Dominating cycles in graphs with high connectivity

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Abstract

Let G be a graph and let $\sigma_k(G)$ be the minimum degree sum of an independent set of k vertices. For $S \subset V(G)$ with $|S| \geq k$, let $\Delta_k(S)$ denote the maximum value among the degree sums of the subset of k vertices in S. A cycle C of a graph G is said to be a dominating cycle if $V(G \setminus C)$ is an independent set. In [2], Bondy showed that if G is a 2-connected graph with $\sigma_3(G) \geq |V(G)| + 2$, then any longest cycle of G is a dominating cycle. In this paper, we improve it as follows: if G is a 2-connected graph with $\Delta_3(S) \geq |V(G)| + 2$ for every independent set S of order $\kappa(G) + 1$, then any longest cycle of G is a dominating cycle.

Keywords: degree sum, longest cycle, dominating cycle

1 Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. We denote the degree of a vertex x in a graph G by $d_G(x)$. Let $\delta(G)$, $\alpha(G)$ and $\kappa(G)$ be the minimum degree, the independence number and the connectivity of a graph G, respectively. If $\alpha(G) \geq k$, we define

$$\sigma_k(G) = \min\{\sum_{x \in X} d_G(x) \colon X \text{ is an independent set of } G \text{ with } |X| = k\};$$

if $\alpha(G) < k$, we set $\sigma_k(G) = +\infty$. For $S \subset V(G)$ with $|S| \geq k$, let $\Delta_k(S) = \max\{\sum_{x \in X} d_G(x) : X \subset S, |X| = k\}$. If $\alpha(G) \geq r$, we define

$$\sigma_k^r(G) = \min\{\Delta_k(S) \colon S \text{ is an independent set of } G \text{ with } |S| = r\};$$

if $\alpha(G) < r$, we set $\sigma_k^r(G) = +\infty$. Note $\sigma_k(G) = \sigma_k^k(G)$. If no ambiguity can arise, we often simply write δ, κ and σ_k^r instead of $\delta(G)$, $\kappa(G)$ and $\sigma_k^r(G)$, respectively.

The following is a classical result due to Dirac (1953) in hamiltonian graph theory.

Theorem 1 (Dirac [5]) *Let G be a graph of order $n \geq 3$. If $\delta \geq n/2$, then G is hamiltonian.*

In 1960, Ore introduced a degree sum condition for a graph to be hamiltonian.

Theorem 2 (Ore [7]) *Let G be a graph of order $n \geq 3$. If $\sigma_2 \geq n$, then G is hamiltonian.*

In [9], the second author gave a degree sum condition for a graph with high connectivity.

Theorem 3 *Let G be a connected graph on n vertices. If $\sigma_2^{\kappa(G)+1} \geq n$, then G is hamiltonian.*

A cycle C of a graph G is said to be a dominating cycle if $V(G \backslash C)$ is an independent set. In 1971, Nash-Williams gave a minimum degree condition for a dominating cycle.

Theorem 4 (Nash-Williams [6]) *Let G be a 2-connected graph on n vertices. If $\delta \geq (n+2)/3$, then any longest cycle of G is a dominating cycle.*

In 1980, Bondy gave a degree sum condition.

Theorem 5 (Bondy [2]) *Let G be a 2-connected graph on n vertices. If $\sigma_3 \geq n+2$, then any longest cycle is a dominating cycle.*

In 2005, Lu, Liu and Tian showed the extension of Theorem 5.

Theorem 6 (Lu et al. [8]) *Let G be a 3-connected graph on n vertices. If $\sigma_4 \geq \frac{4}{3}n + \frac{5}{3}$, then any longest cycle is a dominating cycle.*

In this paper, we prove the following result.

Theorem 7 *Let G be a 2-connected graph on n vertices. If $\sigma_3^{\kappa(G)+1} \geq n+2$, then any longest cycle of G is a dominating cycle.*

Theorem 7 is best possible in a sense. Let $k \geq 2$ be an integer. Consider the graph $G_1 = K_k + (k+1)K_2$. Then $\kappa(G_1) = k$ and $\sigma_3^{k+1}(G_1) = |V(G_1)| + 1$, but no longest cycle of G_1 is a dominating cycle.

