The 2-color Rado Number of $x + y + kz = 3w$

Dan Saracino and Brian Wynne
Colgate University

Abstract

For any positive integer k there exists a smallest positive integer N, depending on k, such that every 2-coloring of 1,2,...,N contains a monochromatic solution of the equation $x+y+kz = 3w$. Based on computer checks, Robertson and Myers in [5] conjectured values for N depending on the congruence class of k (mod 9). In this note we establish the values of N and find that in some cases they depend on the congruence class of k (mod 27).

Introduction

If N is a positive integer, let $[1, N]$ denote the set $\{1, 2, ..., N\}$. A *2-coloring* of $[1, N]$ is a function $\chi : [1, N] \to \{0, 1\}$. If $\Sigma_{i=1}^{n} c_i x_i = 0$ is a homogeneous linear equation with integer coefficients then a solution $a_1, ..., a_n$ is called *monochromatic* if $\chi(a_i) = \chi(a_j)$ for all $1 \leq i, j \leq n$.

It follows from a theorem of R. Rado [4] that if $n \geq 3$ and at least one $c_i > 0$ and at least one $c_j < 0$ then there exists an N such that every 2-coloring of $[1, N]$ admits a monochromatic solution. The smallest such N is called the *2-color Rado number* of the equation $\Sigma_{i=1}^{n} c_i x_i = 0$ and denoted here by $RR(\Sigma_{i=1}^{n} c_i x_i = 0)$.

It was shown in [3] that for every positive integer k,

$$RR(x + y + kz = w) = k^2 + 5k + 5.$$

In [5], Robertson and Myers showed that $RR(x + y + kz = 2w)$ depends on the congruence class of k (mod 4), and determined its value for all k. Based on computer calculations for $k \leq 23$, they conjectured values for $RR(x + y + kz = 3w)$, depending on the congruence class of k (mod 9). (We remark that when $k = 17$ their conjectured formula does not yield the computed value.) In this paper we confirm their conjectures for $k \equiv 0, 1, 5, 6$, or 7 (mod 9). We show that for $k \equiv 8$ (mod 9) with $k > 8$ the conjectured value is too high, and we determine the correct value. For $k \equiv 2, 3$ or 4 (mod 9) we find that the true value is given by one of three formulas, depending on the congruence class of k (mod 27). So here the value of $RR(x + y + kz = \ell w)$ depends on the congruence class

ARS COMBINATORIA 90(2009), pp. 119-127

of k (mod ℓ^3).This finding invalidates another conjecture in [5], to the effect that $RR(x+y+kz=\ell w)$ depends only on the congruence class of k (mod ℓ^2).

For the convenience of the reader we restate the values for $RR(x+y+kz=3w)$ conjectured in [5]:

Conjecture [5]: For $k \geq 5$,

$$RR(x+y+kz=3w) = \begin{cases} \lfloor \frac{k+4}{3} \rfloor^2 - \frac{k}{9} = \frac{k^2+5k+9}{9} & \text{if } k \equiv 0 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 = \frac{k^2+4k+4}{9} & \text{if } k \equiv 1 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 - \frac{k+16}{9} = \frac{k^2+7k}{9} & \text{if } k \equiv 2 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 - 1 = \frac{k^2+6k}{9} & \text{if } k \equiv 3 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 + 1 = \frac{k^2+4k+13}{9} & \text{if } k \equiv 4 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 - \frac{k+4}{9} = \frac{k^2+7k+12}{9} & \text{if } k \equiv 5 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 = \frac{k^2+6k+9}{9} & \text{if } k \equiv 6 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 + \frac{k+2}{9} = \frac{k^2+5k+6}{9} & \text{if } k \equiv 7 \pmod 9 \\ \lfloor \frac{k+4}{3} \rfloor^2 - 1 = \frac{k^2+8k+7}{9} & \text{if } k \equiv 8 \pmod 9). \end{cases}$$

The values of $RR(x+y+kz=3w)$ for $k=2,\ 3$ and 8 were computed in [5] to be 4, 5, and 15, respectively. Our formulas yield the values for all other positive k.

