', e or g. If $cz \in E(G)$, let $S = S_v \cup (S' - \{c'\})$. If $ez \in E(G)$, let $S = S_v \cup (S' - S_u) \cup \{c, g, h\}$. If $gz \in E(G)$, let $S = S_v \cup (S' - S_u) \cup \{c, d, e\}$. In all three cases, S is a TDS of G containing

U with $|S| = F(n,k) - 1$, a contradiction. Hence the link vertex z of G_v must be adjacent to the link vertex c' of G_u. The desired result now follows readily: Take the units of G to be the unit G_v and the units of G', take the link vertices of G to be the vertex z and the link vertices of G', and take the link edges of G to be the edge zc' and the link edges of G'. With this selection of units and link vertices of G, the resulting set of pivot vertices is precisely the set U.

Suppose $G_u = C_6$. If $dz \in E(G)$, let $S = S_v \cup (S' - \{c,e\}) \cup \{g\}$. If $fz \in E(G)$, let $S = S_v \cup (S' - \{e\})$. If $gz \in E(G)$, let $S = S_v \cup (S' - \{f\})$. If $hz \in E(G)$, let $S = S_v \cup (S' - \{c\})$. In all four cases, S is a TDS of G containing U with $|S| = F(n,k) - 1$, a contradiction. Hence the link vertex z of G_v must be adjacent to either c or e, both of which are link vertices of G'. The desired result now follows readily.

Case 2. $G_v = C_6$.

Then, $n' = n - 6$. Let G_v be the 6-cycle $1,2,3,4,5,6,1$ and let the vertex 1 be incident with the link edge of G joining G_v to a vertex of G'. If $r(G_v, U_v, \gamma_t) < F(6, k-k')$, then $r(G,U,\gamma_t) \leq r(G_v,U_v,\gamma_t) + r(G',U',\gamma_t) < F(6,k-k') + F(n',k') = F(n,k)$. Hence we may assume that $r(G_v,U_v,\gamma_t) \geq F(6,k-k')$, for otherwise $r(G,U,\gamma_t) < F(n,k)$ as desired. Thus by Observation 18, we may assume that $k - k' = |U_v| \leq 1$. If $U_v = \emptyset$, let S_v be a $\gamma_t(G_v)$-set; otherwise if $|U_v| = 1$, let S_v be a minimum TDS of G_v containing the set U_v. In either case, $|S_v| = 4$ and we may choose S_v to contain the vertex 1.

Case 2.1. $|U_v| = 1$.

Then, $k' = k - 1$ and $|S_v| = 4 = F(6,1)$. If $r(G',U',\gamma_t) < F(n',k-1)$, then $r(G,U,\gamma_t) \leq r(G_v,U_v,\gamma_t) + r(G',U',\gamma_t) < F(6,1) + F(n',k-1) = F(n,k)$. Hence we may assume $r(G',U',\gamma_t) = F(n',k') = F(n',k-1)$. Thus, $|S_v| + |S'| = F(n,k)$.

Suppose $G_u = L_{6,1}$. As shown in Case 1, the link vertex 1 of G_v must be adjacent to the link vertex c' of G_u. If $U_v \subset \{1,3,4\}$, let $S = S' \cup \{1,3,4\}$. If $U_v = \{5\}$, let $S = S' \cup \{1,4,5\}$. In both cases, S is a TDS of G containing U with $|S| = F(n,k) - 1$, a contradiction. Hence $U_v \subset \{2,6\}$. If $U_v = \{2\}$, we select 3 as the other link vertex of G_v, while if $U_v = \{6\}$, we select 5 as the other link vertex of G_v. The desired result now follows readily.

Suppose $G_u = C_6$. If $G' = G_u$, then $G \in \mathcal{G}$ consist of two type (a) units and using Observation 18, the desired result follows readily. Hence we may assume that G' consists of at least two units. Since $G \in \mathcal{G}$ the only possible vertices in G_u adjacent to the vertex 1 are the vertices c, e or g. If $1g \in E(G)$, then $S_v \cup (S' - \{f\})$ is a TDS of G containing U with $|S| = F(n,k) - 1$, a contradiction. Hence the link vertex 1 of G_v must be adjacent to either c or e, both of which are link vertices of G'. Proceeding now as in the previous paragraph, $U_v \subset \{2,6\}$ and the desired result follows readily.

Case 2.2. $U_v = \emptyset$.

Then, $k' = k$ and $|S_v| = 4$.

Case 2.2.1. $G_u = L_{6,1}$.

Suppose G_u be obtained from the 6-cycle c,d,e,f,g,h,c by adding the pendant edge cc' to the vertex c (and so, c' is the link vertex of G_u). Since $G \in \mathcal{G}$, the only possible vertices in G_u adjacent to the vertex 1 are c, c', e or g. If $G' = G_u$, then $G \in \mathcal{G}$ consist of a type (a) unit and a type (b) unit, and using Observations 18 and 19, the desired result follows readily. Hence we may assume that G' consists of at least two units.

Suppose $1c' \in E(G)$. If $U_u \cap \{c,d,e,f,g,h\} = \emptyset$, then S_u can be chosen so that $S_u = \{c,c',e,f\}$. In particular, $c' \in S'$. Thus, $S' \cup \{1,3,4\}$ is a TDS of G containing U of cardinality $|S'| + 3 \leq F(n',k) + 3 < F(n',k) + F(6,0) = F(n,k)$. On the other hand, suppose $|U_u \cap \{c,d,e,f,g,h\}| \geq 1$. We now consider the graph $H = G - \{c,d,e,f,g,h\}$. Then, $H \cong G'$ (we have simply replaced the type (b) unit G_u in G' by the type (b) unit obtained from the 6-cycle G_v by adding the vertex c' and the edge $1c'$). Thus, $H \in \mathcal{G}$. If $|U \cap V(H)| \geq 1$, then proceeding in an identical manner as in the first paragraph of Case 2 and as in Case 2.1, the desired result follows readily. Suppose therefore that $U \cap V(H) = \emptyset$. Then since H is not a cycle, it follows from Theorem 1, that $\gamma_t(H) \leq 4|V(H)|/7 = F(n-6,0)$. Let $H_u = G - V(H)$, and so H_u is the 6-cycle c,d,e,f,g,h,c. Since $U \subseteq V(H_u)$ and $k \geq 1$, Observation 18 implies that $r(H_u,U,\gamma_t) \leq F(6,k)$ with equality if and only if $k = 1$. Thus, $r(G,U,\gamma_t) \leq r(H_u,U,\gamma_t) + \gamma_t(H) \leq F(6,k) + F(n-6,0) = F(n,k)$. Furthermore, if $r(G,U,\gamma_t) = F(n,k)$, then $H \in \mathcal{H}$ (i.e., $H \in \mathcal{G}$ and every unit of H is a type (b) unit) and $k = 1$. If $U \not\subset \{d,h\}$, then we can extend a $\gamma_t(H)$-set that contains the vertex c' to a TDS of G containing U by adding three vertices of H_u, and so $r(G,U,\gamma_t) \leq \gamma_t(H) + 3 = F(n,k) - 1$, a contradiction. Hence, $U \subset \{d,h\}$ and the desired result follows readily. Thus if $1c' \in E(G)$, the desired result follows.

Suppose $1c' \notin E(G)$ (and so, the vertex 1 is adjacent to c, e or g). Let $G^* = G' - (V(G_u) - \{c'\})$ and let $U^* = U \cap V(G^*)$ and $|U^*| = k^*$.

Further let $G'' = G - V(G^*)$ and let $U'' = U \cap V(G'')$ and $|U''| = k''$. Then, $k = k^* + k''$ and possibly $k^* = 0$ or $k'' = 0$ (but not both). Let $|V(G^*)| = n^*$. Then, $n^* = n - 12$. Since G' is connected, so too is G^*. Further, since G is in the family $\mathcal{G}$, so too is G^*. Since $G'' \in \mathcal{G}$ consists of two type (a) units, at least one of which contains no vertex of U, it follows readily using Observation 18 that $r(G'', U'', \gamma_t) < F(12, k'')$ (if $k'' = 0$, then $r(G'', U'', \gamma_t) = \gamma_t(G'') = 6 < F(12, 0)$). If $k^* \geq 1$, then by the inductive hypothesis, $r(G^*, U^*, \gamma_t) \leq F(n^*, k^*)$, and so $r(G, U, \gamma_t) \leq r(G'', U'', \gamma_t) + r(G^*, U^*, \gamma_t) < F(12, k'') + F(n^*, k^*) = F(n, k)$. Hence we may assume $k^* = 0$ (and so, $k = k''$), for otherwise $r(G, U, \gamma_t) < F(n, k)$. Since G' consists of at least two units, and since G_u is a type (b) unit, G^* is not a cycle. Hence, by Theorem 1, $\gamma_t(G^*) \leq 4n^*/7 = F(n^*, 0)$. Thus, $r(G, U, \gamma_t) \leq r(G'', U'', \gamma_t) + \gamma_t(G^*) < F(12, k) + F(n^*, 0) = F(n, k)$. Hence if $1c' \notin E(G)$, the desired result follows.

Case 2.2.2. $G_u = C_6$.

Suppose that G_u is given by c, d, e, f, g, h, c. If $G' = G_u$, then $G \in \mathcal{G}$ consist of two type (a) units, and using Observation 18, the desired result follows readily. Hence we may assume that G' consists of at least two units. Among all link edges of G' incident with a link vertex of G_u, let e' be chosen, if possible, so that $G - e'$ does not contain a C_6-component with no vertex of U. Renaming vertices if necessary, we may assume that $e' = cc'$.

Let G_1 and G_2 be the two components of $G - cc'$, where $c \in V(G_1)$. Then, $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}$. Since there is an edge joining the two 6-cycles G_v and G_u, both G_v and G_u are type (a) units of G_1. For $i = 1, 2$, let $n_i = |V(G_i)|$ and let $U_i = U \cap V(G_i)$ and $|U_i| = k_i$. Then, $n = n_1 + n_2$ and $k = k_1 + k_2$. Possibly, $k_1 = 0$ or $k_2 = 0$ (but not both).

We show first that $r(G_1, U_1, \gamma_t) < F(n_1, k_1)$. If $k_1 = 0$, then it follows from Theorem 12 that $\gamma_t(G_1) < 4n_1/7 = F(n_1, 0)$. If $k_1 \geq 1$, then it follows from the inductive hypothesis that since G_1 has a type (a) unit, namely G_v, that contains no vertex of U, $r(G_1, U_1, \gamma_t) < F(n_1, k_1)$. Hence irrespective of whether $k_1 = 0$ or $k_1 \geq 1$, $r(G_1, U_1, \gamma_t) < F(n_1, k_1)$.

If $k_2 \geq 1$, then by the inductive hypothesis, $r(G_2, U_2, \gamma_t) \leq F(n_2, k_2)$, whence $r(G, U, \gamma_t) \leq r(G_1, U_1, \gamma_t) + r(G_2, U_2, \gamma_t) < F(n_1, k_1) + F(n_2, k_2) = F(n, k)$. Hence we may assume that $k_2 = 0$ (and so, $k = k_1$), for otherwise $r(G, U, \gamma_t) < F(n, k)$.

Suppose G_2 is not a cycle. Then either $G_2 = L_{6,1}$, in which case $\gamma_t(G_2) = 4 = 4n_2/7$, or $\delta(G_2) \geq 2$, in which case $\gamma_t(G_2) \leq 4n_2/7$ by Theorem 1. In both cases, $\gamma_t(G_2) \leq F(n_2, 0)$. Hence, $r(G, U, \gamma_t) \leq r(G_1, U_1, \gamma_t) +$

$\gamma_t(G_2) < F(n_1, k) + F(n_2, 0) = F(n, k)$. Thus if G_2 is not a cycle, then $r(G, U, \gamma_t) < F(n, k)$.

Suppose, finally, that G_2 is a cycle. Then, $G_2 = C_6$ and by our choice of the link edge cc', the removal of any link edge of G' incident with a link vertex of G_u, produces a C_6-component containing no vertex of U. It follows that T is a star of order at least 3 each leaf of which corresponds to a type (a) unit of G which contains no vertex of U. Thus, $U = U_u$. We may assume that e is the other link vertex of G_u in G. A TDS S of G containing U can now be constructed with $|S| < F(n, k)$: For each unit G_w of G corresponding to a leaf w of T, let S contain the link vertex of G_w that is joined to G_u, and two other vertices, one at distance 2 and the other at distance 3 from this link vertex in G_w. Let S contain both link vertices c and e (note that $c' \in S$), and depending on the value of k, $1 \le k \le 6$, additional vertices of G_u can easily be chosen so that $|S| < F(n, k)$. □

6 Proof of Theorem 4

The sufficiency follows from Theorem 3. To prove the necessary condition of Theorem 4, we proceed by induction on the order $n \ge 3$ of a $\frac{4}{7}$-k-minimal graph. Suppose $3 \le n \le 6$. If G is not hamiltonian, then either $G \in \{K_{2,3}, K_{2,4}, D(3,3), D(3,4), D_b(3,3)\}$ or G is obtained from $K_{2,3}$ by subdividing one edge once. In all cases, it is straightforward to check that $r_k(G, \gamma_t) < F(n, k)$, a contradiction. Hence, G is hamiltonian, and so C_n is a subgraph of G. It now follows readily from Theorem 14 and Observation 18 that $G = C_6$ and $k = 1$, i.e., $G \in \mathcal{G}$. This establishes the base cases.

Let $n \ge 7$ and assume the result is true for all $\frac{4}{7}$-k'-minimal graphs G' of order n', where $n' < n$ and $1 \le k' \le n'$. Since the restricted total domination number of a graph cannot decrease if edges are removed, our induction hypothesis implies that for all connected graphs G^* of order n^*, where $n^* < n$ and $1 \le k^* \le n^*$, with $\delta(G^*) \ge 2$, that $r_k(G^*, \gamma_t) \le 4(n^* + k^*)/7$.

For $1 \le k \le n$, let $G = (V, E)$ be a $\frac{4}{7}$-k-minimal graph of order n. Let U be a set of $k \ge 1$ vertices in G for which $r(G, U, \gamma_t) = r_k(G, \gamma_t) \ge F(n, k)$. If $G = C_n$ (and still $n \ge 7$), then by Theorem 14, $r_k(G, \gamma_t) < F(n, k)$, a contradiction. Hence, G *is not a cycle.*

If e is an edge of G, then $r(G - e, U, \gamma_t) \ge r(G, U, \gamma_t)$. Thus, by the minimality of G, we have the following observation.

Observation 20 *If $e \in E$, then either e is a bridge of G or $\delta(G-e) = 1$.*

The next result is a consequence of the inductive hypothesis.

Observation 21 *If G' is a connected subgraph of G of order $n' < n$ with $\delta(G') \geq 2$, then, for $1 \leq k' \leq n'$, either $G' \in \mathcal{G}_{k'}$ or $r_{k'}(G', \gamma_t) < F(n', k')$.*

The following three lemmas, proofs of which are given in Subsection 6.1, will be useful in what follows.

Lemma 22 *For $i = 1, 2$, let G_i be a connected graph of order $n_i < n$ and let $v_i \in V(G_i)$. Further, let $U_i \subseteq V(G_i)$ where $k_i = |U_i|$ and $k_2 \geq 1$ (possibly, $k_1 = 0$). Let G' be a graph obtained from $G_1 \cup G_2$ by adding the edge v_1v_2 and possibly other edges joining G_1 and G_2. Suppose there exists an $r(G_1, U_1, \gamma_t)$-set that contains v_1 and suppose $G_2 \in \mathcal{G}$ and $r(G_2, U_2, \gamma_t) = F(n_2, k_2)$. By Theorem 3, there is a selection of units and link vertices of G_2 so that the resulting set of pivot vertices is precisely the set U_2. If v_2 is not a link vertex of G_2, then $r(G', U_1 \cup U_2, \gamma_t) \leq r(G_1, U_1, \gamma_t) + r(G_2, U_2, \gamma_t) - 1$.*

Lemma 23 *If x, x_1, x_2, x_3, x_4, y is an induced path in G every internal vertex of which has degree two in G, then $|U \cap \{x_1, x_2, x_3, x_4\}| \geq 1$.*

Lemma 24 *If x, x_1, x_2, y is an induced path in G every internal vertex of which has degree two in G, then $|U \cap \{x_1, x_2\}| \leq 1$.*

Since G is not a cycle, G contains at least one vertex of degree at least 3. Let $S = \{v \in V \mid \deg v \geq 3\}$. Each vertex of $V - S$ therefore has degree 2. For each $v \in S$, we define the **2-graph of** v to be the component of $G - (S - \{v\})$ that contains v. So each vertex of the 2-graph of v has degree 2 in G, except for v. Furthermore, the 2-graph of v consists of edge-disjoint cycles through v, which we call **2-graph cycles**, and paths emanating from v, which we call **2-graph paths**.

Using the inductive hypothesis, and the structure and properties of graphs in the families $\mathcal{H}$ and $\mathcal{G}$ established earlier, we shall prove the following lemma, a proof of which is given in Subsection 6.2.

Lemma 25 *If S is not an independent set, then $G \in \mathcal{G}_k$.*

By Lemma 25, we may assume that S **is an independent set**, for otherwise $G \in \mathcal{G}_k$. With this assumption we shall prove the following lemma, a proof of which is given in Subsection 6.3.

Lemma 26 *There is at least one* 2*-graph cycle in* G.

By Lemma 26, G contains a 2-graph cycle. Among all 2-graph cycles of G, let C be chosen to contain as few vertices of U as possible. Let H be the (connected) graph obtained from G by deleting all the vertices of C except for the vertex of C that belongs to S. We shall prove the following two lemmas, proofs of which are given in Subsections 6.4 and 6.5.

Lemma 27 *If* $\delta(H) \geq 2$, *then* $G \in \mathcal{G}_k$.

Lemma 28 *If* $\delta(H) = 1$, *then* $G \in \mathcal{G}_k$.

It follows from Lemmas 27 and 28 that $G \in \mathcal{G}_k$. This completes the proof of Theorem 4.

6.1 Proof of Lemmas 22, 23 and 24

6.1.1 Proof of Lemma 22

For $i = 1, 2$, let S_i be an $r(G_i, U_i, \gamma_t)$-set where S_1 contains v_1 and where S_2 is chosen as follows: In each type (a) unit, let S_2 contain the pivot vertex of the unit, its two neighbors (i.e., the two link vertices), and one vertex in the unit at distance 2 from the pivot vertex. In each type (b) unit, let S_2 contain the link vertex of the unit, its neighbor in the unit, one vertex in the unit at distance 3 from the link vertex, and the vertex in the unit at distance 4 from the link vertex. Let F denote the unit of G_2 containing v_2 and let $T = S_2 \cap V(F)$.