On the other hand, Theorem 7 implies Theorems 5 and 6. If $\sigma_3 \geq n+2$, then $\sigma_3^{\kappa(G)+1} \geq \sigma_3 \geq n+2$. If $\sigma_4 \geq \lceil \frac{4}{3}n + \frac{5}{3} \rceil$, then $\sigma_3^{\kappa+1} \geq \sigma_3^4 \geq \frac{3}{4}\sigma_4 \geq n + \frac{5}{4}$ and so $\sigma_3^{\kappa+1} \geq n+2$ since $\sigma_3^{\kappa+1}$ is an integer. Actually Theorem 7 is strictly stronger than Theorems 5 and 6. There exist many graphs which satisfy the condition of Theorem 7 not that of Theorems 5 and 6. For example, let $k \geq 3$, $m \geq 3$ and let $G_2 = K_k + (K_1 \cup (k-1)K_2 \cup K_m)$. Then $\sigma_3^{k+1}(G_2) = 3k + m + 1 = |V(G_2)| + 2$ and so any longest cycle of G_2 is dominating, but G_2 does not satisfy the condition of Theorems 5 and 6, since $\sigma_3(G_2) = 3k + 2 < |V(G_2)| + 2$, $\sigma_4(G_2) = 4k + 3 < \frac{4}{3}|V(G_2)| + \frac{5}{3}$ if $k \geq 4$ and $\sigma_4(G_2) = m + 13 < \frac{4}{3}|V(G_2)| + \frac{5}{3}$ if $k = 3$.

2 Proof of Theorem 7

For standard graph-theoretic terminology not explained in this paper, we refer the reader to [3]. We denote by $N_G(x)$ the neighborhood of a vertex x in a graph G. For a subgraph H of G and a vertex $x \in V(G) \backslash V(H)$, we also denote $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. For $X \subset V(G)$, $N_G(X)$ denote the set of vertices in $G \setminus X$ which are adjacent to some vertex in X. Furthermore, for a subgraph H of G and $X \subset V(G) \setminus V(H)$, we sometimes write $N_H(X) := N_G(X) \cap V(H)$. If there is no fear of confusion, we often identify a subgraph H of a graph G with its vertex set $V(H)$. For example, we often write $G \setminus H$ instead of $G \setminus V(H)$. We write a cycle C with a given orientation by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of u on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. We abbreviate x^{+1} and x^{-1} by x^+ and x^-, respectively. A path P with endvertices x and y is denotes by xPy. For a subgraph H of G, a path xPy is called an H-path if $V(xPy) \cap V(H) = \{x, y\}$. For a cycle $\overrightarrow{C}$ and $X \subset V(C)$, we define $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$.

Proof of Theorem 7. Let C be a longest cycle in G. If C is a dominating cycle, then there is nothing to prove, and so we may assume that there exists a component H of $G \setminus C$ with $|V(H)| \geq 2$. Let $N_C(H) = \{u_1, \ldots, u_k\}$. Note that $k \geq \kappa(G)$. Without loss of generality, we may assume that $u_1, \ldots, u_k$ appear in this order along $\overrightarrow{C}$. Let $u'_i \in N_H(u_i)$ for $1 \leq i \leq k$. A vertex $u \in u_i^+\overrightarrow{C}u_{i+1}^-$ is *insertible* if there exist vertices $v, v^+ \in u_{i+1}\overrightarrow{C}u_i$ such that $uv, uv^+ \in E(G)$; then $vv^+ \in E(C)$ is an *insertion edge* of u. Let $I(u) = \{e \in E(C) : e \text{ is an insertion edge of } u\}$.

Claim 1 *There exists a non-insertible vertex in $u_i^+\overrightarrow{C}u_{i+1}^-$ for $1 \leq i \leq k$.*

Proof. Suppose not, that is, every vertex of $u_i^+\overrightarrow{C}u_{i+1}^-$ is insertible. Let $u'_iPu'_{i+1}$ be a path in H. Set $C_0 := u_i\overleftarrow{C}u_{i+1}u'_{i+1}Pu'_iu_i$ and $v_0 := u_i^+$. For $j \geq 1$, define the graph C_j such that

$$\begin{aligned} V(C_j) &:= V(C_{j-1}) \cup V(v_{j-1}\overrightarrow{C}v_j^-) \text{ and} \\ E(C_j) &:= E(C_{j-1}) \setminus \{w_{j-1}w_{j-1}^+\} \cup E(w_{j-1}v_{j-1}\overrightarrow{C}v_j^-w_{j-1}^+), \end{aligned}$$

where $w_{j-1}w_{j-1}^+ \in I(v_{j-1})$ and $v_j^- \in v_{j-1}\overrightarrow{C}u_{i+1}^-$ such that (i) $w_{j-1}w_{j-1}^+ \in I(v_j^-)$ and (ii) $|v_{j-1}\overrightarrow{C}v_j^-|$ is as large as possible, subject to (i). By the choice of v_j, $w_jw_j^+ \neq w_hw_h^+$ for any h, $0 \leq h \leq j-1$. Hence we can easily see C_j is a cycle for any j. Since there exists r such that $v_r = u_{i+1}$, C_r is a longer cycle than C, a contradiction. □

Let x_i be the first non-insertible vertex along $u_i^+ \overrightarrow{C} u_{i+1}^-$ for $1 \le i \le k$. Without loss of generality, we may assume that $d_G(x_1) = \max\{d_G(x_i) : 1 \le i \le k\}$. Since $|V(H)| \ge 2$, we can choose $x_{k+1} \in V(H) \setminus \{u_1'\}$. Suppose $N_H(u_1) = \{u_1'\}$. Then $N_C(H \setminus u_1') \cup \{u_1'\}$ is a cut set, and so $|N_C(H \setminus u_1') \cup \{u_1'\}| \ge \kappa$. Let $X = \{x_i : u_i \in N_C(H \setminus u_1')\} \cup \{x_1, x_{k+1}\}$. Since $u_1 \notin N_C(H \setminus u_1')$, $|X| \ge \kappa + 1$. If $|N_H(u_1)| \ge 2$, then let $X := \{x_1, x_2, x_3, \dots, x_{k+1}\}$. In either cases, X is an independent set by the following claim (see [1],[4]).

Claim 2 *There exists no C-path joining a vertex of $u_i^+ \overrightarrow{C} x_i$ and a vertex of $u_j^+ \overrightarrow{C} x_j$ for $1 \le i \ne j \le k$.*

Hence there exist $x_s, x_t \in X$ $(2 \le s < t \le k+1)$ such that

$$d_G(x_1) + d_G(x_s) + d_G(x_t) = \Delta_3(X) \ge |V(G)| + 2. \tag{1}$$

Later, we use the following claims.

Claim 3 *For each $v \in \{u_s', u_t', x_{k+1}\}$, there exists $u_1'' \in N_H(u_1)$ such that $u_1'' \ne v$.*

Proof. Let $v \in \{u_s', u_t', x_{k+1}\}$. Suppose $|N_H(u_1)| \ge 2$. Since $N_H(u_1) \setminus \{v\} \ne \emptyset$, there exists $u_1'' \in N_H(u_1)$ such that $u_1'' \ne v$. If $|N_H(u_1)| = 1$, then $u_1' \ne u_s', u_t', x_{k+1}$ by the definition of X. Hence u_1' is a desired vertex. □

Claim 4 *For $1 \le i \ne j \le k$, the following statements hold.*

(i) *For any $u \in V(u_i^+ \overrightarrow{C} x_i)$ and $v \in V(u_j^+ \overrightarrow{C} x_j)$, $N_C(u)^- \cap N_C(v) \cap V(x_i^+ \overrightarrow{C} u_j) = \emptyset$.*

(ii) *For any $w \in V(H)$, $N_G(x_i)^- \cap N_G(w) \cap V(x_i^+ \overrightarrow{C} u_i) = \emptyset$.*

(iii) *If $u_i' \ne u_j'$, then $N_C(x_i)^- \cap N_C(x_j)^+ \cap V(x_i^+ \overrightarrow{C} u_j) = \emptyset$.*

(iv) *If $w \in V(H) \setminus \{u_i'\}$, then $N_G(x_i)^- \cap N_G(w)^+ \cap V(x_i^+ \overrightarrow{C} u_i) = \emptyset$.*

Proof. The statements (i) and (ii) are proved in [1],[4], and so we omit the proofs. We show the statement (iii). Let $u_i' P u_j'$ be a path in H. Suppose, to the contrary, that $a \in N_C(x_i)^- \cap N_C(x_j)^+ \cap V(x_i^+ \overrightarrow{C} u_j)$. Then

$u_i \overleftarrow{C} x_j a^- \overleftarrow{C} x_i a^+ \overrightarrow{C} u_j u_j' P u_i' u_i$ is a cycle. By the statement (i), we may assume $a^-a, aa^+ \notin I(u)$ for any $u \in V(u_i^+ \overrightarrow{C} x_i \cup u_j^+ \overrightarrow{C} x_j)$, then we can obtain a cycle C' containing $V(C) \setminus \{a\} \cup \{u_i', u_j'\}$. Then C' is a longer cycle than C since $u_i' \neq u_j'$. Hence we obtain the statement (iii). By the similar argument, we can prove the statement (iv). □

Now we divide our argument into two cases.

Case 1. $x_t \neq x_{k+1}$.

Let $C_1 := u_1^+ \overrightarrow{C} x_1$, $C_2 := x_1^+ \overrightarrow{C} u_s$, $C_3 := u_s^+ \overrightarrow{C} x_s$, $C_4 := x_s^+ \overrightarrow{C} u_t$, $C_5 := u_t^+ \overrightarrow{C} x_t$, and $C_6 := x_t^+ \overrightarrow{C} u_1$. By Claim 2, $N_{C_1}(x_s) = \emptyset$ and $N_{C_1}(x_t) = \emptyset$. Clearly, $N_{C_1}(x_1) \subset V(C_1) \setminus \{x_1\}$. Therefore $d_{C_1}(x_1) + d_{C_1}(x_s) + d_{C_1}(x_t) \leq |V(C_1)| - 1$. Similarly, for $i = 1, 3, 5$, we have

$$d_{C_i}(x_1) + d_{C_i}(x_s) + d_{C_i}(x_t) \leq |V(C_i)| - 1. \tag{2}$$

Clearly, $N_{C_2}(x_1)^- \cup N_{C_2}(x_s)^+ \cup N_{C_2}(x_t) \subset V(C_2) \cup \{x_1, u_s^+\}$. On the other hand, by Claim 4 (i), $N_{C_2}(x_1)^- \cap N_{C_2}(x_t) = \emptyset$ and $N_{C_2}(x_s)^+ \cap N_{C_2}(x_t) = \emptyset$. By Claims 3 and 4 (iii), $N_{C_2}(x_1)^- \cap N_{C_2}(x_s)^+ = \emptyset$. Similarly, by considering $N_{C_6}(x_t)^-$, $N_{C_6}(x_1)^+$ and $N_{C_6}(x_s)$, for $i = 2, 6$ we obtain

$$d_{C_i}(x_1) + d_{C_i}(x_s) + d_{C_i}(x_t) \leq |V(C_i)| + 2. \tag{3}$$

If $u_s' \neq u_t'$, then we also have $d_{C_4}(x_1) + d_{C_4}(x_s) + d_{C_4}(x_t) \leq |V(C_4)| + 2$. However, it is possible that $u_s' = u_t'$, and so we especially prove the following claim.

Claim 5 $d_{C_4}(x_1) + d_{C_4}(x_s) + d_{C_4}(x_t) \leq |V(C_4)| + 2$.

Proof. Assume that $N_{C_4}(x_1) = \emptyset$. Then $N_{C_4}(x_s) \cup N_{C_4}(x_t)^+ \subset V(C_4) \cup \{u_t^+\}$. This leads the desired inequality by Claim 4 (i). Hence $N_{C_4}(x_1) \neq \emptyset$, and let $N_{C_4}(x_1) := \{w_1, w_2, \dots, w_r\}$. Without loss of generality, we may assume that $w_1, \dots, w_r$ appear in this order along $\overrightarrow{C}$. If $w_1 = x_s^+$, then $u_s \overleftarrow{C} x_1 x_s^+ \overrightarrow{C} u_1 u_1' P u_s' u_s$ is a cycle, and we can obtain a cycle containing $V(C) \setminus \{x_s\} \cup \{u_1', u_s'\}$, a contradiction. Therefore $w_1 \neq x_s^+$. Let $w_0 := x_s^+$ and $w_{r+1} := u_t^+$ and let $D_i := w_i \overrightarrow{C} w_{i+1}^-$ for each i, $0 \leq i \leq r$. Then, by Claim 4 (i), $N_{D_i}(x_s) \cap N_{D_i}(x_t)^+ = \emptyset$ for each i, $1 \leq i \leq r$. Since $N_{D_r}(x_s) \cup N_{D_r}(x_t)^+ \subset V(D_r) \cup \{u_t^+\}$, we obtain

$$d_{D_r}(x_s) + d_{D_r}(x_t) \leq |V(D_r)| + 1.$$

By Claim 4 (i), $w_{i+1} \notin N_{D_i}(x_t)^+$ for every i, $0 \le i \le r-1$. Therefore $N_{D_0}(x_s) \cup N_{D_0}(x_t)^+ \subset V(D_0)$, and so we get

$$d_{D_0}(x_s) + d_{D_0}(x_t) \le |V(D_0)|.$$

Since x_1 is a non-insertible vertex, $|V(D_i)| \ge 2$ for all i, $1 \le i \le r-1$. Let $I_2 := \{i : 1 \le i \le r-1, |V(D_i)| = 2\}$ and $I_3 := \{i : 1 \le i \le r-1, |V(D_i)| \ge 3\}$. By Claim 4 (i), $w_i^+ \notin N_{D_i}(x_s)$ for $i \in I_2$. By Claims 4 (i) and (iii), $w_{i+1}^{-2}, w_{i+1}^- \notin N_{D_i}(x_t)$ for $i \in I_2$. Hence $d_{D_i}(x_s) + d_{D_i}(x_t) \le 1 = |V(D_i)| - 1$ for all $i \in I_2$.

By Claims 4 (i) and (iii), $w_i^+, w_i^{+2} \notin N_{D_i}(x_s)$ for $i \in I_3$. Since x_t is a non-insertible vertex, $\{w_i^+, w_i^{+2}\} \not\subset N_{D_i}(x_t)^+$ for $i \in I_3$. For each $i \in I_3$, we define

$$w_i^* = \begin{cases} w_i^+ & \text{if } w_i^+ \notin N_{D_i}(x_t)^+, \\ w_i^{+2} & \text{if } w_i^+ \in N_{D_i}(x_t)^+. \end{cases}$$

Then $N_{D_i}(x_s) \cup N_{D_i}(x_t)^+ \subset V(D_i) \setminus \{w_i^*\}$ and so $d_{D_i}(x_s) + d_{D_i}(x_t) \le |V(D_i)| - 1$ for all $i \in I_3$. Therefore, for all i, $1 \le i \le r-1$, we get

$$d_{D_i}(x_s) + d_{D_i}(x_t) \le |V(D_i)| - 1.$$

Thus we have

$$\begin{aligned} d_{C_4}(x_s) + d_{C_4}(x_t) &\le |V(D_0)| + \sum_{i=1}^{r-1}(|V(D_i)| - 1) + |V(D_r)| + 1 \\ &= |V(C_4)| - r + 2. \end{aligned}$$

From $d_{C_4}(x_1) = r$, we obtain the desired inequality. □

By Claim 2, $N_{G \setminus C}(x_1)$, $N_{G \setminus C}(x_s)$ and $N_{G \setminus C}(x_t)$ are pairwise disjoint. Since $|V(H)| \ge 2$ and $N_H(x_1) = N_H(x_s) = N_H(x_t) = \emptyset$, we obtain

$$\begin{aligned} d_{G \setminus C}(x_1) + d_{G \setminus C}(x_s) + d_{G \setminus C}(x_t) &\le |V(G \setminus C)| - |V(H)| \\ &\le |V(G \setminus C)| - 2. \qquad (4) \end{aligned}$$

By (2)–(4) and Claim 5, we have

$$d_G(x_1) + d_G(x_s) + d_G(x_t) \le |V(G)| + 1.$$

This contradicts (1) and completes the proof of Case 1.

Case 2. $x_t = x_{k+1}$.

Let $C_1 := u_1^+ \overrightarrow{C} x_1$, $C_2 := x_1^+ \overrightarrow{C} u_s$, $C_3 := u_s^+ \overrightarrow{C} x_s$ and $C_4 := x_s^+ \overrightarrow{C} u_1$. By Claim 2, for $i = 1, 3$, we have

$$d_{C_i}(x_1) + d_{C_i}(x_s) + d_{C_i}(x_t) \leq |V(C_i)| - 1. \tag{5}$$

By Claims 3 and 4, we obtain

$$d_{C_2}(x_1) + d_{C_2}(x_s) + d_{C_2}(x_t) \leq |V(C_2)| + 2. \tag{6}$$

Since $N_C(x_{k+1}) \cap N_C(x_{k+1})^+ = \emptyset$, we can regard x_{k+1} as a non-insertible vertex. Suppose that $N_{C_4}(x_1) \neq \emptyset$. Let $N_{C_4}(x_1) := \{w_1, w_2, \ldots, w_r\}$. Without loss of generality, we may assume that $w_1, \ldots, w_r$ appear in this order along $\overrightarrow{C}$. Using the same argument as the proof of Claim 5, we have $w_1 \neq x_s^+$. Let $w_0 := x_s^+$ and $w_{r+1} := u_1^+$ and let $D_i := w_i \overrightarrow{C} w_{i+1}^-$ for each i, $0 \leq i \leq r$. Then as in the proof of Claim 5, we can show that the following inequalities hold.

$$d_{D_0}(x_s) + d_{D_0}(x_t) \leq |V(D_0)|,$$

$$d_{D_r}(x_s) + d_{D_r}(x_t) \leq |V(D_r)| + 1,$$

and

$$d_{D_i}(x_s) + d_{D_i}(x_t) \leq |V(D_i)| - 1 \text{ for all } i,\ 1 \leq i \leq r - 1,$$

and so we can obtain

$$d_{C_4}(x_1) + d_{C_4}(x_s) + d_{C_4}(x_t) \leq |V(C_4)| + 2. \tag{7}$$

If $N_{C_4}(x_t) = \emptyset$, then we also obtain the above inequality, since $N_{C_4}(x_s) \cup N_{C_4}(x_t)^+ \subset V(C_4) \cup \{u_t^+\}$.

Clearly, $N_H(x_1) = N_H(x_s) = \emptyset$ and $N_H(x_t) \subset V(H) \setminus \{x_t\}$. By Claim 2, $N_{G \setminus C}(x_1)$, $N_{G \setminus C}(x_s)$ and $N_{G \setminus C}(x_t)$ are pairwise disjoint. Hence we obtain

$$\begin{aligned} d_{G\setminus C}(x_1) + d_{G\setminus C}(x_s) + d_{G\setminus C}(x_t) &\leq |V(G \setminus C)| - |\{x_t\}| \\ &= |V(G \setminus C)| - 1. \end{aligned} \tag{8}$$

By (5)–(8), we have

$$d_G(x_1) + d_G(x_s) + d_G(x_t) \leq |V(G)| + 1,$$

contradicting (1). This completes the proofs of Case 2 and Theorem 7. □

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