ESSENTIAL NORM OF AN OPERATOR FROM THE WEIGHTED HILBERT-BERGMAN SPACE TO THE BLOCH-TYPE SPACE

STEVO STEVIĆ
Mathematical Institute of the Serbian Academy of Science,
Knez Mihailova 36/III, 11000 Beograd, Serbia
E-mail: sstevic@ptt.rs

Abstract

This note calculates the essential norm of a recently introduced integral-type operator from the Hilbert-Bergman weighted space $A^2_\alpha(\mathbb{B})$, $\alpha \geq -1$ to a Bloch-type space on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$, $S = \partial\mathbb{B}$ its boundary, $dV(z)$ the Lebesgue measure on $\mathbb{B}$, $dV_\alpha(z) = c_{\alpha,n}(1-|z|^2)^\alpha dV(z)$, $\alpha > -1$ and where the constant $c_{\alpha,n}$ is chosen such that $V_\alpha(\mathbb{B}) = 1$, $d\sigma$ the normalized rotation invariant measure on S and $H(\mathbb{B})$ the class of all holomorphic functions on the unit ball. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, $\langle z, w\rangle = \sum_{k=1}^n z_k \bar{w}_k$ and $|z| = \sqrt{\langle z, z\rangle}$. For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let $\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$.

For $p > 0$ the Hardy space $H^p = H^p(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$

Recall that for $f \in H^p$ the radial limit $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists a. e. on S.

The weighted Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{B})$, $p > 0$, $\alpha > -1$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) < \infty.$$

Since for every $f \in H^p$, $\lim_{\alpha\to -1+0} \|f\|_{A^p_\alpha} = \|f\|_p$, we will also use the notation A^p_{-1} for the Hardy space H^p.

A positive continuous function ϕ on $[0, 1)$ is called normal ([14]) if there is $\delta \in [0, 1)$ and a and b, $0 < a < b$ such that

$$\phi(r)/(1-r)^a \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \phi(r)/(1-r)^a = 0;$$

$$\phi(r)/(1-r)^\beta \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \phi(r)/(1-r)^\beta = \infty.$$

From now on if we say that a function $\mu : \mathbb{B} \to [0, \infty)$ is normal we will also assume that it is radial, that is, $\mu(z) = \mu(|z|)$, $z \in \mathbb{B}$.

The class of all $f \in H(\mathbb{B})$ such that $B_\mu(f) = \sup_{z\in\mathbb{B}} \mu(z)|\Re f(z)| < \infty$, where μ is normal, is called the Bloch-type space and is denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{B})$. With the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + B_\mu(f)$, $\mathcal{B}_\mu$ becomes a Banach space.

In [18] (see also [19]) we extended a recently introduced product of integral and composition operators on $H(\mathbb{D})$ (see [11] and [12]) in the unit ball settings, by introducing the following operator on $H(\mathbb{B})$

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}, \tag{1}$$

where $g \in H(\mathbb{B})$, $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$. For some results on related integral operators on spaces of holomorphic functions in $\mathbb{C}^n$, see [1]-[10], [13, 15, 16, 17, 20] and the references therein.

In this note we calculate the essential norm of the operator $P_\varphi^g : A_\alpha^2(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$. The result partially solve an open problem posed in [18].

In the proof of the main result we need the following known lemmas.

Lemma 1. ([21]) *Suppose $p \in (0, \infty)$ and $\alpha \geq -1$. Then for all $f \in A_\alpha^p(\mathbb{B})$ and $z \in \mathbb{B}$, the following inequality holds*

$$|f(z)| \leq \frac{\|f\|_{A_\alpha^p}}{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}. \tag{2}$$

Lemma 2. ([21]) *Suppose $0 < p < \infty$, $\alpha > -1$, then*

$$\|f\|_{A_\alpha^p}^p \asymp |f(0)|^p + \int_{\mathbb{B}} |\nabla f(z)|^p (1 - |z|^2)^{p+\alpha} dV(z),$$

for every $f \in A_\alpha^p$.

Lemma 3. ([18]) *Let $f, g \in H(\mathbb{B})$ and $g(0) = 0$. Then $\Re P_\varphi^g(f)(z) = f(\varphi(z))g(z)$.*

Throughout the paper C denotes a positive constant not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leq B \leq CA$.

2. Essential norm of $P_\varphi^g : A_\alpha^2 \to \mathcal{B}_\mu$

Let X and Y be Banach spaces, and $L : X \to Y$ be a bounded linear operator. The essential norm of the operator, $\|L\|_{e, X\to Y}$, is defined as follows

$$\|L\|_{e, X\to Y} = \inf\{\|L + K\|_{X\to Y} \ : \ K \text{ is compact from } X \text{ to } Y\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm.

From this and since the set of compact operators is a closed subset of the set of bounded operators it follows that L is compact if and only if $\|L\|_{e, X\to Y} = 0$.

In [18], among others, we proved the following result.

Theorem A. *Assume* $p \in (1,\infty)$, $\alpha \geq -1$, $g \in H(\mathbb{B})$, $g(0) = 0$, μ *is normal,* φ *is a holomorphic self-map of* $\mathbb{B}$ *and* $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ *is bounded. Then*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \leq \|P_\varphi^g\|_{e,A_\alpha^p\to\mathcal{B}_\mu} \leq 2\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}. \quad (3)$$

Motivated by Theorem A, in [18] we posed the following open problem.

Open problem. *Find the exact value of the essential norm of* $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$.

Here we partially solve the open problem by calculating the essential norm of the operator $P_\varphi^g : A_\alpha^2 \to \mathcal{B}_\mu$.

Theorem 1. *Assume* $\alpha \geq -1$, $g \in H(\mathbb{B})$, $g(0) = 0$, μ *is normal,* φ *is a holomorphic self-map of* $\mathbb{B}$ *and* $P_\varphi^g : A_\alpha^2 \to \mathcal{B}_\mu$ *is bounded. Then*

$$\|P_\varphi^g\|_{e,A_\alpha^2\to\mathcal{B}_\mu} = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{2}}}. \quad (4)$$

Proof. We follow the lines of the proof of Theorem 7.1 in [18]. A complete proof is given for the benefit of the reader. Assume that $(\varphi(z_k))_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then $P_\varphi^g : A_\alpha^2 \to \mathcal{B}_\mu$ is compact and (4) is vacuously satisfied).

For $w \in \mathbb{B}$ fixed, set

$$f_w(z) = \frac{(1-|w|^2)^{\frac{n+1+\alpha}{2}}}{(1-\langle z,w\rangle)^{\frac{2(n+1+\alpha)}{2}}}, \quad z \in \mathbb{B}. \quad (5)$$

It is known that $\|f_w\|_{A_\alpha^2} = 1$, for each $w \in \mathbb{B}$. Note that the sequence $(f_{\varphi(z_k)})_{k\in\mathbb{N}}$ is such that $\|f_{\varphi(z_k)}\|_{A_\alpha^2} = 1$, for each $k \in \mathbb{N}$, and it converges to zero uniformly on compacts of $\mathbb{B}$. From this and by Theorems 2.12 in [21] it follows that $f_{\varphi(z_k)} \to 0$ weakly in A_α^2, as $k \to \infty$. Hence, for every compact operator $K : A_\alpha^2 \to \mathcal{B}_\mu$ we have that $\|Kf_{\varphi(z_k)}\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$. Thus, for every such sequence and for every compact operator $K : A_\alpha^2 \to \mathcal{B}_\mu$ we have that

$$\begin{aligned}
\|P_\varphi^g + K\|_{A_\alpha^2\to\mathcal{B}_\mu} &\geq \limsup_{k\to\infty} \frac{\|P_\varphi^g f_{\varphi(z_k)}\|_{\mathcal{B}_\mu} - \|Kf_{\varphi(z_k)}\|_{\mathcal{B}_\mu}}{\|f_{\varphi(z_k)}\|_{A_\alpha^2}} \\
&= \limsup_{k\to\infty} \|P_\varphi^g f_{\varphi(z_k)}\|_{\mathcal{B}_\mu} \\
&\geq \limsup_{k\to\infty} \mu(z_k)|g(z_k) f_{\varphi(z_k)}(\varphi(z_k))| \\
&= \limsup_{n\to\infty} \frac{\mu(z_k)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{2}}}. \qquad (6)
\end{aligned}$$

Taking the infimum in (6) over the set of all compact operators $K : A_\alpha^2 \to \mathcal{B}_\mu$ we obtain

$$\|P_\varphi^g\|_{e,A_\alpha^2\to\mathcal{B}_\mu} \geq \limsup_{n\to\infty} \frac{\mu(z_k)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{2}}},$$

from which one inequality in (4) follows.

In the sequel we prove the reverse inequality. Assume that $(r_l)_{l\in\mathbb{N}}$ is a sequence which increasingly converges to 1. Consider the operators defined by

$$(P^g_{r_l\varphi}f)(z) = \int_0^1 g(tz)f(r_l\varphi(tz))\frac{dt}{t}, \quad l\in\mathbb{N}.$$

It is easy to see that these operators are compact (see Theorem 5.1 in [18]).

Since $P^g_\varphi : A^2_\alpha \to \mathcal{B}_\mu$ is bounded then for $f(z) = 1 \in A^2_\alpha$, we have that $\|g\|_{H^\infty_\mu} := \sup_{z\in\mathbf{B}} \mu(z)|g(z)| < \infty$. Let $\rho \in (0,1)$ be fixed. By Lemma 3, we have

$$\begin{aligned}
\|P^g_\varphi - P^g_{r_l\varphi}\|_{A^2_\alpha\to\mathcal{B}_\mu} &= \sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{z\in\mathbf{B}}\mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\le \sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}\mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\quad + \sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{|\varphi(z)|>\rho}\mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\le \|g\|_{H^\infty_\mu}\sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}|f(\varphi(z)) - f(r_l\varphi(z))| \qquad (7)\\
&\quad + \sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{|\varphi(z)|>\rho}\mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \qquad (8)
\end{aligned}$$

By using the polar coordinates and Parseval formula, we have

$$\begin{aligned}
\|f - f_r\|^2_{A^2_\alpha} &= c_{\alpha,n}\int_{\mathbf{B}}\Big|\sum_\beta a_\beta z^\beta(1 - r^{|\beta|})\Big|^2(1-|z|^2)^\alpha dV(z) \\
&= c_{\alpha,n}V(B)\int_0^1\sum_\beta |a_\beta|^2\rho^{2|\beta|+2n-1}(1-r^{|\beta|})^2(1-\rho^2)^\alpha d\rho \\
&\le c_{\alpha,n}V(B)\int_0^1\sum_\beta |a_\beta|^2\rho^{2|\beta|+2n-1}(1-\rho^2)^\alpha d\rho = \|f\|^2_{A^2_\alpha}. \qquad (9)
\end{aligned}$$

Lemma 1 along with (9) and the fact that $f(z) - f(rz) \in A^2_\alpha$, implies that

$$|f(\varphi(z)) - f(r_l\varphi(z))| \le \frac{\|f\|_{A^2_\alpha}}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{2}}}. \qquad (10)$$

Let $I_l := \sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}|f(\varphi(z)) - f(r_l\varphi(z))|$. If $\alpha > -1$, then by using the mean value theorem, the subharmonicity of the partial derivatives of f and Lemma 2, we have

$$\begin{aligned}
I_l &\le \sup_{\|f\|_{A^2_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}(1-r_l)|\varphi(z)|\sup_{|w|\le\rho}|\nabla f(w)| \qquad (11)\\
&\le C_\rho(1-r_l)\sup_{\|f\|_{A^2_\alpha}\le 1}\left(\int_{|w|\le\frac{1+\rho}{2}}|\nabla f(w)|^2(1-|w|^2)^{2+\alpha}dV(w)\right)^{1/2} \\
&\le C_\rho(1-r_l)\sup_{\|f\|_{A^2_\alpha}\le 1}\left(\int_{\mathbf{B}}|f(w)|^2dV_\alpha(w)\right)^{1/2} \to 0, \text{ as } l\to\infty. \qquad (12)
\end{aligned}$$

If $\alpha = -1$, then applying in (11) the known fact that for each compact $K \subset \mathbb{B}$, there is a positive constant C independent of f such that $\sup_{w \in K} |\nabla f(w)| \leq C\|f\|_2$ (see [21]), we obtain that (12) also holds in this case.

Using (10) in (8), letting $l \to \infty$ in (7), using (12), and then letting $\rho \to 1$ the reverse inequality follows, finishing the proof of the theorem. □

References

[1] D. C. Chang, S. Li and S. Stević, On some integral operators on the unit polydisk and the unit ball, *Taiwan. J. Math.* **11** (5) (2007), 1251-1286.

[2] D. C. Chang and S. Stević, Estimates of an integral operator on function spaces, *Taiwanese J. Math.* **7** (3) (2003), 423-432.

[3] Z. Hu, Extended Cesàro operators on mixed norm spaces, *Proc. Amer. Math. Soc.* **131** (7) (2003), 2171-2179.

[4] Z. Hu, Extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$, *Acta Math. Sci. Ser. B Engl. Ed.* **23** (4) (2003), 561-566.

[5] Z. Hu, Extended Cesàro operators on Bergman spaces, *J. Math. Anal. Appl.* **296** (2004), 435-454.

[6] S. Li and S. Stević, Integral type operators from mixed-norm spaces to α-Bloch spaces, *Integral Transform. Spec. Funct.* **18** (7) (2007), 485-493.

[7] S. Li and S. Stević, Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), 621-628.

[8] S. Li and S. Stević, Riemann-Stieltjes type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Variables Elliptic Functions* **52** (6) (2007), 495-517.

[9] S. Li and S. Stević, Compactness of Riemann-Stieltjes operators between $F(p,q,s)$ and α-Bloch spaces, *Publ. Math. Debrecen* **72** (1-2) (2008), 111-128.

[10] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), 1282-1295.

[11] S. Li and S. Stević, Products of composition and integral type operators from H^∞ to the Bloch space, *Complex Variables Elliptic Functions* **53** (5) (2008), 463-474.

[12] S. Li and S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to the Zygmund space, *J. Math. Anal. Appl.* **345** (2008), 40-52.

[13] S. Li and S. Stević, Riemann-Stieltjes operators between mixed norm spaces, *Indian J. Math.* **50** (1) (2008), 177-188.

[14] A. L. Shields and D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* **162** (1971), 287-302.

[15] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.* **1** (2005), 81-88.

[16] S. Stević, Boundedness and compactness of an integral operator on a weighted space on the polydisc, *Indian J. Pure Appl. Math.* **37** (6) (2006), 343-355.

[17] S. Stević, Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc, *Sibirsk. Mat. Zh.* **48** (3) (2007), 694-706.

[18] S. Stević, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, *Discrete Dyn. Nat. Soc.* Vol. 2008, Article ID 154263, (2008), 14 pages.

[19] S. Stević, On a new operator from H^∞ to the Bloch-type space on the unit ball, *Util. Math.* (to appear).

[20] X. Tang, Extended Cesàro operators between Bloch-type spaces in the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **326** (2) (2007), 1199-1211.

[21] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, New York, Springer (2005).