Suppose that F is a type (a) unit. If v_2 is the pivot vertex of F, then let T' consist of v_2, one vertex in F at distance 2 from v_2, and the vertex in F at distance 3 from v_2. If v_2 is not the pivot vertex, then let T' consist of v_2, the pivot vertex of F, and a neighbor of the pivot vertex in F that is not adjacent to v_2. On the other hand, if F is a type (b) unit, then let T' consist of v_2, one vertex at distance 2 from v_2 on the 6-cycle in F, and the vertex at distance 3 from v_2 on this 6-cycle. Then in all cases, $S_1 \cup (S_2 - T) \cup T'$ is a TDS of G' of cardinality $|S_1| + |S_2| - 1 = r(G_1, U_1, \gamma_t) + r(G_2, U_2, \gamma_t) - 1$ that contains $U_1 \cup U_2$. The desired result follows. □

6.1.2 Proof of Lemma 23

Suppose that $U \cap \{x_1, x_2, x_3, x_4\} = \emptyset$. Let $F' = (G - \{x_1, x_2, x_3, x_4\}) + xy$. Then, F' is a connected graph with $\delta(F') \geq 2$. By the inductive hypothesis, $r(F', U, \gamma_t) \leq F(n-4, k)$. Let T' be a $r(F', U, \gamma_t)$-set. If $x, y \in T$, let $T = T' \cup \{x_1, x_4\}$. If $x \in T'$ and $y \notin T'$, let $T = T' \cup \{x_3, x_4\}$. If $x \notin T'$ and $y \in T'$, let $T = T' \cup \{x_1, x_2\}$. If $x, y \notin T'$, let $T = T' \cup \{x_2, x_3\}$. Then, $r(G, U, \gamma_t) \leq |T| = r(F', U, \gamma_t) + 2 \leq F(n-4, k) + 2 < F(n, k)$, a contradiction. □

6.1.3 Proof of Lemma 24

Suppose that $x_1, x_2 \in U$. Let $F' = (G - \{x_1, x_2\}) + xy$, and let $U' = U - \{x_1, x_2\}$. Then, F' is a connected graph with $\delta(F') \geq 2$. Since G is not a cycle, neither is F'. If $k = 2$, then $U' = \emptyset$, and by Theorem 1, $r(F', U', \gamma_t) = \gamma_t(F') \leq 4|V(F')|/7 = F(n-2, k-2)$. If $k \geq 3$, then by the inductive hypothesis, $r(F', U', \gamma_t) \leq F(n-2, k-2)$. In both cases, $r(F', U', \gamma_t) \leq F(n-2, k-2)$. Let T' be a $r(F', U', \gamma_t)$-set. Then, $r(G, U, \gamma_t) \leq |T'| + |\{x_1, x_2\}| \leq F(n-2, k-2) + 2 < F(n, k)$, a contradiction. □

6.2 Proof of Lemma 25

Let $e = uv$ be an edge, where $u, v \in S$. By Observation 20, e must be a bridge of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$, and so $n = n_1 + n_2$. Further, for $i = 1, 2$, let $U_i = U \cap V(G_i)$ and let $k_i = |U_i|$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. Hence, by Observation 21, for $i = 1, 2$, if $k_i \geq 1$, then $r_{k_i}(G_i, \gamma_t) \leq F(n_i, k_i)$.

Claim 29 *If $k_1 \geq 1$ and $k_2 \geq 1$, then $G \in \mathcal{G}_k$.*

Proof. Since $k_1 \geq 1$ and $k_2 \geq 1$, $r(G, U, \gamma_t) \leq r(G_1, U_1, \gamma_t) + r(G_2, U_2, \gamma_t) \leq F(n_1, k_1) + F(n_2, k_2) = F(n, k)$. Since $r(G, U, \gamma_t) \geq F(n, k)$, we must have equality throughout this inequality chain. In particular for $i = 1, 2$, $r(G_i, U_i, \gamma_t) = F(n_i, k_i)$, and so by Observation 21, $G_i \in \mathcal{G}_{k_i}$. For $i = 1, 2$, choose the units of G_i and select the link vertices of G_i so that U_i is precisely the resulting set of pivot vertices of G_i (this is possible by Theorem 3). It now follows readily by Lemma 22 that u must be a link vertex of G_1 and v a link vertex of G_2, whence $G \in \mathcal{G}_k$. □

By Claim 29, we may assume that $k_1 = 0$, and so $k = k_2 \geq 1$ and $U = U_2$, for otherwise $G \in \mathcal{G}_k$. By Observation 21, $G_2 \in \mathcal{G}_{k_2}$ or $r(G_2, U_2, \gamma_t) < F(n_2, k_2)$. Let S_2 be a minimum TDS of G_2 containing $U_2 \cup \{v\}$. Then, $|S_2| = r(G_2, U \cup \{v\}, \gamma_t)$, and so, by the inductive hypothesis, $|S_2| \leq F(n_2, k+1)$.

If $\gamma_t(G_1) < 4n_1/7$, then $r(G, U, \gamma_t) \leq \gamma_t(G_1) + r(G_2, U, \gamma_t) < 4n_1/7 + F(n_2, k) = F(n, k)$, a contradiction. Hence, $\gamma_t(G_1) \geq 4n_1/7$. Since $\delta(G_1) \geq 2$ and G_1 is connected, it follows that G_1 is a $\frac{4}{7}$-minimal graph. Hence, by Theorem 12, $G_1 \in \{C_3, C_5, C_6, C_7, C_{10}, C_{14}, H_1\} \cup \mathcal{H}$. Since $k_1 = 0$, Lemma 23 implies that $G \notin \{C_7, C_{10}, C_{14}\}$. The desired result of the lemma now follows from Claims 30, 31, 32, and 33.

Claim 30 $G_1 \notin \{C_3, C_5\}$.

Proof. If $G_1 = C_3$, then $S_2 \cup \{u\}$ is a TDS of G containing U, and so $r(G, U, \gamma_t) \leq F(n_2, k+1)+1 = F(n-3, k+1)+1 < F(n, k)$, a contradiction. If $G_1 = C_5$, let S_1 be the set of two vertices at distance 2 from u in G_1. Then, $S_1 \cup S_2$ is a TDS of G containing U, and so $r(G, U, \gamma_t) \leq 2 + F(n_2, k+1) = 2 + F(n-5, k+1) < F(n, k)$, a contradiction. □

Claim 31 $G_1 \neq H_1$.

Proof. Suppose $G_1 = H_1$. Let x and y denote the two vertices of degree 3 in G_1, and let x, x_1, x_2, x_3, y denote the x–y path of length 4 in G_1. Let w and z denote the two common neighbors of x and y in G_1. If $u = w$, then xw is a cycle edge and $\delta(G - xw) \geq 2$, contradicting Observation 20. Hence, $u \neq w$. Similarly, $u \neq z$. Now let S_1 consists of u, a vertex at distance 2 from u in G_1 and a vertex at distance 3 from u in G_1. Then $S_1 \cup S_2$ is a TDS of G containing U with $|S_1| + |S_2| \leq 3 + F(n_2, k+1) = 3 + F(n-7, k+1) < F(n, k)$, and so $r(G, U, \gamma_t) < F(n, k)$, a contradiction. □

Claim 32 *If* $G_1 = C_6$, *then* $G \in \mathcal{G}_k$.

Proof. Let G_1 be the cycle $u = u_1, u_2, \ldots, u_6, u_1$. Let $S_1 = \{u, u_3, u_4\}$. Then, $D = S_1 \cup S_2$ is a TDS of G containing U. If $|S_2| \leq F(n_2, k)$ (in particular if v belongs to a minimum TDS of G_2 that contains U_2), then $|D| \leq 3 + F(n_2, k) = 3 + F(n-6, k) < F(n, k)$, a contradiction. Hence, $|S_2| > F(n_2, k)$. In particular, $v \notin U$.

Let H be the graph obtained from G_2 by adding the path v_1, v_2, v_3, v_4, v_5 and joining v to v_1 and v_5. Then, H is a connected graph of order $n-1$

with $\delta(H) \geq 2$ (and in which $\deg_H v \geq 4$). By the inductive hypothesis, $r(H, U \cup \{v_1\}, \gamma_t) \leq F(n-1, k+1) = F(n,k)$. Let T_2 be a minimum TDS of H containing $U \cup \{v_1\}$. Then, $|T_2| \leq F(n,k)$. Let C denote the 6-cycle $v, v_1, v_2, \ldots, v_5, v$. Since $(U \cup \{v_1\}) \cap V(C) = \{v_1\}$, we may assume that $T_2 \cap V(C) = \{v, v_1, v_3, v_4\}$.