Theorem If k is a positive integer and $k \neq 2, 3, 8$ then

$$RR(x+y+kz=3w) = \begin{cases} \frac{k^2+5k+9}{9} & \text{if } k \equiv 0 \pmod 9 \\ \frac{k^2+4k+4}{9} & \text{if } k \equiv 1 \pmod 9 \\ \frac{k^2+7k+12}{9} & \text{if } k \equiv 5 \pmod 9 \\ \frac{k^2+6k+9}{9} & \text{if } k \equiv 6 \pmod 9 \\ \frac{k^2+5k+6}{9} & \text{if } k \equiv 7 \pmod 9 \\ \frac{k^2+6k+14}{9} & \text{if } k \equiv 8 \pmod 9). \end{cases}$$

If $k \equiv 2 \pmod 9$ then

$$RR(x+y+kz=3w) = \begin{cases} \frac{3k^2+19k+31}{27} & \text{if } k \equiv 2 \pmod{27} \\ \frac{3k^2+19k+22}{27} & \text{if } k \equiv 11 \pmod{27} \\ \frac{3k^2+19k+40}{27} & \text{if } k \equiv 20 \pmod{27}). \end{cases}$$

If $k \equiv 3 \pmod 9$ then

$$RR(x+y+kz=3w) = \begin{cases} \frac{3k^2+16k+33}{27} & \text{if } k \equiv 3 \pmod{27} \\ \frac{3k^2+16k+24}{27} & \text{if } k \equiv 12 \pmod{27} \\ \frac{3k^2+16k+42}{27} & \text{if } k \equiv 21 \pmod{27}). \end{cases}$$

If $k \equiv 4 \pmod 9$ then

$$RR(x+y+kz=3w)=\begin{cases}\frac{3k^2+13k+8}{27} & \text{if } k\equiv 4 \pmod{27}\\ \frac{3k^2+13k+26}{27} & \text{if } k\equiv 13 \pmod{27}\\ \frac{3k^2+13k+17}{27} & \text{if } k\equiv 22 \pmod{27}.\end{cases}$$

We prove the theorem by proving in Section 1 that the asserted values are lower bounds for the true values, and in Section 2 that they are upper bounds.

For ease of exposition we refer to the two colors as red and blue. We sometimes write $j \in R$ or $j \in B$ to indicate that j is colored red or blue, respectively.

1. Lower Bounds

To show that our asserted values M for $RR(x+y+kz=3w)$ are lower bounds for the true values we must show that in every case there is a 2-coloring of $[1, M-1]$ that contains no monochromatic solution to $x+y+kz=3w$. The colorings we use depend on the congruence class of $k \pmod 9$. The arguments in this section do not require the restriction $k \notin \{2,3,8\}$.

$k \equiv 0, 1,$ or $8 \pmod 9$:

If $k \equiv 0 \pmod 9$ then $M-1=\frac{k^2+5k}{9}$ and we color $[1, \frac{k}{3}]$ red and $[\frac{k+3}{3}, \frac{k^2+5k}{9}]$ blue. From $x+y+kz=3w$ we obtain $w \geq \frac{k+2}{3}$ for every solution, so $w \in B$ and all monochromatic solutions are blue. But if x, y, z are in B then $3w \geq (\frac{k+3}{3})(k+2)$ so $w \geq \frac{k^2+5k+6}{9} > M-1$, which is impossible.

If $k \equiv 1 \pmod 9$ then $M-1=\frac{k^2+4k-5}{9}$ and we color $[1, \frac{k-1}{3}]$ red and $[\frac{k+2}{3}, \frac{k^2+4k-5}{9}]$ blue. If $k \equiv 8 \pmod 9$ then $M-1=\frac{k^2+6k+5}{9}$ and we color $[1, \frac{k+1}{3}]$ red and $[\frac{k+4}{3}, \frac{k^2+6k+5}{9}]$ blue. The arguments to show that these colorings work are virtually identical to that for $k \equiv 0 \pmod 9$.

$k \equiv 5, 6,$ or $7 \pmod 9$:

If $k \equiv 5 \pmod 9$ then $M-1=\frac{k^2+7k+3}{9}$ and we color $[1, \frac{k+1}{3}]$ red, $[\frac{k+4}{3}, \frac{k^2+6k-1}{9}]$ blue, and $[\frac{k^2+6k+8}{9}, \frac{k^2+7k+3}{9}]$ red. Then for any solution with x, y, z all blue, $3w \geq (\frac{k+4}{3})(k+2)$ so w must be red. Thus every monochromatic solution is red.

For every such solution, $w \geq \frac{k^2+6k+8}{9}$ (since $w \geq \frac{k+2}{3}$ as above). We must have $z \leq \frac{k+1}{3}$, so if both x and y are $\leq \frac{k+1}{3}$ then

$$\frac{k^2+6k+8}{3} \leq 3w \leq \left(\frac{k+1}{3}\right)(k+2)$$

and therefore $6k+8 \leq 3k+2$. Since this is impossible, at least one of x or y, say x, must be at least $\frac{k^2+6k+8}{9}$.

Since $y+kz = 3w-x$,

$$3\left(\frac{k^2+6k+8}{9}\right)-\frac{k^2+7k+3}{9} \leq y+kz \leq 3\left(\frac{k^2+7k+3}{9}\right)-\frac{k^2+6k+8}{9},$$

i.e.,

$$\frac{2k^2+11k+21}{9} \leq y+kz \leq \frac{2k^2+15k+1}{9}.$$

If $z \geq \frac{2k+17}{9}$ then $y \leq \frac{2k^2+15k+1}{9} - \frac{2k^2+17k}{9} = \frac{-2k+1}{9} < 0$, which is impossible. So $z \leq \frac{2k+8}{9}$ and therefore $y \geq \frac{2k^2+11k+21}{9} - \frac{2k^2+8k}{9} = \frac{3k+21}{9} > \frac{k+1}{3}$. Since we are assuming $y \in R$, this implies $y \geq \frac{k^2+6k+8}{9}$.

Write $x = \frac{k^2+6k+a}{9}, y = \frac{k^2+6k+b}{9}, w = \frac{k^2+6k+c}{9}$, with $a,b,c \in [8, k+3]$. Since $x+y+kz = 3w$, k must divide $3c-a-b$. But

$$18-2k \leq 3c-a-b \leq 3k-7$$

and therefore $\frac{3c-a-b}{k}$ must be in $[-1, 2]$. Since $z = \frac{k+6+\frac{3c-a-b}{k}}{9}$ and $k \equiv 5 \pmod 9$, this implies that z is not an integer. So there are no monochromatic solutions.

For $k \equiv 6 \pmod 9$ we color $[1, \frac{k}{3}]$ red, $[\frac{k+3}{3}, \frac{k^2+5k-3}{9}]$ blue, and $[\frac{k^2+5k+6}{9}, \frac{k^2+6k}{9}]$ red. For $k \equiv 7 \pmod 9$ we color $[1, \frac{k-1}{3}]$ red, $[\frac{k+2}{3}, \frac{k^2+4k-5}{9}]$ blue, and $[\frac{k^2+4k+4}{9}, \frac{k^2+5k-3}{9}]$ red. Arguments essentially the same as that for $k \equiv 5 \pmod 9$ show that these colorings work.

$k \equiv 2, 3$, or $4 \pmod 9$:

If $k \equiv 20 \pmod{27}$ we color $[1, \frac{k+1}{3}]$ red, $[\frac{k+4}{3}, \frac{k^2+6k+2}{9}]$ blue, and $[\frac{k^2+6k+11}{9}, \frac{3k^2+19k+13}{27}]$ red. As in the case $k \equiv 5 \pmod 9$ we see that every monochromatic solution must be red.

For every such solution, $w \geq \frac{k^2+6k+11}{9}$ (again since $w \geq \frac{k+2}{3}$). As for $k \equiv 5 \pmod 9$ we argue that at least one of x or y, say x, is at least $\frac{k^2+6k+11}{9}$. We then have $3\left(\frac{k^2+6k+11}{9}\right) - \frac{3k^2+19k+13}{27} \leq y+kz \leq 3\left(\frac{3k^2+19k+13}{27}\right) - \frac{k^2+6k+11}{9}$, so

$$\frac{6k^2+35k+86}{27} \leq y+kz \leq \frac{6k^2+39k+6}{27}.$$

If $z \geq \frac{2k+14}{9}$ then $y \leq \frac{6k^2+39k+6}{27} - \frac{2k^2+14k}{9} = \frac{-3k+6}{27} < 0$, which is impossible. So $z \leq \frac{2k+5}{9}$ and therefore $y \geq \frac{6k^2+35k+86}{27} - \frac{2k^2+5k}{9} = \frac{20k+86}{27} > \frac{k+1}{3}$. Thus $y \geq \frac{k^2+6k+11}{9}$.

Write $x = \frac{k^2+6k+a}{9}, y = \frac{k^2+6k+b}{9}, w = \frac{k^2+6k+c}{9}$, with $a, b, c \in [11, \frac{k+13}{3}]$. Since $x + y + kz = 3w$, k must divide $3c - a - b$. Since

$$\frac{73 - 2k}{3} \le 3c - a - b \le k - 9$$

it follows that $3c - a - b = 0$. But $z = \frac{k+6+\frac{3c-a-b}{k}}{9}$ and therefore $z = \frac{k+6}{9}$ is not an integer, because $k \equiv 2 \pmod 9$. Thus there are no monochromatic solutions.

The arguments for $k \equiv 2$ or $11 \pmod{27}$ are virtually the same, assuming that $k > 11$ so that $\frac{k^2+6k+11}{9}$ is no greater than $\frac{3k^2+19k+4}{27}$ or $\frac{3k^2+19k-5}{27}$. This assumption is permissible since a computer check reported in [5] establishes the asserted value when $k = 11$.

If $k \equiv 3 \pmod 9$ we color $[1, \frac{k}{3}]$ red, $[\frac{k+3}{3}, \frac{k^2+5k+3}{9}]$ blue, and $[\frac{k^2+5k+12}{9}, M - 1]$ red, where M is the asserted value depending on the congruence class of $k \pmod{27}$. If $k \equiv 4 \pmod 9$ we color $[1, \frac{k-1}{3}]$ red, $[\frac{k+2}{3}, \frac{k^2+4k-5}{9}]$ blue, and $[\frac{k^2+4k+4}{9}, M - 1]$ red. The arguments are much the same as for $k \equiv 2 \pmod 9$.

2. Upper Bounds

To show that our asserted values M for $RR(x + y + kz = 3w)$ are upper bounds for the true values we must show for each M that every 2-coloring of the interval $[1, M]$ admits a monochromatic solution to $x+y+kz = 3w$. In each case we will assume that a 2-coloring of $[1, M]$ that does not admit a monochromatic solution has been given, letting R and B be the associated sets of red and blue numbers, and then derive a contradiction. We will also assume that $1 \in R$, which is no loss of generality because interchanging the sets of red and blue numbers does not affect the existence of a monochromatic solution. Finally, since for $k \le 23$ and not equal to 2, 3, or 8 our values of $RR(x + y + kz = 3w)$ agree with those established by computer in [5], we assume that $k > 23$ for the remainder of this section.

The following result will be of use in every case. Quadruples always represent solutions to $x + y + kz = 3w$.

Lemma. If M is one of our asserted values and a 2-coloring of $[1, M]$ admits no monochromatic solution to $x + y + kz = 3w$ and $1 \in R$ with respect to that 2-coloring, then $1, 2, ..., j + 4 \in R$ and $k, 2k, ..., jk \in B$ whenever j is a positive integer such that $jk \le M$.

Proof. Note that, since $k > 23$, $jk \le M$ implies $j + 4 < k$ for each of our asserted values M so that the sets $\{1, 2, ..., j + 4\}$ and $\{k, 2k, ..., jk\}$ are disjoint.

Suppose a 2-coloring satisfying the hypotheses of the Lemma has been given. First we show that $2 \in R$. If $2 \in B$ then the solution $(2, k + 1, 2, k + 1)$ would show that $k+1 \in R$, for otherwise it would be an all blue solution. Similarly, $(k+1, k+2, 1, k+1)$ would show that $k+2 \in B$ and $(3,3,3,k+2)$ would show that $3 \in R$ and $(2k, 2k, 2, 2k)$ would show that $2k \in R$, so we would get $(2k, 3, 1, k+1)$ as an all red solution, which is a contradiction. Thus $2 \in R$.

Now suppose j is a positive integer such that $jk \leq M$. If $2 \leq i \leq j$, $i \in R$, and $(i-1)k \in B$ then $i, ik \leq M$ and (ik, ik, i, ik) and $(ik, (i-1)k, i+1, ik)$ show that $ik \in B$ and $i+1 \in R$. Since $2 \in R$ and $(k, k, 1, k)$ shows that $k \in B$, it follows that $1, 2, \ldots, j+1 \in R$ and $k, 2k, \ldots, jk \in B$. To conclude the proof it suffices to deal with the case $j > 2$, for then since $3k \leq M$ the result for $j = 3$ yields the result for $j \leq 2$. But if $j > 2$ the solutions $((j-1)k, (j-1)k, j+2, jk)$, $((j-1)k, (j-2)k, j+3, jk)$, and $((j-2)k, (j-2)k, j+4, jk)$ show that we also have $j+2, j+3, j+4 \in R$. □

In most of the following cases, the full strength of the Lemma will not be needed. In fact it is only in the the case $k \equiv 21 \pmod{27}$ that we will use all of the red numbers that the Lemma provides.

$k \equiv 0, 1,$ or $8 \pmod 9$:

If $k \equiv 0 \pmod 9$ then $M = \frac{k^2+5k+9}{9}$. We see from the Lemma that $1, 2, \ldots, \frac{k}{9} + 4 \in R$ and that $k, 2k, \ldots, \frac{k^2}{9} \in B$. Thus, since $k > 23$, we have $1, 2, 3 \in R$ and $\frac{k^2-4k}{9}, \frac{k^2-9k}{9} \in B$. So $(1, 2, 1, \frac{k}{3}+1)$ and $(3, 3, 1, \frac{k}{3}+2)$ show that $\frac{k}{3}+1, \frac{k}{3}+2 \in B$ and $(\frac{k}{3}+1, \frac{k}{3}+2, \frac{k}{3}+1, \frac{k^2+5k+9}{9})$ gives $\frac{k^2+5k+9}{9} \in R$.

Next we show $\frac{k}{3} \in R$ and $\frac{2k}{3} \in B$. Were $\frac{k}{3} \in B$ then $(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}+1, \frac{k^2+5k}{9})$ and $(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}, \frac{k^2+2k}{9})$ would show that $\frac{k^2+5k}{9}, \frac{k^2+2k}{9} \in R$ and $(\frac{k^2+4k}{9}, \frac{k^2+2k}{9}, \frac{k}{9}+1, \frac{k^2+5k}{9})$ would show that $\frac{k^2+4k}{9} \in B$ and so we would have the all blue solution $(k, \frac{k}{3}, \frac{k}{3}, \frac{k^2+4k}{9})$. Thus $\frac{k}{3} \in R$. That $\frac{2k}{3} \in B$ then follows from $(\frac{k}{3}, \frac{2k}{3}, 1, \frac{2k}{3})$.

Now we show that $\frac{k}{3}-2 \in R$ and $\frac{k^2-4k}{9} \in B$. From $(k, k, \frac{k}{3}-2, \frac{k^2}{9})$ we see that $\frac{k}{3}-2 \in R$ and $\frac{k^2-4k}{9} \in B$ because of $(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}-2, \frac{k^2-4k}{9})$.

Finally, we show that $(\frac{k^2-4k}{9}, \frac{4k}{9}, \frac{2k}{9}+1, \frac{k^2+3k}{9})$ is an all blue solution: $(2, \frac{k}{3}-2, 1, \frac{4k}{9})$ shows that $\frac{4k}{9} \in B$, $(\frac{k^2+5k+9}{9}, \frac{k}{9}+2, \frac{2k}{9}+1, \frac{k^2+5k+9}{9})$ shows that $\frac{2k}{9}+1 \in B$, and $(\frac{k^2}{9}, \frac{2k}{3}, \frac{2k}{9}+1, \frac{k^2+5k}{9})$ and $(\frac{k^2+3k}{9}, \frac{k^2+3k}{9}, \frac{k}{9}+1, \frac{k^2+5k}{9})$ show that $\frac{k^2+5k}{9} \in R$ and $\frac{k^2+3k}{9} \in B$. Thus we have a contradiction.

If $k \equiv 1 \pmod 9$ then $M = \frac{k^2+4k+4}{9}$. We see from the Lemma that $1, 2, \ldots, \frac{k-1}{9}+4 \in R$ and that $k, 2k, \ldots, \frac{k^2-k}{9} \in B$. So $(1, 1, 1, \frac{k+2}{3})$ and $(2, 2, 2, \frac{2k+4}{3})$ show that $\frac{k+2}{3}, \frac{2k+4}{3} \in B$ and $(\frac{k+2}{3}, \frac{k+2}{3}, \frac{k+2}{3}, \frac{k^2+4k+4}{9})$ shows that $\frac{k^2+4k+4}{9} \in R$.

Next we show that $\frac{k-1}{3}, \frac{k-7}{3}, \frac{k-16}{3} \in R$. Now $\frac{k^2+2k+6}{9} \in B$ because of $(\frac{k^2+2k+6}{9}, \frac{k^2+2k+6}{9}, \frac{k+8}{9}, \frac{k^2+4k+4}{9})$, so $(\frac{2k+4}{3}, \frac{k+2}{3}, \frac{k-1}{3}, \frac{k^2+2k+6}{9})$ shows that

$\frac{k-1}{3} \in R$. On the other hand $(k, k, \frac{k-7}{3}, \frac{k^2-k}{9})$ and $(2k, 3k, \frac{k-16}{3}, \frac{k^2-k}{9})$ show that $\frac{k-7}{3}, \frac{k-16}{3} \in R$ (recall $k > 23$).

Finally, $(\frac{k^2+4k+4}{9}, \frac{k+8}{9}, \frac{2k+7}{9}, \frac{k^2+4k+4}{9})$, $(\frac{k-1}{3}, \frac{k-1}{3}, \frac{k-1}{3}, \frac{k^2+k-2}{9})$ and $(\frac{k-16}{3}, 1, \frac{k-7}{3}, \frac{k^2-6k-13}{9})$ show that $(\frac{k^2-6k-13}{9}, \frac{2k+7}{9}, \frac{2k+7}{9}, \frac{k^2+k-2}{9})$ is an all blue solution.

If $k \equiv 8 \pmod 9$ then $M = \frac{k^2+6k+14}{9}$. We see from the Lemma that $1, 2, ..., \frac{k+1}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2+k}{9} \in B$. Thus, since $k > 23$, $1, ..., 5 \in R$ and $(2, 2, 1, \frac{k+4}{3})$ and $(5, 5, 1, \frac{k+10}{3})$ show that $\frac{k+4}{3}, \frac{k+10}{3} \in B$. Then $(\frac{k+10}{3}, \frac{k+4}{3}, \frac{k+4}{3}, \frac{k^2+6k+14}{9})$ and $(\frac{k^2+6k+14}{9}, \frac{k+28}{9}, \frac{2k+11}{9}, \frac{k^2+6k+14}{9})$ give $\frac{k^2+6k+14}{9} \in R$ and $\frac{2k+11}{9} \in B$ making $(k, \frac{k^2-17k}{9}, \frac{2k+11}{9}, \frac{k^2+k}{9})$ an all blue solution.

$k \equiv 5, 6,$ or $7 \pmod 9$:

If $k \equiv 5 \pmod 9$ then $M = \frac{k^2+7k+12}{9}$. We see from the Lemma that $1, 2, ..., \frac{k+4}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2+4k}{9} \in B$. Therefore, since $k > 23$, $(2, 2, 1, \frac{k+4}{3})$ and $(4, 4, 2, \frac{2k+8}{3})$ show that $\frac{k+4}{3}, \frac{2k+8}{3} \in B$. Then the solutions $(\frac{k+4}{3}, \frac{k+4}{3}, \frac{k+4}{3}, \frac{k^2+6k+8}{9})$ and $(\frac{k+4}{3}, \frac{2k+8}{3}, \frac{k+4}{3}, \frac{k^2+7k+12}{9})$ show that $\frac{k^2+6k+8}{9}$, $\frac{k^2+7k+12}{9} \in R$. Thus $(\frac{k^2+7k+12}{9}, \frac{k^2+7k+12}{9}, \frac{k+4}{9}, \frac{k^2+6k+8}{9})$ is an all red solution.

If $k \equiv 6 \pmod 9$ then $M = \frac{k^2+6k+9}{9}$. We see from the Lemma that $1, 2, ...,$ $\frac{k+3}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2+3k}{9} \in B$. Thus, since $k > 23$, $(1, 2, 1, \frac{k+3}{3})$, $(3, 3, 1, \frac{k+6}{3})$, and $(1, 2, 2, \frac{2k+3}{3})$ show that $\frac{k+3}{3}, \frac{k+6}{3}, \frac{2k+3}{3} \in B$. Then we have $(\frac{k+3}{3}, \frac{k+3}{3}, \frac{k+3}{3}, \frac{k^2+5k+6}{9})$ and $(\frac{k+6}{3}, \frac{2k+3}{3}, \frac{k+3}{3}, \frac{k^2+6k+9}{9})$ showing that $\frac{k^2+5k+6}{9}$ and $\frac{k^2+6k+9}{9}$ are red. Hence $(\frac{k^2+6k+9}{9}, \frac{k^2+6k+9}{9}, \frac{k+3}{9}, \frac{k^2+5k+6}{9})$ is an all red solution.

If $k \equiv 7 \pmod 9$ then $M = \frac{k^2+5k+6}{9}$. We see from the Lemma that $1, 2, ...,$ $\frac{k+2}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2+2k}{9} \in B$. Therefore $(1, 1, 1, \frac{k+2}{3})$ and $(2, 2, 2, \frac{2k+4}{3})$ show that $\frac{k+2}{3}, \frac{2k+4}{3} \in B$. Then $(\frac{k+2}{3}, \frac{k+2}{3}, \frac{k+2}{3}, \frac{k^2+4k+4}{9})$ and $(\frac{2k+4}{3}, \frac{k+2}{3}, \frac{k+2}{3}, \frac{k^2+5k+6}{9})$ show that $\frac{k^2+4k+4}{9}, \frac{k^2+5k+6}{9} \in R$. So we see that $(\frac{k^2+5k+6}{9}, \frac{k^2+5k+6}{9}, \frac{k+2}{9}, \frac{k^2+4k+4}{9})$ is an all red solution.

$k \equiv 2, 3,$ or $4 \pmod 9$:

If $k \equiv 2 \pmod{27}$ then $M = \frac{3k^2+19k+31}{27}$. We see from the Lemma that $1, 2, ..., \frac{k-2}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2-2k}{9} \in B$. Therefore, since $k > 23$, $(2, 2, 1, \frac{k+4}{3})$, $(5, 5, 1, \frac{k+10}{3})$, and $(3, 4, 1, \frac{k+7}{3})$ show that $\frac{k+4}{3}, \frac{k+10}{3}, \frac{k+7}{3} \in B$. Then $(\frac{k+10}{3}, \frac{k+10}{3}, \frac{k+4}{3}, \frac{k^2+6k+20}{9})$ and $(\frac{k+7}{3}, \frac{k+4}{3}, \frac{k+4}{3}, \frac{k^2+6k+11}{9})$ show that $\frac{k^2+6k+11}{9}, \frac{k^2+6k+20}{9} \in R$. The solutions $(\frac{k+25}{9}, 4, 1, \frac{10k+61}{27})$ and $(\frac{k+16}{9}, \frac{k+16}{9}, 1,$

$\frac{11k+32}{27}$) show that $\frac{10k+61}{27}, \frac{11k+32}{27} \in B$ so $(\frac{10k+61}{27}, \frac{11k+32}{27}, \frac{k+4}{3}, \frac{3k^2+19k+31}{27})$ shows that $\frac{3k^2+19k+31}{27} \in R$. Hence we have the all red solution $(\frac{k^2+6k+11}{9}, \frac{k^2+6k+20}{9}, \frac{k+7}{9}, \frac{3k^2+19k+31}{27})$.

If $k \equiv 11 \pmod{27}$ then $M = \frac{3k^2+19k+22}{27}$. As before $1, ..., \frac{k+34}{9} \in R$, $\frac{k+4}{3} \in B$, and $\frac{k^2+6k+11}{9} \in R$ and so $(\frac{k+7}{9}, 5, 1, \frac{10k+52}{27})$, $(\frac{k+7}{9}, \frac{k+7}{9}, 1, \frac{11k+14}{27})$, and $(\frac{10k+52}{27}, \frac{11k+14}{27}, \frac{k+4}{3}, \frac{3k^2+19k+22}{27})$ lead to the all red solution $(\frac{k^2+6k+11}{9}, \frac{k^2+6k+11}{9}, \frac{k+7}{9}, \frac{3k^2+19k+22}{27})$.

If $k \equiv 20 \pmod{27}$ then $M = \frac{3k^2+19k+40}{27}$. As before $1, ..., \frac{k+34}{9} \in R$, $\frac{k+4}{3} \in B$, and $\frac{k^2+6k+20}{9} \in R$ and so $(\frac{k+25}{9}, 5, 1, \frac{10k+70}{27})$, $(\frac{k+25}{9}, \frac{k+25}{9}, 1, \frac{11k+50}{27})$, and $(\frac{10k+70}{27}, \frac{11k+50}{27}, \frac{k+4}{3}, \frac{3k^2+19k+40}{27})$ provide us with the all red solution $(\frac{k^2+6k+20}{9}, \frac{k^2+6k+20}{9}, \frac{k+7}{9}, \frac{3k^2+19k+40}{27})$.

If $k \equiv 3 \pmod{27}$ then $M = \frac{3k^2+16k+33}{27}$. We see from the Lemma that $1, 2, ..., \frac{k-3}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2-3k}{9} \in B$. Therefore, since $k > 23$, $(1, 2, 1, \frac{k+3}{3})$, $(4, 5, 1, \frac{k+9}{3})$, and $(6, 6, 1, \frac{k+12}{3})$ show that $\frac{k+3}{3}, \frac{k+9}{3}, \frac{k+12}{3} \in B$ and $(\frac{k+9}{3}, \frac{k+12}{3}, \frac{k+3}{3}, \frac{k^2+5k+21}{9})$ shows that $\frac{k^2+5k+21}{9} \in R$. Then $(\frac{k+3}{3}, \frac{k+9}{3}, \frac{k+3}{3}, \frac{k^2+5k+12}{9})$ shows that $\frac{k^2+5k+12}{9} \in R$ and $(\frac{k+24}{9}, 6, 1, \frac{10k+78}{27})$, $(\frac{k+24}{9}, \frac{k-3}{9}, 1, \frac{11k+21}{27})$, and $(\frac{10k+78}{27}, \frac{11k+21}{27}, \frac{k+3}{3}, \frac{3k^2+16k+33}{27})$ provide the all red solution $(\frac{k^2+5k+12}{9}, \frac{k^2+5k+21}{9}, \frac{k+6}{9}, \frac{3k^2+16k+33}{27})$.

If $k \equiv 12 \pmod{27}$ then $M = \frac{3k^2+16k+24}{27}$. As before $1, ..., \frac{k+33}{9} \in R$, $\frac{k+3}{3} \in B$, and $\frac{k^2+5k+12}{9} \in R$ so $(\frac{k+15}{9}, 3, 1, \frac{10k+42}{27})$, $(\frac{k+15}{9}, \frac{k+15}{9}, 1, \frac{11k+30}{27})$, and $(\frac{10k+42}{27}, \frac{11k+30}{27}, \frac{k+3}{3}, \frac{3k^2+16k+24}{27})$ show that $(\frac{k^2+5k+12}{9}, \frac{k^2+5k+12}{9}, \frac{k+6}{9}, \frac{3k^2+16k+24}{27})$ is an all red solution.

If $k \equiv 21 \pmod{27}$ then $M = \frac{3k^2+16k+42}{27}$. As before $1, 2, ..., \frac{k+33}{9} \in R$, $\frac{k+3}{3} \in B$, and $\frac{k^2+5k+21}{9} \in R$ so $(\frac{k+6}{9}, 6, 1, \frac{10k+60}{27})$, $(\frac{k+33}{9}, \frac{k+33}{9}, 1, \frac{11k+66}{27})$, and $(\frac{10k+60}{27}, \frac{11k+66}{27}, \frac{k+3}{3}, \frac{3k^2+16k+42}{27})$ provide the all red solution $(\frac{k^2+5k+21}{9}, \frac{k^2+5k+21}{9}, \frac{k+6}{9}, \frac{3k^2+16k+42}{27})$.

If $k \equiv 4 \pmod{27}$ then $M = \frac{3k^2+13k+8}{27}$. We see from the Lemma that $1, 2, ..., \frac{k-4}{9} + 4 \in R$ and that $k, 2k, ..., \frac{k^2-4k}{9} \in B$. Therefore $(1, 1, 1, \frac{k+2}{3})$ shows that $\frac{k+2}{3} \in B$, $(\frac{k+2}{3}, \frac{k+2}{3}, \frac{k+2}{3}, \frac{k^2+4k+4}{9})$ gives $\frac{k^2+4k+4}{9} \in R$ and hence $(\frac{k+14}{9}, 2, 1, \frac{10k+32}{27})$, $(\frac{k-4}{9}, \frac{k-4}{9}, 1, \frac{11k-8}{27})$, and $(\frac{10k+32}{27}, \frac{11k-8}{27}, \frac{k+2}{3}, \frac{3k^2+13k+8}{27})$ lead to the all red solution $(\frac{k^2+4k+4}{9}, \frac{k^2+4k+4}{9}, \frac{k+5}{9}, \frac{3k^2+13k+8}{27})$.

If $k \equiv 13 \pmod{27}$ then $M = \frac{3k^2+13k+26}{27}$. As before $1, 2, ..., \frac{k+32}{9} \in R$ and $\frac{k+2}{3} \in B$. Thus $(2, 3, 1, \frac{k+5}{3})$ and $(4, 4, 1, \frac{k+8}{3})$ give $\frac{k+5}{3}, \frac{k+8}{3} \in B$ and $(\frac{k+8}{3}, \frac{k+5}{3}, \frac{k+2}{3}, \frac{k^2+4k+13}{9})$ gives $\frac{k^2+4k+13}{9} \in R$ and hence $(\frac{k+14}{9}, 5, 1, \frac{10k+59}{27})$, $(\frac{k+5}{9}, \frac{k+14}{9}, 1, \frac{11k+19}{27})$, and $(\frac{10k+59}{27}, \frac{11k+19}{27}, \frac{k+2}{3}, \frac{3k^2+13k+26}{27})$ yield the all red solution $(\frac{k^2+4k+13}{9}, \frac{k^2+4k+13}{9}, \frac{k+5}{9}, \frac{3k^2+13k+26}{27})$.

If $k \equiv 22 \pmod{27}$ then $M = \frac{3k^2+13k+17}{27}$. As before $1, \ldots, \frac{k+32}{9} \in R$, $\frac{k+2}{3} \in B$, and $\frac{k^2+4k+4}{9}, \frac{k^2+4k+13}{9} \in R$. Therefore $(\frac{k+5}{9}, 3, 1, \frac{10k+32}{27})$, $(\frac{k+5}{9}, \frac{k+14}{9}, 1, \frac{11k+19}{27})$, and $(\frac{10k+32}{27}, \frac{11k+19}{27}, \frac{k+2}{3}, \frac{3k^2+13k+17}{27})$ provide us with the all red solution $(\frac{k^2+4k+4}{9}, \frac{k^2+4k+13}{9}, \frac{k+5}{9}, \frac{3k^2+13k+17}{27})$.

References

[1] S. Guo and Z-W. Sun, Determination of the 2-color Rado number for $a_1x_1 + \cdots + a_mx_m = x_0$, to appear in *JCTA*, preprint available at arXiv://math.CO/0601409.

[2] B. Hopkins and D. Schaal, On Rado numbers for $\Sigma_{i=1}^{m-1} a_ix_i = x_m$, *Adv. Applied Math* **35** (2005) 433-441.

[3] S. Jones and D. Schaal, Some 2-color Rado numbers, Congr. Numer. 152 (2001), pp. 197-199.

[4] R. Rado, Studien zur Kombinatorik, *Mathematische Zeitschrift* **36** (1933), 424-480.

[5] A. Robertson and K. Myers, Some Two Color, Four Variable Rado Numbers, to appear in *Adv. Applied Math.*