HARMONIC BLOCH AND BESOV SPACES ON THE UNIT BALL

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Abstract
We correct and improve results from a recent paper by G. Ren and U. Kähler, which characterizes the Bloch, the little Bloch and Besov space of harmonic functions on the unit ball $B \subset \mathbb{R}^n$.

1. INTRODUCTION

Throughout this paper $B(a,r) = \{x \in \mathbb{R}^n \,|\, |x-a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \mathbb{R}^n$, B the open unit ball in $\mathbb{R}^n$, $rB = B(0,r)$, $S = \partial B = \{x \in \mathbb{R}^n \,|\, |x| = 1\}$ is the boundary of B, $S_r = \{x \in \mathbb{R}^n \mid |x| = r\}$, dV the Lebesgue measure on $\mathbb{R}^n$, $dV_s(x) = (1-|x|^2)^s dV(x)$, $d\tau(x) = dV_{-n}(x)$, $d\sigma$ the surface measure on S, and $\mathcal{H}(B)$ the set of all harmonic functions on B (see, e.g., [2]-[7], [10]-[15], [18]-[20]).

Let $\mathbb{Z}_n^+$ be the set of all ordered n-tuples of nonnegative integers, and for each $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_n^+$, let $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $\gamma! = \gamma_1! \cdots \gamma_n!$. For a harmonic function u we denote $\partial^\gamma u = \frac{\partial^{|\gamma|} u}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}$.

The harmonic α-Bloch space $\mathcal{BH}^\alpha(B)$, $\alpha > 0$ consists of $u \in \mathcal{H}(B)$ such that

$$\sup_{|x|<1} (1-|x|^2)^\alpha |\nabla u(x)| < \infty,$$

and the harmonic little α-Bloch space $\mathcal{BH}_0^\alpha(B)$ consists of all $u \in \mathcal{H}(B)$ such that $\lim_{|x| \to 1} (1-|x|^2)^\alpha |\nabla u(x)| = 0$.

The harmonic Besov space $\mathcal{B}_p$ (see [8] for analytic) consists of all $u \in \mathcal{H}(B)$ such that

$$\int_B (1-|x|^2)^p |\nabla u(x)|^p d\tau(x) < \infty.$$

In paper [11] the authors anticipated and formulated the following result:

Theorem A. *Assume that $u \in \mathcal{H}(B)$. Then the following statements are true.*

(a) $u \in \mathcal{BH}^1(B)$ if and only if

$$\sup_{x,y \in B, x \neq y} (1-|x|^2)^{1/2} (1-|y|^2)^{1/2} \frac{|u(x)-u(y)|}{|x-y|} < \infty.$$

(b) $u \in \mathcal{BH}_0^1(B)$ *if and only if*

$$\lim_{|x|\to 1, y\in B, x\neq y} (1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|u(x)-u(y)|}{|x-y|} = 0.$$

(c) If $p \in (2(n-1),\infty)$. *Then* $u \in \mathcal{B}_p$ *if and only if*

$$\int_B\int_B (1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\frac{|u(x)-u(y)|^p}{|x-y|^p}d\tau(x)d\tau(y). \qquad (1)$$

Our aim here is to improve and correct Theorem A. Namely, in [11] is used the technique related to hyper-harmonic functions, which is not quite suitable for harmonic functions. For example, in the proof of Theorem A *(a)* ([11, Theorem 3.1]), it is claimed that for $\delta \in (0,1)$ fixed, if u is harmonic, from [9, p. 504] it follows that there is a positive constant C such that

$$(1-|x|^2)|\nabla u(x)| \leq C\int_{E(x,\delta)} |u(y)|d\tau(y), \qquad (2)$$

where $E(x,\delta)$ is the image of the ball $B(0,\delta)$ mapped by

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{||x|a - x/|x||^2}, \quad a, x \in \mathbb{R}^n, \qquad (3)$$

the involutive Möbius transformation of B such that $\varphi(0) = \lim_{x\to 0}\varphi_a(x) = a$ (see [1] for more details on the topic). However, paper [9] is devoted to the space of holomorphic functions on $\mathbb{C}^n$, so that these Möbius transformations are not the same as those in the real n-dimensional space $\mathbb{R}^n$. Hence, the paper [11] confuses these two classes of Möbius transformations. In fact, inequality (2) can be obtained from a similar estimate for harmonic functions on Euclidean balls and by using Möbius transformations. To avoid such confusion, it is more convenient to use Euclidean balls only, as we do in our Theorem 1. Further, there is a gap in the proof of Lemma 4.2 in [11], so that the proof is not complete. Namely, the function $h(\rho) = (\rho^{(n-1)/p}M_p(\rho, |\hat{\nabla} f|)/(1-\rho)$ is replaced in a Hardy inequality and obtained the integral

$$\int_B |\nabla f(a)|^p dV_{\alpha+1}(x) \quad \text{instead of the requested} \quad \int_B |\nabla f(a)|^p dV_{p+\alpha}(x),$$

which is not appropriate for proving Theorem 4.1 in [11].

Another gap in [11] appears in the proof of Theorem 4.1. Namely, if we assume that Lemma 4.2 in [11] is true, then we must note that in the proof of the lemma is used the harmonicity of the function f which appear therein (see inequality (4.3) in [11]). On the other hand, in the proof of Theorem 4.1 the function f is replaced by $f \circ \varphi_x$ which need not be a harmonic function for the case of $n \geq 3$. Indeed, for the case of the conformal metric $ds = \rho|dV(x)|$, the Laplace-Beltrami operator is

$$\Delta_2 f(x) = \rho^{-n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\rho^{n-2} \frac{\partial f}{\partial x_i}(x) \right),$$

which for the weight $\rho = 2/(1-|x|^2)$ becomes

$$\Delta_2 f(x) = \frac{(1-|x|^2)^2}{4} \left(\Delta f(x) + \frac{2(n-2)}{1-|x|^2} \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) \right),$$

from which the claim easily follows.

Throughout the remainder of the paper, C will denote a constant not necessarily the same at each occurrence.

2. Auxiliary results

In this section we quote three auxiliary results which we use in the proofs of the main results.

Lemma 1. *Assume that $\beta \in (0,1)$. Then for every $x, y \geq 0$, the following inequality holds*

$$x + y \geq \left(\frac{1}{\beta}\right)^{\beta} \left(\frac{1}{1-\beta}\right)^{1-\beta} x^{\beta} y^{1-\beta}.$$

Proof. If, in Hölder's inequality

$$ab \leq a^p/p + b^{p'}/p',$$

where $a, b \geq 0$, $p, p' > 1$ and $1/p + 1/p' = 1$, we choose $p = 1/\beta$, $a = (x/\beta)^\beta$ and $y = (y/(1-\beta))^{1-\beta}$, we obtain the desired inequality. □

Lemma 2. *Assume that $u \in \mathcal{H}(B)$ and $p \in (0,\infty)$. Then there is a positive constant C independent of u such that*

$$\int_B |u(x)|^p dV_\alpha(x) \leq C \left(|u(0)|^p + \int_B |\nabla u(x)|^p (1-|x|)^p dV_\alpha(x) \right). \qquad (4)$$

For a proof of the result, see, for example, [7] and [18]. Closely related results on the mixed norm spaces and weighted Bergman spaces of analytic or harmonic functions of one or several variables can be found, for example, in [2, 3, 6, 9, 10, 12, 13, 16, 17, 19, 20, 21] (see, also, the references therein).

We say that a locally integrable function f on B possesses the $HL-$property, with a constant $c > 0$ if

$$f(a) \leq \frac{c}{r^n} \int_{B(a,r)} f(x) dV(x) \quad \text{whenever } \overline{B}(a,r) \subset B.$$

For example, every subharmonic function ([4]) possesses the $HL-$property with $c = 1/v_n$. In [5] Fefferman and Stein proved that $|u|^p$, $p > 0$, also possesses the $HL-$property whenever u is a harmonic function in B.

Using Fefferman-Stein's result it follows that the following result holds true:

Lemma 3. *Let $p \in (0, \infty)$ and γ be a multi-index. Suppose u is harmonic on a proper open subset G of $\mathbb{R}^n$. Then, we have*

$$|\partial^\gamma u(x)|^p \le \frac{C}{d^{n+p|\gamma|}(x, \partial G)} \int_{B(x,d(x,\partial G)/2)} |u(y)|^p dV(y) \qquad (x \in G),$$

where $d(x, \partial G)$ denotes the distance from x to the boundary ∂G. The constant C depends only on n, p and γ.

3. Main results

In this section we prove the main results in this paper. The first is an extension of [11, Theorem 3.1] for the case of harmonic α-Bloch functions.

Theorem 1. *Assume that $u \in \mathcal{H}(B)$ and $\alpha < \min\{1/\beta, 1/(1-\beta)\}$, for some $\beta \in (0,1)$. Then the following statements hold true.*

(a) $u \in \mathcal{BH}^\alpha(B)$ if and only if

$$\sup_{x,y \in B, x \ne y} (1-|x|^2)^{\alpha\beta}(1-|y|^2)^{\alpha(1-\beta)} \frac{|u(x)-u(y)|}{|x-y|} < \infty. \tag{5}$$

(b) $u \in \mathcal{BH}_0^\alpha(B)$ if and only if

$$\lim_{|x| \to 1, y \in B, x \ne y} (1-|x|^2)^{\alpha\beta}(1-|y|^2)^{\alpha(1-\beta)} \frac{|u(x)-u(y)|}{|x-y|} = 0. \tag{6}$$

Proof. First, assume that $u \in \mathcal{BH}^\alpha(B)$. Let $l_{x,y}(t) = tx + (1-t)y$. We have

$$u(x) - u(y) = \int_0^1 \frac{d}{dt}(u(l_{x,y}(t))dt = \int_0^1 \sum_{k=1}^n (x_k - y_k) \frac{\partial u}{\partial x_k}(l_{x,y}(t))dt. \tag{7}$$

By the Cauchy-Schwarz inequality and the definition of α-Bloch function, we have

$$|u(x) - u(y)| \le |x-y| \int_0^1 |(\nabla u)(l_{x,y}(t))|dt \le \|u\|_{\mathcal{B}^\alpha} |x-y| \int_0^1 \frac{dt}{(1-|l_{x,y}(t)|^2)^\alpha}. \tag{8}$$

Since $1 - |tx + (1-t)y| \ge 1 - t|x| - (1-t)|y| = t(1-|x|) + (1-t)(1-|y|)$, by Lemma 1, it follows that, for every $t \in [0,1]$, $x, y \in B$ and $\beta \in (0,1)$

$$\left(\frac{1}{\beta}\right)^{\alpha\beta} \left(\frac{1}{1-\beta}\right)^{\alpha(1-\beta)} [(1-t)(1-|y|)]^{\alpha(1-\beta)}[t(1-|x|)]^{\alpha\beta} \le (1-|l_{x,y}(t)|)^\alpha. \tag{9}$$

From (8) and (9), when $x \neq y$, we obtain

$$(1-|x|)^{\alpha\beta}(1-|y|)^{\alpha(1-\beta)}\frac{|u(x)-u(y)|}{|x-y|} \leq \|u\|_{\mathcal{B}^{\alpha}}\frac{B(1-\alpha(1-\beta),1-\alpha\beta)}{\beta^{-\alpha\beta}(1-\beta)^{\alpha(\beta-1)}}. \quad (10)$$

Since $\alpha < \min\{1/\beta, 1/(1-\beta)\}$, it follows that the integral in (10) converges. Taking the supremum in (10) over all $x, y \in B$, $x \neq y$, we obtain that the condition (5) holds.

Now, assume that (5) holds. By Cauchy's estimate ([4]) we have

$$|\nabla u(x)| \leq \frac{C}{1-|x|}\sup_{y\in B(x,(1-|x|)/2)}|u(y)-u(x)|, \quad (11)$$

for some positive constant C independent of u.

From (11) and since

$$|x-y| < \frac{1}{2}(1-|x|) \leq 1-|y| \leq \frac{3}{2}(1-|x|), \quad (12)$$

when $y \in B(x,(1-|x|)/2)$, we have

$$(1-|x|^2)^{\alpha}|\nabla u(x)| \leq C \sup_{y\in B(x,(1-|x|)/2)}|u(x)-u(y)|\frac{(1-|x|^2)^{\alpha\beta}(1-|y|^2)^{\alpha(1-\beta)}}{|x-y|}. \quad (13)$$

Taking the supremum in (13) over $x \in B$, we obtain that $u \in \mathcal{BH}^{\alpha}$. as desired.

(b) The proof of this part of the theorem is similar to the proof of Theorem 3.2 in [11], hence, it will be omitted. □

Remark. Note that in the proof of necessity of Theorem 1 we only use the fact $u \in C^{(1)}(B)$, hence, the harmonicity does not play any important role in the part of the proof. Note also that $\max_{\beta\in(0,1)}\min\{1/\beta, 1/(1-\beta)\} = 2$ which implies that $\alpha \in (0,2)$.

Now, we address the mistake made in the proof of Lemma 4.2 in [11]. Actually, they essentially wanted to prove a more direct inequality contained in the following theorem (see the last inequality in [11, p. 753]).

Theorem 2. *Assume that $u \in \mathcal{H}(B)$ and $p \in (0,\infty)$. Then there is a positive constant independent of u such that*

$$\int_B \frac{|u(x)-u(0)|^p}{|x|^p}dV_{\alpha}(x) \leq C\int_B (1-|x|^2)^p|\nabla u(x)|^p dV_{\alpha}(x). \quad (14)$$

Proof. Denote the first integral in (14) by I. Then, by Lemma 2 and some simple estimates (see, [18, Lemma 4]), we have

$$\begin{aligned} I &= \int_{|x|\leq 1/2}\frac{|u(x)-u(0)|^p}{|x|^p}dV_{\alpha}(x) + \int_{1/2<|x|<1}\frac{|u(x)-u(0)|^p}{|x|^p}dV_{\alpha}(x) \\ &\leq C\sup_{|x|<1/2}|\nabla u(x)|^p + 2^p\int_B |u(x)-u(0)|^p dV_{\alpha}(x) \\ &\leq C\int_B (1-|x|^2)^p|\nabla u(x)|^p dV_{\alpha}(x). \quad □ \end{aligned}$$

Although Theorem 2 surmounts the gap in [11, Lemma 4.2], we will not prove Theorem 4.1 in [11], i.e., Theorem A *(c)*. Hence the statement is left unconfirmed and is a good conjecture for the experts in the research area. Instead of that we prove the following closely related result.

Theorem 3. *Assume that* $u \in \mathcal{H}(B)$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$ *and* $p \in ((n-1)/\beta, \infty)$. *Then* $u \in \mathcal{B}_p$ *if and only if*

$$\int_B \int_{B\left(x, \frac{1-|x|}{2}\right)} (1-|x|^2)^{p\alpha}(1-|y|^2)^{p\beta} \frac{|u(x)-u(y)|^p}{|x-y|^p} d\tau(y) d\tau(x). \tag{15}$$

Proof. By Lemma 3 applied to the function $u(y) - u(x)$, and (12), we have

$$\begin{aligned} (1-|x|)^p |\nabla u(x)|^p &\leq \frac{C}{(1-|x|)^n} \int_{B\left(x, \frac{1-|x|}{2}\right)} |u(x)-u(y)|^p dV(y) \\ &\leq C \int_{B\left(x, \frac{1-|x|}{2}\right)} (1-|x|^2)^{p\alpha}(1-|y|^2)^{p\beta} \frac{|u(x)-u(y)|^p}{|x-y|^p} d\tau(y). \end{aligned} \tag{16}$$

Multiplying (16) by $d\tau(x)$, then integrating over B, we obtain

$$\begin{aligned} &\int_B (1-|x|)^p |\nabla u(x)|^p d\tau(x) \\ \leq \; & C \int_B \int_{B\left(x, \frac{1-|x|}{2}\right)} (1-|x|^2)^{p\alpha}(1-|y|^2)^{p\beta} \frac{|u(x)-u(y)|^p}{|x-y|^p} d\tau(y) d\tau(x), \end{aligned}$$

from which it follows that (15) implies $u \in \mathcal{B}_p$.

Let I be the integral in (15). Then from (12), by applying Lemma 3 to the partial derivatives of u with $\alpha = 0$, and some simple estimates, we have

$$\begin{aligned} I &\leq C \int_B \sup_{y \in B\left(x, \frac{1-|x|}{2}\right)} |u(x)-u(y)|^p \int_{B\left(x, \frac{1-|x|}{2}\right)} \frac{(1-|x|^2)^{p-n}}{|x-y|^p} dV(y) d\tau(x) \\ &\leq C \int_B \sup_{y \in B\left(x, \frac{1-|x|}{2}\right)} (1-|y|)^p |\nabla u(y)|^p d\tau(x) \\ &\leq C \int_B \int_{y \in B\left(x, \frac{3(1-|x|)}{4}\right)} (1-|y|)^p |\nabla u(y)|^p d\tau(y) d\tau(x) \\ &\leq C \int_B (1-|y|)^p |\nabla u(y)|^p \int_{x \in A(y)} d\tau(x) d\tau(y) \\ &\leq C \int_B (1-|y|)^p |\nabla u(y)|^p d\tau(y), \end{aligned}$$

where in the last inequality we have used the fact that the set $A(y) \subset \{y \mid |x - y| < 1 - |y|\}$, from which it follows that the quantity $\int_{x \in A(y)} d\tau(x)$ is bounded and where in the second inequality we have used the formula

$$\int_{B\left(x,\frac{1-|x|}{2}\right)} \frac{dV(y)}{|x-y|^p} = \sigma(S)\int_0^{(1-|x|)/2} r^{n-1-p}dr = C(1-|x|)^{n-p}. \quad \square$$

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