The set $(T_2 - \{v_1, v_3, v_4\}) \cup S_1$ is a TDS of G containing U of cardinality $|T_2|$. Hence if $|T_2| < F(n,k)$, then $r(G, U, \gamma_t) < F(n,k)$, a contradiction. Therefore, $|T_2| \geq F(n,k)$. Consequently, $|T_2| = F(n,k)$. In particular, it follows that $r(H, U \cup \{v_1\}, \gamma_t) = F(n-1, k+1)$, and so, by the inductive hypothesis, $H \in \mathcal{G}_{k+1}$. Furthermore, we can choose the units of H and select the link vertices of H so that $U \cup \{v_1\}$ is the resulting set of pivot vertices of H. Thus the subgraph C is a type (a) unit of H with link vertices v and v_2. Replacing this type (a) unit of H with the type (b) unit obtained from G_1 by adding v and the edge uv (and with resulting link vertex v), and keeping all other units and link vertices of H unchanged, shows that $G \in \mathcal{G}_k$, as desired. □

Claim 33 *If $G_1 \in \mathcal{H}$, then $G \in \mathcal{G}_k$.*

Proof. Since $\delta(G_1) \geq 2$, it follows from the way in which the family $\mathcal{H}$ is defined that G_1 consists of $\ell \geq 2$ type (b) units and the underlying tree of G_1 is precisely the subgraph of G_1 induced by the ℓ link vertices of these type (b) units. Let S_1 be a $\gamma_t(G_1)$-set chosen as follows: For each (type (b)) unit F of G_1, let $S_1 \cap V(F)$ be a $\gamma_t(F)$-set (of cardinality 4) that contains the link vertex of F (and therefore also its neighbor). Let F_u be the (type (b)) unit of G_1 containing u.

Suppose u is not the link vertex of F_u. Then replace the four vertices of S_1 in F_u by u, a vertex at distance 2 from u on the 6-cycle in F_u, and the vertex at distance 3 from u on the 6-cycle in F_u. Let S_1' denote the resulting adjusted set S_1 (note that the link vertex of F_u is dominated by S_1' since G_1 contains at least two units). Then, $S_1' \cup S_2$ is a TDS of G containing U of cardinality $(|S_1|-1)+|S_2| = (F(n_1,0)-1)+F(n_2,k+1) = F(n,k+1)-1 < F(n,k)$, a contradiction. Hence, u is the link vertex of F_u.

If $r(G_2, U, \gamma_t) < F(n_2, k)$, then $r(G, U, \gamma_t) \leq \gamma_t(G_1) + r(G_2, U, \gamma_t) < F(n_1, 0) + F(n_2, k) = F(n,k)$, a contradiction. Hence, $r(G_2, U, \gamma_t) = F(n_2, k)$, and so, by Observation 21, $G_2 \in \mathcal{G}_k$. By Theorem 3, there is a selection of units and link vertices of G_2 so that the resulting set of pivot vertices is precisely the set U. If v is not a link vertex of G_2, then, by Lemma 22, $r(G, U, \gamma_t) \leq r(G_1, U_1, \gamma_t) + r(G_2, U_2, \gamma_t) - 1 = F(n,k) - 1$, a contradiction. Hence, v is a link vertex of G_2. It follows that $G \in \mathcal{G}_k$. □

6.3 Proof of Lemma 26

Suppose, to the contrary, that G has no 2-graph cycle. Then, $|S| \geq 2$ and for every $w \in S$, the 2-graph of w consists of 2-graph paths.

Let $w \in S$ and let P_w be a 2-graph path of w. By assumption, S is an independent set, and so P_w has length at least one. Let P_w be a w–v path and let z be the neighbor of v not on P_w. Then, $z \in S$. Let $P = P_w - w$ have order $m \geq 1$. If $m = 1$, then P is the trivial path consisting of the vertex v. If $m \geq 2$, let P be a u–v path (and so u is the neighbor of w in P_w) given by $u = u_1, u_2, \ldots, u_m = v$.

Let $F = G - V(P)$ and let F have order n'. Then, $n' = n - m$ and $\delta(F) \geq 2$. Possibly, F is disconnected in which case F has two components, one containing w and the other z. Further since G has no 2-graph cycle, neither of these two components of F is a cycle. Thus applying Theorem 1 and Observation 21 to the two components of F, $r_{k'}(F, \gamma_t) \leq F(n', k')$ for $0 \leq k' \leq n'$. On the other hand if F is connected, then by Observation 21, $r_{k'}(F, \gamma_t) \leq F(n', k')$ for $1 \leq k' \leq n'$.

Claim 34 *If $U \cap V(P) = \emptyset$, then $m = 3$ and $w, z \notin U$.*

Proof. By Lemma 23, $1 \leq m \leq 3$.

Suppose $m = 1$. Then, $r(F, U, \gamma_t) \leq F(n-1, k)$. If w or z belong to a $r(F, U, \gamma_t)$-set, then $r(G, U, \gamma_t) \leq F(n-1, k)$, a contradiction. Hence no $r(F, U, \gamma_t)$-set contains w or z. Suppose that F is connected. By the inductive hypothesis, $F(n, k) \leq r(G, U, \gamma_t) \leq r(F, U \cup \{w\}, \gamma_t) \leq F(n-1, k+1) = F(n, k)$. Hence we must have equality throughout this inequality chain. In particular, $r(F, U \cup \{w\}, \gamma_t) = F(n-1, k+1)$, and so, by the inductive hypothesis, $F \in \mathcal{G}_{k+1}$. By Theorem 3, there is a selection of units and link vertices of F so that the resulting set of pivot vertices is precisely the set $U \cup \{w\}$. In particular, w is a pivot vertex in a type (a) unit of F and therefore has degree 2 in F (and degree 3 in G). Since $|U \cup \{w\}| \geq 2$, F consists of at least two type (a) units. Therefore at least one of the two link vertices of F adjacent to w has degree at least 3. This contradicts our assumption that S is an independent set in G. If F is disconnected (with two components, one containing w and the other z), then applying a similar argument to the component of F containing w shows that either w has a 2-graph cycle in G (which contradicts our assumption that G has a 2-graph cycle) or w is adjacent to some other vertex of S (which contradicts our assumption that S is independent). Hence, $m = 2$ or $m = 3$.

Suppose $m = 2$. Then, $r(F, U, \gamma_t) \leq F(n-2, k)$. If w or z belong to a

$r(F,U,\gamma_t)$-set, then $r(G,U,\gamma_t) \le F(n-2,k)+1 < F(n,k)$, a contradiction. Hence no $r(F,U,\gamma_t)$-set contains w or z. Suppose F is connected. By the inductive hypothesis, $F(n,k) \le r(G,U,\gamma_t) \le r(F,U \cup \{w,z\},\gamma_t) \le F(n-2,k+2) = F(n,k)$. Hence we must have equality throughout this inequality chain. In particular, $r(F,U \cup \{w,z\},\gamma_t) = F(n-2,k+2)$, and so, by the inductive hypothesis, $F \in \mathcal{G}_{k+2}$. By Theorem 3, there is a selection of units and link vertices of F so that the resulting set of pivot vertices is precisely the set $U \cup \{w,z\}$. In particular, each of w and z is a pivot vertex in a type (a) unit of F. This contradicts our assumption that S is an independent set in G. Similarly, if F is disconnected, we produce a contradiction. Hence, $m = 3$. Further suppose $w \in U$. Then, $r(G,U,\gamma_t) \le |\{v\}| + r(F,U \cup \{z\},\gamma_t) \le 1 + F(n-3,k+1) < F(n,k)$, a contradiction. Hence, $w \notin U$. Similarly, $z \notin U$. □

In what follows, let $U_1 = U \cap V(P)$ and $U_2 = U - U_1$. Further, let $k_1 = |U_1|$ and $k_2 = |U_2|$ (and so, $k = k_1 + k_2$).

Claim 35 $U \cap S = \emptyset$.

Proof. Suppose $|U \cap S| \ge 1$. Let $w \in U \cap S$. We shall use the notation introduced in the paragraph preceding Claim 34. By Claim 34, $k_1 \ge 1$. Since $w \in U_2$, $k_2 \ge 1$.

Suppose that $m + k_1 \le 8$. Then it is straightforward (though tedious) to show that $r(G,U,\gamma_t) < F(n,k)$. For example, consider the case when $m = 5$ and $k_1 = 2$. By Lemma 24, no two adjacent vertices of P are both in U. If $U_1 = \{u_1,u_3\}$ or $U_1 = \{u_1,u_4\}$, then $r(G,U,\gamma_t) \le |\{u_1,u_3,u_4\}| + r(F,U_2,\gamma_t) \le 3 + F(n-5,k-2) < F(n,k)$. If $U_1 = \{u_1,u_5\}$, then $r(G,U,\gamma_t) \le |\{u_1,u_4,u_5\}| + r(F,U_2,\gamma_t) < F(n,k)$. If $U_1 = \{u_2,u_4\}$, then $r(G,U,\gamma_t) \le |\{u_2,u_3,u_4\}| + r(F,U_2,\gamma_t) < F(n,k)$. If $U_1 = \{u_2,u_5\}$ or $U_1 = \{u_3,u_5\}$, then $r(G,U,\gamma_t) \le |\{u_2,u_3,u_5\}| + r(F,U_2 \cup \{z\},\gamma_t) \le 3 + F(n-5,k-1) < F(n,k)$. In all cases, $r(G,U,\gamma_t) < F(n,k)$.

Hence $m + k_1 \ge 9$ (for otherwise, $r(G,U,\gamma_t) < F(n,k)$, a contradiction). Let C be obtained from the 2-graph path P_w by adding the edge vw. Thus, C is the cycle $w, u_1, u_2, \ldots, u_m, w$ of length $m+1$ and $U \cap V(C) = U_1 \cup \{w\}$. By Theorem 14, $r(C,U_1 \cup \{w\},\gamma_t) \le H(m+1,k_1+1) = (m+k_1+3)/2$.

Suppose $r(C,U_1 \cup \{w\},\gamma_t) \le (m+k_1+2)/2$. Let S_1 be a $r(C,U_1 \cup \{w\},\gamma_t)$-set and let S_2 be a $r(F,U_2 \cup \{z\},\gamma_t)$-set. Then, $S_1 \cap S_2 = \{w\}$ and $S_1 \cup S_2$ is a TDS of G containing U. Hence, $r(G,U,\gamma_t) \le |S_1 \cup S_2| = |S_1| + |S_2| - 1 \le (m+k_1+2)/2 + r(F,U_2 \cup \{z\},\gamma_t) - 1 \le (m+k_1+2)/2 + F(n-m,k_2+1) - 1 = F(n,k) + (8-m-k_1)/14 < F(n,k)$, a contradiction. Hence,

$r(C, U_1 \cup \{w\}, \gamma_t) = (m + k_1 + 3)/2$, and so, by Theorem 14, $k_1 + 1 \geq 3$ is odd and every two consecutive vertices x and y of U on C have ℓ vertices between them (not including x and y) where $\ell \equiv_4 1$. It therefore follows from Lemma 23, that $m = 2k_1 + 1$ and $U_1 = \{u_{2i} \mid 1 \leq i \leq k_1\}$. Thus, $r(P, U_1, \gamma_t) \leq 3k_1/2 = (m + k_1 - 1)/2$. Consequently, $r(G, U, \gamma_t) \leq r(P, U_1, \gamma_t) + r(F, U_2, \gamma_t) \leq (m + k_1 - 1)/2 + F(n - m, k - k_1) < F(n, k)$, a contradiction. Hence, $U \cap S = \emptyset$. □

Claim 36 $r(F, U_2, \gamma_t) \leq F(n', k_2)$.

Proof. If F is disconnected or if F is connected and $1 \leq k_2 \leq n'$, then, as observed earlier (see the third paragraph of §6.3), the desired result follows. Suppose, then, that F is connected and $k_2 = 0$. If F is not a cycle, then, by Theorem 1, $r(F, U_2, \gamma_t) = \gamma_t(F) \leq 4n'/7 = F(n', k_2)$. If F is a cycle, then it follows from Claim 34 that $F = C_8$ (with w and z at distance 4 apart on this cycle), and so $\gamma_t(F) = 4 < F(8, 0) = F(n', k_2)$. Hence if F is connected and $k_2 = 0$, then $r(F, U_2, \gamma_t) \leq F(n', k_2)$. □

Claim 37 *If* $|U \cap V(P)| \geq 1$, *then* $m = 3$ *and* $U \cap V(P) = \{u, v\}$.

Proof. By Lemma 24, no two adjacent vertices of P are both in U. By Claim 35, $w, z \notin U$.

Suppose that $m + k_1 \geq 10$. Let C be obtained from the 2-graph path P_w by adding the edge vw. Thus, C is the cycle $w, u_1, u_2, \ldots, u_m, w$ of length $m + 1$ and $U \cap V(C) = U_1$. By Theorem 14, $r(C, U_1, \gamma_t) \leq H(m + 1, k_1) = (m + k_1 + 2)/2$.

Suppose $r(C, U_1, \gamma_t) \leq (m + k_1 + 1)/2$. Let S_1 be a $r(C, U_1, \gamma_t)$-set. By Claim 36, $r(F, U_2, \gamma_t) \leq F(n - m, k_2)$. Thus if $w \notin S_1$, then $r(G, U, \gamma_t) \leq |S_1| + r(F, U_2, \gamma_t) \leq (m + k_1 + 1)/2 + F(n - m, k_2) = F(n, k) + (7 - m - k_1)/14 < F(n, k)$. Hence, $w \in S_1$. Let S_2 be a $r(F, U_2 \cup \{w, z\}, \gamma_t)$-set. Then, $S_1 \cap S_2 = \{w\}$ and $S_1 \cup S_2$ is a TDS of G containing U. Hence, $r(G, U, \gamma_t) \leq |S_1 \cup S_2| = |S_1| + |S_2| - 1 \leq (m + k_1 + 1)/2 + r(F, U_2 \cup \{w, z\}, \gamma_t) - 1 \leq (m + k_1 - 1)/2 + F(n - m, k_2 + 2) = F(n, k) + (9 - m - k_1)/14 < F(n, k)$, a contradiction. Hence, $r(C, U_1, \gamma_t) = (m + k_1 + 2)/2$.

It follows from Theorem 14 and Lemma 23 that $k_1 \geq 3$ is odd, $m = 2k_1 - 1$ and $U_1 = \{u_{2i-1} \mid 1 \leq i \leq k_1\}$. Thus, $r(G, U, \gamma_t) \leq |U_1| + |\{u_{4i-2} \mid 1 \leq i \leq (k_1 - 1)/2\}| + r(F, U_2 \cup \{z\}, \gamma_t) \leq k_1 + (k_1 - 1)/2 + F(n - m, k_2 + 1) \leq (m + k_1)/2 + F(n - m, k_2 + 1) = F(n, k) + (8 - m - k_1) < F(n, k)$, a contradiction. Hence, $m + k_1 \leq 9$.

The desired result now follows from the fact that if $m \neq 3$ or if $m = 3$ and $k_1 = 1$, then it is straightforward (though tedious) to show that $r(G, U, \gamma_t) < F(n, k)$. For example, consider the case when $m = 4$. By Lemma 24, either $k_1 = 1$ or $k_1 = 2$. If $U_1 = \{u\}$, then $r(G, U, \gamma_t) \leq |\{u, u_2\}| + r(F, U_2 \cup \{z\}, \gamma_t) \leq 2 + F(n-4, k) < F(n, k)$. If $U_1 = \{u_2\}$ or $U_1 = \{u_3\}$, then $r(G, U, \gamma_t) \leq |\{u_2, u_3\}| + r(F, U_2, \gamma_t) \leq 2 + F(n-4, k-1) < F(n, k)$. If $U_1 = \{v\}$, then $r(G, U, \gamma_t) \leq |\{u_3, v\}| + r(F, U_2 \cup \{w\}, \gamma_t) \leq 2 + F(n-4, k) < F(n, k)$. If $U_1 = \{u, u_3\}$, then $r(G, U, \gamma_t) \leq |\{u, u_2, u_3\}| + r(F, U_2, \gamma_t) \leq 3 + F(n-4, k-2) < F(n, k)$. If $U_1 = \{u, v\}$, then $r(G, U, \gamma_t) \leq |\{u, v\}| + r(F, U_2 \cup \{w, z\}, \gamma_t) \leq 2 + F(n-4, k) < F(n, k)$. If $U_1 = \{u_2, v\}$, then $r(G, U, \gamma_t) \leq |\{u_2, u_3, v\}| + r(F, U_2, \gamma_t) \leq 3 + F(n-4, k-2) < F(n, k)$. Hence if $m = 4$, then $r(G, U, \gamma_t) < F(n, k)$. □

By Claims 34, 35, and 37, $U \cap S = \emptyset$ and every 2-graph path in G has length 3 and either contains no vertex of U or exactly two vertices of U (namely, the end-vertex of the 2-graph path and the vertex at distance 2 from it on this path). In particular, G has girth at least 8. Since $k = |U| \geq 1$, there must exist a 2-graph path that contains two vertices of U. Using our notation introduced earlier, we may assume that $P_w\colon w, u, u_2, v$ is such a 2-graph path (and so $u, v \in U$). As defined earlier, $N(v) = \{u_2, z\}$ where $z \in S$.

Claim 38 $|S| \geq 3$.

Proof. Suppose $S = \{w, z\}$. Let r (respectively, s) be the number of 2-graph paths of w that contain (respectively, do not contain) vertices of U. Then, $\deg_G w = r+s$, $k = 2r$, $n = 2+3(r+s)$ and G is obtained from $K_{2,r+s}$ by subdividing each edge exactly once. Let $D = U \cup N[w] \cup \{z\}$. Then, D is a TDS of G containing U of cardinality $k + s + 2 = 2r + s + 2 < F(n, k)$, a contradiction. Hence, $|S| \geq 3$. □

By Claim 38, $|S| \geq 3$. Let G' be the graph obtained from $G - \{w, u, u_2, v, z\}$ by adding as few edges as possibly joining vertices in $(N(w) \cup N(z)) - \{u, v\}$ so that G' is connected, $\delta(G') \geq 2$ and G' has girth as large as possible. Since G has girth at least 8, and since $|S| \geq 3$, it follows that G' has girth at least 7, and so $G' \notin \mathcal{G}_{k-2}$. Hence, by the inductive hypothesis, $r(G', U - \{u, v\}, \gamma_t) < F(n-5, k-2)$. Let S_w be a $r(G', U - \{u, v\}, \gamma_t)$-set. Then, $S_w \cup \{u, v, w, z\}$ is a TDS of G containing U of cardinality $|S_w| + 4 < F(n-5, k-2) + 4 = F(n, k)$, a contradiction. This completes the proof of Lemma 26.

6.4 Proof of Lemma 27

Suppose C is a 2-graph cycle of $w \in S$. Let $P = C - w$ be a u–v path on $m \geq 2$ vertices, and so $H = G - V(P)$. Since $\delta(H) \geq 2$, we note that $\deg_G w \geq 4$. Let H have order h, and so $n = h + m$. Let U_1 contain the vertices of U in P, and let $U_2 = U - U_1$. Let $|U_1| = k_1$, and let $k_2 = k - k_1$. By Observation 21, if $k_2 \geq 1$, then either $H \in \mathcal{G}_{k_2}$ or $r_{k_2}(H, \gamma_t) < F(h, k_2)$.

Claim 39 $k_1 \geq 1$.

Proof. Suppose $k_1 = 0$, and so $U = U_2$ and $k = k_2$. By Lemma 23, $m \leq 5$. If $m = 2$, then $r(G, U, \gamma_t) \leq r(H, U \cup \{w\}, \gamma_t) \leq F(h, k+1) < F(h+2, k) = F(n, k)$. If $m = 3$, then $r(G, U, \gamma_t) \leq |\{u\}| + r(H, U \cup \{w\}, \gamma_t) \leq 1 + F(n-3, k+1) < F(n, k)$. If $m = 4$, then $r(G, U, \gamma_t) \leq 2 + r(H, U, \gamma_t) \leq 2 + F(h, k) = 2 + F(n-4, k) < F(n, k)$. If $m = 5$, then $r(G, U, \gamma_t) \leq 2 + r(H, U \cup \{w\}, \gamma_t) \leq 2 + F(n-5, k+1) < F(n, k)$. In all cases, $r(G, U, \gamma_t) < F(n, k)$, a contradiction. □

Claim 40 $r(H, U_2, \gamma_t) \leq F(h, k_2)$.

Proof. If $k_2 \geq 1$, then the desired result follows from the inductive hypothesis. If $k_2 = 0$, then since C is a 2-graph cycle of G that contains as few vertices of U as possible, H is not a cycle, and so by Theorem 1, $r(H, U_2, \gamma_t) = \gamma_t(H) \leq 4h/7 = F(h, k_2)$. □

Claim 41 *If* $r(P, U_1, \gamma_t) \leq H(m, k_1)$, *then* $r(G, U, \gamma_t) < F(n, k)$.

Proof. If $r(P, U_1, \gamma_t) < H(m, k_1)$, then, using Claim 40, $r(G, U, \gamma_t) \leq r(P, U_1, \gamma_t) + r(H, U_2, \gamma_t) \leq (m+k_1)/2 + F(h, k_2) = F(n, k) - (m+k_1)/14 < F(n, k)$. Hence we may assume $r(P, U_1, \gamma_t) = H(m, k_1)$.

Suppose first that $m + k_1 \leq 6$. If $k_1 = 1$ (and so, $m \leq 5$), then, since $r(P, U_1, \gamma_t) = H(m, k_1)$, U_1 must consist of an end-vertex of P and either $m = 2$ or $m = 4$. It follows that $r(G, U, \gamma_t) \leq r(H, U_2 \cup \{w\}, \gamma_t) + m/2 \leq F(h, k) + m/2 = F(n, k) - m/14 < F(n, k)$. If $k_1 = 2$ (and so, $m \leq 4$), then, since $r(P, U_1, \gamma_t) = H(m, k_1)$, $U_1 = \{u, v\}$ and $m = 3$. It follows that $r(G, U, \gamma_t) \leq r(H, U_2 \cup \{w\}, \gamma_t) + 2 \leq F(h, k-1) + 2 = F(n, k) - 2/7 < F(n, k)$. Suppose $k_1 = 3$ (and so, $m = 3$). Then using Claim 40, $r(G, U, \gamma_t) \leq r(H, U_2, \gamma_t) + 3 \leq F(h, k_2) + 3 = F(n-3, k-3) + 3 = F(n, k) - 3/7 < F(n, k)$. Hence if $m + k_1 \leq 6$, then $r(G, U, \gamma_t) < F(n, k)$. Thus we may assume that $m + k_1 \geq 7$. In particular, $m \geq 4$.

Using Claim 40, $r(G,U,\gamma_t) \le r(P,U_1,\gamma_t) + r(H,U_2,\gamma_t) \le H(m,k_1) + F(h,k_2) = F(n,k) + (7-m-k_1)/14$. If $m+k_1 > 7$ or if $m+k_1 = 7$ and $r(H,U_2,\gamma_t) < F(h,k_2)$, then $r(G,U,\gamma_t) < F(n,k)$. Suppose then that $m+k_1 = 7$ and $r(H,U_2,\gamma_t) = F(h,k_2)$. Since $m+k_1 = 7$ and $r(P,U_1,\gamma_t) = H(m,k_1)$, it follows that either $k_1 = 1$ and $m = 6$ or $k_1 = 2$ and $m = 5$. If $k_2 = 0$, then $r(H,U_2,\gamma_t) = \gamma_t(H) = F(h,0) = 4h/7$. As observed earlier, H is not a cycle, and so by Theorem 12, $H \in \mathcal{H} \cup \{H_1\}$. If $k_2 \ge 1$, then $H \in \mathcal{G}$ by Observation 21. Hence by Theorem 3 if $k_2 \ge 1$ or by Observation 11 if $k_2 = 0$, there exists a TDS D_2 of H containing $U_2 \cup \{w\}$ of cardinality $F(h,k_2)$. It follows that D_2 can be extended to a TDS of G containing U by adding three vertices of P, and so $r(G,U,\gamma_t) \le |D_2| + 3 = F(h,k_2) + 3 < F(n,k)$. □

By Theorem 15, $r_{k_1}(P_m,\gamma_t) \le H(m+1,k_1)$. Consequently, by Claim 41, $r(P,U_1,\gamma_t) = H(m+1,k_1)$.

If $k_1 = 1$ and $m \le 4$ or if $k_1 = 2$ and $m \le 3$, then $r(P,U_1,\gamma_t) \le H(m,k_1)$, a contradiction. If $m = 4$ and $k_1 \ge 2$, then, since $r(P,U_1,\gamma_t) = H(m+1,k_1)$, we must have $U_1 = \{u,v\}$, and so $r(G,U,\gamma_t) \le r(H,U_2 \cup \{w\},\gamma_t) + 2 \le F(h,k-1) + 2 < F(n,k)$, a contradiction. Hence, $m \ge 5$.

Claim 42 $|U \cap \{u,v\}| \ge 1$.

Proof. Suppose $u,v \notin U$. Suppose first that $m+k_1 \le 8$. If $k_1 = 1$ and $m = 5$ or $k_1 = 2$ and $5 \le m \le 6$ or $k_1 = 3$ and $m = 5$, then $r(P,U_1,\gamma_t) \le H(m,k_1)$, a contradiction. Hence $k_1 = 1$ and $m = 6$ or $m = 7$. In both cases, $r(G,U,\gamma_t) \le r(H,U_2 \cup \{w\},\gamma_t) + 3 \le F(h,k) + 3 < F(n,k)$, a contradiction. Hence, $m + k_1 \ge 9$. Since $v \notin U$, Corollary 16 implies that $r(P-v,U_1,\gamma_t) \le H(m-1,k_1)$, and so $r(G,U,\gamma_t) \le r(P-v,U_1,\gamma_t) + r(H,U_2 \cup \{w\},\gamma_t) \le H(m-1,k_1) + F(h,k_2+1) = F(n,k) + (8-m-k_1)/14 < F(n,k)$, a contradiction. □

By Claim 42, we may assume **v** $\in$ **U**.

Corollary 16 implies there exists a set S' containing U_1 such that $|S'| \le H(m-1,k_1)$ and every vertex of $P-v$ is adjacent to some vertex of S'. Hence, $r(G,U,\gamma_t) \le |S'| + r(H,U_2 \cup \{w\},\gamma_t) \le H(m-1,k_1) + F(h,k_2+1) = F(n,k) + (8-m-k_1)/14$. If $m+k_1 \ge 9$, then $r(G,U,\gamma_t) < F(n,k)$. Hence we may assume $6 \le m+k_1 \le 8$ (recall that $m \ge 5$).

Let the path P be given by $v = v_1, v_2, \ldots, v_m = u$. Since $r(P,U_1,\gamma_t) = H(m+1,k_1)$, it follows that either $(m,k_1) = (5,1)$ and $U_1 = \{v\}$ or $(m,k_1) = (5,3)$ and $U_1 = \{u,v,v_3\}$ or $(m,k_1) = (6,2)$ and $U_1 = \{v,v_3\}$.

Claim 43 $(m, k_1) \neq (5, 3)$.

Proof. Suppose that $(m, k_1) = (5, 3)$, and so $U_1 = \{u, v, v_3\}$. Then, $r(G, U, \gamma_t) \leq |U_1 \cup \{v_2\}| + r(H, U_2 \cup \{w\}, \gamma_t) \leq 4 + F(n-5, k-2) = F(n, k)$. Hence we must have equality throughout this inequality chain. In particular, $r(H, U_2 \cup \{w\}, \gamma_t) = F(n-5, k-2)$, and so, by the inductive hypothesis, $H \in \mathcal{G}_{k-2}$. By Theorem 3, there is a selection of units and link vertices of H so that the resulting set of pivot vertices is precisely the set $U_2 \cup \{w\}$. In particular, w is a pivot vertex in a type (a) unit of H and therefore has degree 2 in H (and so, $\deg_G w = 4$). If H consists of only one unit, then $H = C_6$ and $k_2 = 0$, contradicting our choice of C. Hence, H consists of at least two units. Therefore at least one of the two link vertices of H adjacent to w has degree at least 3. This contradicts our assumption that S is an independent set in G. Hence, $(m, k_1) \neq (5, 3)$. □

Claim 44 $(m, k_1) \neq (6, 2)$.

Proof. Suppose that $(m, k_1) = (6, 2)$, and so $U_1 = \{v, v_3\}$. Let $K = (G - \{u, v_5\}) + v_4w$. Then K is a connected graph of order $n-2$ with $\delta(K) \geq 2$. By the inductive hypothesis, $r(F, U, \gamma_t) \leq F(n-2, k)$. Let S_K be a $r(K, U, \gamma_t)$-set. In particular, $v, v_3 \in S_K$. We may assume that $v_2 \in S_K$. But then $r(G, U, \gamma_t) \leq |(S_K - \{v_2\}) \cup \{v_4, w\}| \leq F(n-2, k) + 1 < F(n, k)$, a contradiction. The desired result follows. □

By Claims 43 and 44, $(m, k_1) = (5, 1)$ and $U_1 = \{v\}$. Hence, $h = n - 5$ and $k_2 = k - 1$.

Claim 45 $r(H, U_2 \cup \{w\}, \gamma_t) = (4(h+k) - 1)/7$.

Proof. By induction, $r(H, U_2 \cup \{w\}, \gamma_t) \leq F(h, k_2 + 1) = F(h, k)$. If $r(H, U_2 \cup \{w\}, \gamma_t) \leq (4(h+k) - 2)/7$, then $r(G, U, \gamma_t) \leq |\{u, v, v_2\}| + r(H, U_2 \cup \{w\}, \gamma_t) \leq 3 + (4(h+k) - 2)/7 < F(n, k)$, a contradiction. Hence, $r(H, U_2 \cup \{w\}, \gamma_t) \geq (4(h+k) - 1)/7$.

Suppose $r(H, U_2 \cup \{w\}, \gamma_t) = F(h, k_2 + 1) = F(h, k)$. By the inductive hypothesis, $H \in \mathcal{G}_k$. By Theorem 3, there is a selection of units and link vertices of H so that the resulting set of pivot vertices is precisely the set $U_2 \cup \{w\}$. Proceeding now in a similar manner as in the proof of Claim 43, we can show that w is adjacent to some other vertex of S, a contradiction. The desired result follows. □

By Claim 45, $r(H, U_2 \cup \{w\}, \gamma_t) = (4(h+k)-1)/7$. Let D be a minimum TDS of H containing U_2 and, if possible, the vertex w. If $w \in D$, then $r(G, U, \gamma_t) \leq 3 + r(H, U_2, \gamma_t) \leq 3 + F(n-5, k-1) < F(n, k)$, a contradiction. Hence, $w \notin D$, and so w belongs to no minimum TDS of H containing U_2. It follows that $|D| = r(H, U_2, \gamma_t) = r(H, U_2 \cup \{w\}, \gamma_t) - 1 = 4(h+k-2)/7 = F(n, k) - 4$.

Since S is an independent set, each neighbor of w in H has degree 2 in G and therefore degree 1 in $H - w$. Let H^* be the graph obtained from $H - w$ by adding as few edges as possibly joining vertices in $N(w) - \{u, v\}$ so that H^* is connected and, if possible, $\delta(H^*) \geq 2$. Let $|V(H^*)| = n^*$ $(= n - 6)$.

Claim 46 $\delta(H^*) \geq 2$.

Proof. Suppose that $\delta(H^*) = 1$. Then there exists a vertex w^* of degree one in H^*. By the way in which H^* is defined, it follows that $H = K_3$. Since w belongs to no minimum TDS of H containing U_2, it follows that $U_2 = V(H) - \{w\}$. Hence, $n = 8$ and $k = 3$ and $r(G, U, \gamma_t) \leq 6 < F(n, k)$, a contradiction. Hence, $\delta(H^*) \geq 2$. □

By Claim 46, $\delta(H^*) \geq 2$. The desired result of Lemma 27 now follows from Claims 47 and 48.

Claim 47 *If $k \geq 2$, then $G \in \mathcal{G}_k$.*

Proof. Then, $k_2 = k - 1 \geq 1$. By the inductive hypothesis, $r(H^*, U_2, \gamma_t) \leq F(n^*, k_2) = F(n-6, k-1) = F(n, k) - 4$. If $r(H^*, U_2, \gamma_t) < F(n, k) - 4$, then $r(G, U, \gamma_t) \leq |\{u, v, v_2, w\}| + r(H^*, U_2, \gamma_t) < F(n, k)$, a contradiction. Thus, $r(H^*, U_2, \gamma_t) = F(n, k) - 4$, and so, by the inductive hypothesis, $H^* \in \mathcal{G}_{k_2}$. Since $k_2 = k - 1$, $H^* \in \mathcal{G}_{k-1}$. Hence, by Theorem 3, there is a selection of units and link vertices of H^* so that the resulting set of pivot vertices is precisely the set U_2.

We show now that each neighbor of w in H^* is a link vertex of H^*. Suppose that $x \in N(w) \cap V(H)$ is not a link vertex of H^*. Note that $\{u, v, v_2, w\}$ is a $r(C, U_1, \gamma_t)$-set that contains w. It now follows from Lemma 22 (with $G_1 = C$, $G_2 = H^*$, $v_1 = w$ and $v_2 = x$) that $r(G, U, \gamma_t) \leq r(C, U_1, \gamma_t) + r(H^*, U_2, \gamma_t) - 1 = 4 + (F(n, k) - 4) - 1 = F(n, k) - 1$, a contradiction. Hence each neighbor of w in H^* is a link vertex of H^*. Therefore, $G \in \mathcal{G}_k$. □

Claim 48 *If $k = 1$, then $G \in \mathcal{G}_k$.*

Proof. Since $k = 1$, $U \cap V(H^*) = \emptyset$. Since H^* contains a vertex of S, H^* cannot be a cycle. Hence, by Theorem 1, $\gamma_t(H^*) \le 4n^*/7 = F(n-6,0)$. If $\gamma_t(H^*) < F(n-6,0)$, then $r(G,U,\gamma_t) \le |\{u,v,v_2,w\}| + \gamma_t(H^*) < 4 + F(n-6,0) = F(n,1) = F(n,k)$, a contradiction. Thus, $\gamma_t(H^*) = F(n-6,0)$. It follows from the way in which H^* is constructed that H^* is a $\frac{4}{7}$-minimal graph. Hence, by Theorem 12, $H^* \in \mathcal{H} \cup \{H_1\}$.

If $H^* = H_1$ (and so $n = 13$), then since S is independent and $\deg_H w \ge 2$, the vertex w would belongs to a $\gamma_t(H)$-set. This contradicts our earlier observation (see the paragraph following Claim 45) that the vertex w belongs to no $\gamma_t(H)$-set (in our case, $U_2 = \emptyset$). Hence, $H^* \in \mathcal{H}$.

Since $\delta(H^*) \ge 2$, H^* consists of at least two type (b) units and the underlying tree of H^* is precisely the subgraph of H^* induced by the link vertices of these type (b) units. Let S^* be a $\gamma_t(H^*)$-set chosen as follows: For each (type (b)) unit F of H^*, let $S^* \cap V(F)$ be a $\gamma_t(F)$-set (of cardinality 4) that contains the link vertex of F (and therefore also its neighbor).

We show next that each neighbor of w in H^* is a link vertex of H^*. Let $y \in N(w) \cap V(H)$ and let F_y be the (type (b)) unit of H^* containing y. Suppose y is not the link vertex of F_y. Then replace the four vertices of S^* in F_y by y, a vertex at distance 2 from y on the 6-cycle in F_y, and the vertex at distance 3 from y on the 6-cycle in F_y. Let D^* denote the resulting adjusted set S^* (note that the link vertex of F_y is dominated by D^* since H^* contains at least two units). Then, $D^* \cup \{u,v,v_2,w\}$ is a TDS of G containing U of cardinality $(|S^*|-1)+4 = (F(n-6,0)-1)+F(6,1) = F(n,k)-1 < F(n,k)$, a contradiction. Hence, y is the link vertex of F_y. Therefore each neighbor of w in H^* is a link vertex of H^*. It follows that $G \in \mathcal{G}_k$. □

6.5 Proof of Lemma 28

Suppose C is a 2-graph cycle of $w \in S$. Since $\delta(H) = 1$, we note that $\deg_G w = 3$. Let z be the vertex of G of degree at least three that is at minimum distance from w. Since S is an independent set, $d(w,z) = r+1 \ge 2$. Let v denote the vertex adjacent to z on the w–z path and let u denote the neighbor of w on this path (if $r = 1$, then $u = v$). Then $G - vz$ consists of two components, one of which is a key $L_{m,r}$ which contains v as its end-vertex and the other, which we call F, is a connected graph with $\delta(F) \ge 2$ containing z. Let F have order p. Let U_1 contain the vertices of U in $L_{m,r}$, and let $U_2 = U - U_1$. Let $|U_1| = k_1$, and let $k_2 = k - k_1$. Since G has order n, $n = m + r + p$.

Claim 49 $r(F, U_2, \gamma_t) \leq F(p, k_2)$.

Proof. If $k_2 \geq 1$, then the desired result follows by the inductive hypothesis. Hence we may assume that $k_2 = 0$. Suppose that $F = C_p$. Then, G is obtained from the disjoint union, $C_m \cup C_p$, of C_m and C_p by joining a vertex of C_m to a vertex of C_p and subdividing this edge r times. Hence, G has order $n = m + r + p$ and size $n + 1$. By Theorem 2, $r(G, U, \gamma_t) \leq (n+k+2)/2$. Hence if $n+k > 14$, then $r(G, U, \gamma_t) < F(n, k)$, a contradiction. Thus, $n + k \leq 14$. Since the 2-graph cycle C was chosen to contain as few vertices of U as possible, it follows that all vertices of U belong to the u–v path P_r. Furthermore by Lemma 24, no two adjacent vertices of this path P_r are both in U. By Lemma 23, $m \leq 6$ and $p \leq 6$. Since $n + k \leq 14$, it is now straightforward (though tedious) to show that $r(G, U, \gamma_t) < F(n, k)$, a contradiction. Hence, F is not a cycle, and so by Theorem 1, $r(F, U_2, \gamma_t) = \gamma_t(F) \leq 4p/7 = F(p, k_2)$. □

Claim 50 *If* $k_1 \geq 1$, *then* $m + r + k_1 \geq 9$.

Proof. Suppose that $m + r + k_1 \leq 8$. Then it is straightforward (though tedious) to show that either $r(L_{m,r}, U_1, \gamma_t) \leq (m + r + k_1)/2$ or there exists a set D containing $U_1 \cup \{v\}$ such that $|D| \leq (m + r + k_1 - 1)/2$ and every vertex of $L_{m,r} - v$ is adjacent to some (other) vertex of D. In the former case, by Claim 49, $r(G, U, \gamma_t) \leq r(L_{m,r}, U_1, \gamma_t) + r(F, U_2, \gamma_t) \leq (m+r+k_1)/2+F(p, k_2) = F(n, k)-(m+r+k_1)/14 < F(n, k)$. In the latter case, $r(G, U, \gamma_t) \leq |D|+r(F, U_2 \cup \{z\}, \gamma_t) \leq (m+r+k_1-1)/2+F(p, k_2+1) = F(n, k)+(1-m-r-k_1)/14 < F(n, k)$. In both cases, $r(G, U, \gamma_t) < F(n, k)$ a contradiction. The desired result follows. □

Claim 51 *If* $k_1 \geq 1$, *then* $r(L_{m,r}, U_1, \gamma_t) \leq H(m + r, k_1)$.

Proof. By Claim 50, $m + r + k_1 \geq 9$. By Theorem 17, $r(L_{m,r}, U_1, \gamma_t) \leq H(m + r + 1, k_1)$. Suppose that $r(L_{m,r}, U_1, \gamma_t) = H(m + r + 1, k_1)$. Then by Theorem 17, $r(L_{m,r} - v, U_1, \gamma_t) \leq H(m + n - 1, k_1)$ or there exists a set D containing $U_1 \cup \{v\}$ such that $|D| \leq H(m + r - 1, k_1)$ and every vertex of $L_{m,r} - v$ is adjacent to some vertex of D. Suppose $r(L_{m,r} - v, U_1, \gamma_t) \leq H(m+r-1, k_1)$. Then, $r(G, U, \gamma_t) \leq r(L_{m,r}-v, U_1, \gamma_t)+r(F, U_2 \cup \{z\}, \gamma_t) \leq (m+r+k_1-1)/2+F(p, k_2+1) = F(n, k)+(1-m-r-k_1)/14 < F(n, k)$, a contradiction. On the other hand, if there exists a set D containing $U_1 \cup \{v\}$ such that $|D| \leq H(m + r - 1, k_1)$ and every vertex of $L_{m,r} - v$ is adjacent to some vertex of D, then $r(G, U, \gamma_t) \leq |D| + r(F, U_2 \cup \{z\}, \gamma_t) \leq (m + r + k_1 - 1)/2 + F(p, k_2 + 1) < F(n, k)$, a contradiction. Hence, $r(L_{m,r}, U_1, \gamma_t) \leq H(m + r, k_1)$. □

Claim 52 $k_1 = 0$.

Proof. Suppose that $k_1 \geq 1$. By Claim 50, $m + r + k_1 \geq 9$ and by Claim 51, $r(L_{m,r}, U_1, \gamma_t) \leq H(m+r, k_1)$. Hence, by Claim 49, $r(G, U, \gamma_t) \leq r(L_{m,r}, U_1, \gamma_t) + r(F, U_2, \gamma_t) \leq (m + r + k_1 + 1)/2 + F(p, k_2) = F(n, k) + (7 - m - r - k_1)/14 < F(n, k)$, a contradiction. □

By Claim 52, $k_1 = 0$. Thus $k_2 = k \geq 1$ and so, by Observation 21, $r(F, U, \gamma_t) \leq F(p, k)$ with equality if and only if $F \in \mathcal{G}_k$.

Claim 53 $m = 6$ *and* $r = 1$.

Proof. By Lemma 23, $m \leq 6$ and $r \leq 3$. If $(m, r) \in \{(6,3), (6,2), (5,3)\}$, then, by Theorem 8, $\gamma_t(L_{m,r}) \leq (m + r + 1)/2$, whence $r(G, U, \gamma_t) \leq \gamma_t(L_{m,r}) + r(F, U, \gamma_t) \leq (m+r+1)/2 + F(p, k) = F(n, k) + (7 - m - r)/14 < F(n, k)$, a contradiction. Hence, $m + r \leq 7$. Suppose $m + r \leq 6$. Then it is straightforward to check (or see [6]), that $\gamma_t(L_{m,r}) \leq (m + r)/2$, and so $r(G, U, \gamma_t) \leq \gamma_t(L_{m,r}) + r(F, U, \gamma_t) \leq (m + r)/2 + F(p, k) < F(n, k)$, a contradiction. Thus, $m + r = 7$. If $r \geq 2$, then $r(G, U, \gamma_t) \leq \gamma_t(L_{m,r} - v) + r(F, U \cup \{z\}, \gamma_t) \leq 3 + F(p, k + 1) < F(n, k)$, a contradiction. Hence, $m = 6$ and $r = 1$. □

By Claim 53, $m = 6$ and $r = 1$. Now, $F(n, k) \leq r(G, U, \gamma_t) \leq \gamma_t(L_{6,1}) + r(F, U, \gamma_t) \leq 4 + F(p, k) = F(n, k)$. Consequently, we must have equality throughout this inequality chain. In particular, $r(F, U, \gamma_t) = F(p, k)$, and so $F \in \mathcal{G}_k$. Hence, by Theorem 3, there is a selection of units and link vertices of F so that the resulting set of pivot vertices is precisely the set U.

If z is not a link vertex of F, then it follows from Lemma 22 (with $G_1 = L_{6,1}$ and $G_2 = F$) that $r(G, U, \gamma_t) \leq r(L_{6,1}, U_1, \gamma_t) + r(F, U_2, \gamma_t) - 1 = 4 + F(p, k) - 1 = F(n, k) - 1$, a contradiction. Hence z is a link vertex of F, and so $G \in \mathcal{G}_k$. This completes the proof of Lemma 28.

7 Proof of Theorem 5

Since the restricted total domination number of a graph cannot decrease if edges are removed, the upper bound of Theorem 5 is an immediate consequence of Theorems 3 and 4. The family of graphs achieving equality in the upper bound of Theorem 5 is also an immediate consequence of

Theorems 3 and 4 and the observation that for any graph $G \in \mathcal{G}_k$ with U as its set of k pivot vertices and for any edge e in the complement $\overline{G}$ of G, $r(G+e, U, \gamma_t) < r(G, U, \gamma_t)$ if e does not join two link vertices and $r(G+e, U, \gamma_t) = r(G, U, \gamma_t)$ otherwise.

8 Proof of Theorem 6

If $\delta(G) \geq 2$, then the result follows from Theorem 1. Hence we may assume that $\delta(G) = 1$. Let L denote the set of vertices of degree one in G and let U denote the set of all vertices of G adjacent to at least one vertex in L. Let $|L| = \ell$ and $|U| = k$. By assumption, $k \geq 1$. Since the sum of the degrees of any two distinct adjacent vertices in G is at least 4, each vertex of U has degree at least three in G.

Let $G' = G - L$. Then, G' is a connected graph of order $n' = n - \ell$. Since each vertex in U is adjacent to at least one vertex in L, $\ell \geq k$ and so $n' \leq n - k$. If $G' = K_1$, then $n \geq 4$, G is a star, and $\gamma_t(G) = 2 < 4n/7$. Hence we may assume that $n' \geq 2$, and so $\delta(G') \geq 1$.

Suppose $\delta(G') \geq 2$. Then, by Theorem 5, $r(G', U, \gamma_t) \leq 4(n'+k)/7 \leq 4n/7$ and the desired result follows since $\gamma_t(G) \leq r(G', U, \gamma_t)$.

Suppose $\delta(G') = 1$. Each vertex u of degree one in G' belongs to the set U and is adjacent to $\deg u - 1 \geq 2$ vertices of degree one in G. For each such vertex u of degree one in G', we add one of its neighbors in L to G' and join it to u and all the neighbors of u in G'. Let G^* denote the resulting graph of order n^*. By construction, G^* is a connected graph with $\delta(G^*) \geq 2$. Further, each vertex of U in G^* is adjacent to at least one vertex of $L - V(G^*)$, and so $n^* \leq n-k$. By Theorem 5, $r(G^*, U, \gamma_t) \leq 4(n^*+k)/7 \leq 4n/7$ and the desired result follows since $\gamma_t(G) \leq r(G^*, U, \gamma_t)$. □

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