Note on the nullity of bicyclic graphs *

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Abstract. The nullity of a graph is the multiplicity of the eigenvalue zero in its spectrum. In this paper, we give formulae to calculate the nullity of n-vertex bicyclic graphs by means of the maximum matching number.

Key words: Bicyclic graphs; Nullity

1. Introduction

Let G be a simple undirected graph on n vertices. The *adjacency matrix* $A(G)$ of graph G, having vertex set $V(G) = [v_1, \cdots, v_n]$ is the $n \times n$ symmetric matrix $[a_{ij}]$, such that $[a_{ij}] = 1$, if v_i and v_j are adjacent and 0, otherwise. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of $A(G)$ are said to be the *eigenvalues* of G, and to form the *spectrum* of G. The number of zero eigenvalues in the spectrum of G is called its *nullity* and is denoted by $\eta(G)$. Let $r(A(G))$ be the rank of $A(G)$. Then $\eta(G) = n - r(A(G))$.

Collatz and Sinogowitz [1] first posed the problem of characterizing all graphs which satisfy $\eta(G) > 0$. This question is of great interest in chemistry, because, as has been shown in [6], for a bipartite graph G (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates that the molecule which such a graph represents is unstable. Some results on the nullity of graphs in general are known (See [2], [3], [4],[5] and [7]).

Connected graphs in which the number of edges equals the number of vertices plus one are called *bicyclic graphs.* Define a *b-graph* to be a graph

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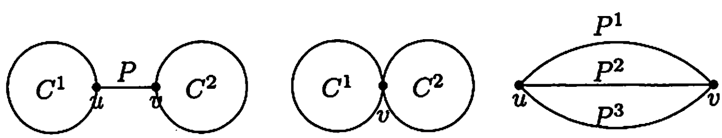
consisting either of two vertex-disjoint cycles C^1 and C^2 and a path P joining them having only its end-vertices u and v in common with the cycles, or two cycles C^1 and C^2 with exactly one vertex v in common. The former is sometimes called b_1-graph and the latter b_2-graph. Define a *θ-graph* to be a graph consisting of two given vertices u and v joined by three paths P^1, P^2 and P^3 with any two of these paths having only the given vertices in common. See Fig. 1, where u and v are called *paste vertices.*



b_1-graph b_2-graph θ-graph

Figure 1

Obviously, a bicyclic graph is a b-graph or a θ-graph with trees attached. In this paper, we give formulae to calculate the nullity of n-vertex bicyclic graphs by means of the maximum matching number. Following the methods employed in [7], we denote by $m(G)$ the maximum matching number of a graph G.

2. Main results

In this section we will first give some preliminary lemmas and then gradually obtain our main results.

Let G be a graph and $v_i \in V(G)$ $(1 \leq i \leq k)$. