Fan-type theorem for a long path passing through a specified vertex

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Abstract

Let G be a 2-connected graph with maximum degree $\Delta(G) \geq d$, x and z be distinct vertices of G, and W be a subset of $V(G)\backslash\{x, z\}$ such that $|W| \leq d-1$. Hirohata proved that if $\max\{d_G(u), d_G(v)\} \geq d$ for every pair of vertices $u, v \in V(G)\backslash(\{x, z\} \cup W)$ such that $d_G(u, v) = 2$, then x and z are joined by a path of length at least $d - |W|$. In this paper, we show that if G satisfies the conditions of Hirohata's theorem, then for any given vertex y such that $d_G(y) \geq d$, x and z are joined by a path of length at least $d-|W|$ which contains y.

Keywords. long path, Fan-type condition

1 Introduction

We use [1] for terminology and notation not defined here and consider only finite undirected graphs without loops or multiple edges. A path whose endvertices are x and z is called an *(x,z)-path.* If an (x,z)-path contains a vertex y, we call it an *(x,y,z)-path.* Let G be a graph and $u,v \in V(G)$. We shall use $d_G(v)$ and $d_G(u,v)$ for the degree of v in G and the distance of u and v in G, respectively. Moreover, $G+uv$ denotes a graph such that $V(G+uv) = V(G)$ and $E(G+uv) = E(G) \cup \{uv\}$. For $V' \subseteq V(G)$, $G[V']$ denotes the graph induced by V'. For simplicity, we sometimes use "$v \in G$" instead of "$v \in V(G)$".

Let G be a graph, x and z be distinct vertices of G, W be a subset of $V(G)\backslash\{x,z\}$, and d be an integer. If

- G is a 2-connected graph with maximum degree $\Delta(G) \geq d$,
- $|W| \leq d-1$, and
- $\max\{d_G(u), d_G(v)\} \geq d$ for every pair of vertices $u,v \in V(G)\backslash(\{x,z\} \cup W)$ such that $d_G(u,v) = 2$,

then we call that G satisfies $DC(x,z,W;d)$.

Saito [4] showed the existence of long paths under the Fan-type condition as follows.

Theorem 1 (Saito [4]) *Let G be a 2-connected graph and x,z be distinct vertices of G. If $\max\{d_G(u), d_G(v)\} \geq d$ for every pair of vertices $u,v \in V(G)\backslash\{x,z\}$ such that $d_G(u,v) = 2$, then there exists an (x,z)-path of length at least $\min\{d, |G|-1\}$.*

Theorem 1 is equivalent to the following corollary.

Corollary 2 *Let G be a graph satisfying $DC(x,z,\emptyset;d)$. Then there exists an (x,z)-path of length at least d.*

In [3], Hirohata gave a generalization of Corollary 2.

Theorem 3 (Hirohata [3]) *Let G be a graph satisfying $DC(x, z, W; d)$. Then there exists an (x, z)-path of length at least $d - |W|$.*

The above theorem allows a graph to have a vertex set W, which is ignored by the degree condition, and the length of the obtained path becomes shorter according to the size of W.

On the other hand, there are some results about long paths containing some specified vertices. The following is one of such results.

Theorem 4 (Egawa, Glas and Locke [2]) *Let $k \geq 3$ and let G be a k-connected graph with minimum degree d and with at least $2d - 1$ vertices. Let x and z be distinct vertices of G and Y be a set of $k - 1$ vertices of G. Then G has an (x, z)-path of length at least $2d - 2$ containing every vertex of Y.*

In this paper, we will put these two types of theorems together, the one showing the existence of a long path using Hirohata-type condition, and the other showing a long path containing some specified vertices. The following theorem is our main result.

Theorem 5 *Let G be a graph which satisfies $DC(x, z, W; d)$. Then for any vertex y such that $d_G(y) \geq d$, there exists an (x, y, z)-path of length at least $d - |W|$.*

The conditions of Theorem 5 are the same as in Theorem 3, and hence Theorem 5 generalizes Theorem 3.

The following example shows that we need a certain degree condition on y. Let H be a complete graph of order $d-1$ such that $v_1, v_2 \in V(H)$, K be a complete graph of order r such that $y \in V(K)$, and G be a graph such that $V(G) = V(H) \cup V(K) \cup \{x, z, u_1, u_2\}$ and $E(G) = E(H) \cup E(K) \cup \{u_i v \mid i = 1, 2,\ v \in V(K)\} \cup \{u_1 x, u_2 z, xz, u_1 v_1,\ u_1 v_2, u_2 v_1, u_2 v_2\}$ (see Figure 1). Now suppose that $W = \{u_1, u_2\}$, $d \geq 7$ and $r \leq d - 6$. Then, G satisfies the conditions of Theorem 3, but the length of a longest (x, y, z)-path in

G is $r+3 \le d-3 < d-|W|$. Since $d_G(y) = r+1 \le d-5$, we need the condition $d_G(y) \ge d-4$.

Moreover, Let H' be a complete graph of order $d-1$, K' be a complete graph of order $d-2$ such that $y \in V(K')$, and G' be a graph such that $V(G') = V(H') \cup V(K') \cup \{x, z\}$ and $E(G') = E(H') \cup E(K') \cup \{uv \mid u = x \text{ or } z,\ v \in V(H') \cup V(K')\}$ (see Figure 2). Now suppose that $W = \emptyset$, then G' satisfies the conditions of Theorem 3, but the length of a longest (x, y, z)-path in G' is $d-1 < d-|W|$. This example shows that the assumption $d_G(y) \ge d$ is sharp in case of $W = \emptyset$. However, it is not known whether this assumption is sharp or not in case of $W \ne \emptyset$.

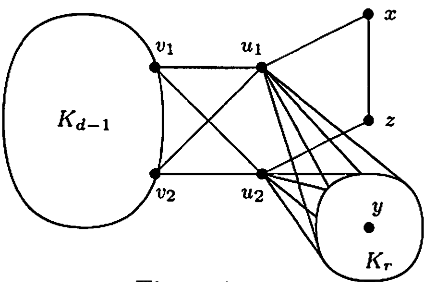

Figure 1:

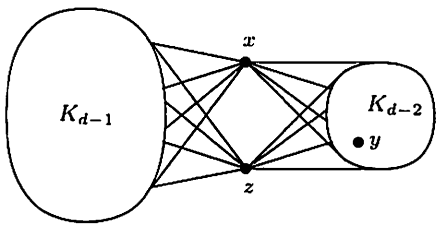

Figure 2:

2 Proof of Theorem 5

If $y \in \{x, z\}$, Theorem 3 implies the assertion. Hence we can assume that $y \notin \{x, z\}$. Let G be a counterexample of the least order to the assertion. By Menger's theorem, we obtain that there exists an (x, y, z)-path of length at least 2. So we can assume that $d-|W| \ge 3$, which implies that $|G| \ge 4$.

Let $G' = G - \{x\}$. If G' is 2-connected, choose a vertex $x' \in N_G(x) \backslash \{z\}$ and let $W' = W \backslash \{x'\}$. Then $d_{G'}(v) \ge d_G(v) - 1$ for all $v \in G'$, and $y \in G'$ implies that $\Delta(G') \ge d-1$. Moreover, since $d - |W| \ge 3$, we have $|W'| \le |W| < (d-1) - 1$. Hence G' satisfies $DC(x', z, W'; d-1)$ and $|G'| < |G|$, therefore there is an (x', y, z)-path P' of length at least $d - |W'| - 1 \ge d - |W| - 1$ in G'. Now $P = xx'P'$ is an (x, y, z)-path of length at least $d - |W|$ in G, a contradiction. Thus we can assume that G' is separable.

Let C be the set of the cutvertices of G'. A maximal connected subgraph without a cutvertex is called a block, and we call a block which contains exactly one cutvertex an endblock. For a block B of G', we will denote $V(B)\backslash C$ by I_B. Moreover, if B is an endblock of G', the only cutvertex in B is denoted by c_B.

Claim 1 *There is a block which contains both y and z. Moreover, $z \notin C$.*

Proof. Assume that there is no block which contains both y and z. Then there exists $c_1 \in C\backslash\{y, z\}$ such that y and z are in the different components of $G' - \{c_1\}$. Let G_y be the component of $G' - \{c_1\}$ which contains y, $G^* = G[V(G_y) \cup \{x, c_1\}] + xc_1$, and $W^* = V(G_y) \cap W$. Then $d_{G^*}(v) = d_G(v)$ for all $v \in G^*\backslash\{x, c_1\}$, and $y \in G^*\backslash\{c_1\}$ implies that $\Delta(G^*) \geq d$. Moreover, we have $|W^*| \leq |W| < d - 1$, so G^* is a 2-connected graph satisfying $DC(x, c_1, W^*; d)$ and $|G^*| < |G|$. Hence we have an (x, y, c_1)-path P_1 of length at least $d - |W^*| \geq d - |W|$ in G^*. It follows from $d - |W| \geq 3$ that P_1 is not the path xc_1, therefore P_1 is a path in G. And we have a (c_1, z)-path P_2 in $G' - V(H_y)$, so $P_1 \cup P_2$ is an (x, y, z)-path of length at least $d - |W|$, a contradiction. Regarding c_1 as z, the second assertion follows by the same argument. □

Let $\mathcal{B}$ be the set of endblocks B such that $\{y, z\} \cap V(B) = \emptyset$ and there exists a (z, y, c_B)-path in $G' - I_B$.

Claim 2 $|\mathcal{B}| \leq 1$.

Proof. Suppose that $B_1, B_2 \in \mathcal{B}$. In case of $V(B_1) \cap W \neq \emptyset$, let $G^* = (G - I_{B_1}) + xc_{B_1}$ and $W^* = (V(G^*) \cap W) \cup \{c_{B_1}\}$. Then, $d_{G^*}(v) = d_G(v)$ for all $v \in G^*\backslash\{x, c_{B_1}\}$, and since $y \in G^*\backslash\{x, c_{B_1}\}$, $\Delta(G^*) \geq d$. Moreover, we have $|W^*| = |W - (V(B_1) \cap W)| + 1 \leq |W| < d - 1$, so G^* is a 2-connected graph satisfying $DC(x, z, W^*; d)$ and $|G^*| < |G|$. Hence we have an (x, y, z)-path P^* of length at least $d - |W^*| \geq d - |W|$ in G^*. If

$xc_{B_1} \notin E(P^*)$, then P^* is a path in G, a contradiction. If $xc_{B_1} \in E(P^*)$, let $P_1 = P^* - xc_{B_1}$ and P_2 be an (x, c_{B_1})-path in $G[V(B_1) \cup \{x\}]$. Then $P_1 \cup P_2$ is an (x, y, z)-path of length at least $d - |W|$, a contradiction. Hence we have $V(B_1) \cap W = \emptyset$, and by the same argument, we have $V(B_2) \cap W = \emptyset$.

Now take two vertices $x_1 \in I_{B_1} \cap N_G(x)$ and $x_2 \in I_{B_2} \cap N_G(x)$. Then $\{x_1, x_2\} \cap W = \emptyset$ and $d(x_1, x_2) = 2$, so we have $\max\{d_G(x_1), d_G(x_2)\} \geq d$. Without loss of generality, we may assume that $d_G(x_1) \geq d$. Let $B_1{}^* = G[V(B_1) \cup \{x\}] + xc_{B_1}$. Then, $d_{B_1{}^*}(v) = d_G(v)$ for all $v \in B_1{}^* \backslash \{x, c_{B_1}\}$, and since $x_1 \in B_1{}^* \backslash \{C_{B_1}\}$, $\Delta(B_1{}^*) \geq d$. So $B_1{}^*$ satisfies $DC(x, c_{B_1}, \emptyset; d)$ and $|B_1{}^*| < |G|$, and hence we have an (x, c_{B_1})-path P_3 of length at least d in $B_1{}^*$. It follows from $d - |W| \geq 3$ that P_3 is not the path xc_{B_1}, therefore P_3 is a path in G. By the definition of $\mathcal{B}$, there exists a (c_{B_1}, y, z)-path P_4 in $G' - I_B$, so $P_3 \cup P_4$ is an (x, y, z)-path of length at least $d - |W|$, a contradiction. □

Claim 3 $y \in C$.

Proof. Suppose that $y \notin C$, and let B_y be the block containing y. By Claim 1, we have $z \in I_{B_y}$. If B_y is not an endblock of G', then we have $|\mathcal{B}| \geq 2$. This contradicts Claim 2. Thus B_y is an endblock of G'. Let $W_y = I_{B_y} \cap W$. Then $d_{B_y}(v) \geq d_G(v) - 1$ for all $v \in I_{B_y}$, and since $y \in I_{B_y}$, $\Delta(B_y) \geq d - 1$. Moreover, since $|W_y| \leq |W|$ and $d - |W| \geq 3$, we have $|W_y| \leq |W| < (d-1) - 1$. So B_y satisfies $DC(c_{B_y}, z, W_y; d-1)$ (note that $\{y, z\} \subseteq I_{B_y}$ implies that B_y is 2-connected) and $|B_y| < |G|$, and hence there is a (c_{B_y}, y, z)-path P_1 of length at least $d - |W_y| - 1 \geq d - |W| - 1$ in B_y. Now we can take an (x, c_{B_y})-path P_2 in $G - I_{B_y}$, so $P_2 \cup P_1$ is an (x, y, z)-path of length at least $d - |W|$, a contradiction. □

Let $\mathcal{B}_y$ be the set of the blocks of G' which contain y, B_z be the block of G' which contains both y and z, $C' = (\bigcup_{B \in \mathcal{B}_y} V(B) \cap C) \backslash \{y\}$, and $S = N_G(y) \backslash (\{x, z\} \cup C' \cup W)$. Then it follows from Claim 2 that $|C'| \leq 1$, and hence $|S| \geq d - |W| - 3$. Moreover, since Claim 1 implies that $z \in I_{B_z}$, B_z is 2-connected.

Claim 4 *There exists $s \in S$ such that $d_G(s) \geq d$.*

Proof. Suppose that $d_G(s) < d$ for all $s \in S$. Then $d(u,v) \neq 2$ for any two distinct vertices $u, v \in S$. Thus, $G[S \cup \{y\}]$ is a complete graph, and so there is a block $B_S \in \mathcal{B}_y$ which contains S. If $B_S = B_z$, there exists a (y,z)-path P_1 with $S \subseteq V(P_1)$ of length at least $|S|+1$ in B_z, and we can take an (x,y)-path P_2 of length at least 2 in $G - I_{B_z}$. Then $P_1 \cup P_2$ is an (x,y,z)-path of length at least $|S|+3 \geq d - |W|$, a contradiction. Otherwise, there exists an (x,y)-path P_3 with $S \subseteq V(P_3)$ of length at least $|S|+1$ in $G - I_{B_z}$. Since B_z is 2-connected, we can take a (y,z)-path P_4 of length at least 2 in B_z. Then $P_3 \cup P_4$ is an (x,y,z)-path of length at least $|S|+3 \geq d - |W|$, a contradiction. □

Suppose that $s \in B_z$, and let $W_z = (W \cap I_{B_z}) \cup (C' \cap B_z)$. Then, $d_{B_z}(v) \geq d_G(v) - 1$ for all $v \in B_z \backslash (\{y\} \cup (C' \cap B_z))$, and $s \in B_z \backslash (\{y\} \cup (C' \cap B_z))$ implies that $\Delta(B_z) \geq d-1$. Moreover, since $|W_z| = |(W \cap I_{B_z}) \cup (C' \cap B_z)| \leq |W| + |C'| \leq |W|+1$ and $|W| \leq d-3$, we have $|W_z| \leq d-2 = (d-1)-1$. So B_z satisfies $DC(y,z,W_z;d-1)$ and $|B_z| < |G|$. Hence there exists a (y,z)-path P_1 of length at least $d-1-|W_z| \geq d-|W|-2$. Since we can take an (x,y)-path P_2 of length at least 2 in $G - I_{B_z}$, there exists an (x,y,z)-path $P_1 \cup P_2$ of length at least $d - |W|$, a contradiction. Hence we have $s \notin B_z$. Let G_s be the component of $G' - \{y\}$ which contains s, $G^* = G[V(G_s) \cup \{x,y\}] + xy$, and $W^* = V(G_s) \cap W$. Then, $d_{G^*}(v) = d_G(v)$ for all $v \in G^* \backslash \{x,y\}$, and $s \in G^* \backslash \{x,y\}$ implies $\Delta(G^*) \geq d$. Moreover, we have $|W^*| \leq |W| < d-1$, so G^* satisfies $DC(x,y,W^*;d)$ and $|G^*| < |G|$. Hence there exists an (x,y)-path P_3 of length at least $d - |W^*| \geq d - |W|$ in G^*. Since $d - |W| \geq 3$, P_3 is not the path xy, thus P_3 is a path in G. Now we can take a (y,z)-path P_4 in $G - V(G_s)$, and hence $P_3 \cup P_4$ is an (x,y,z)-path of length at least $d - |W|$, a contradiction. This completes the proof of Theorem 5. □

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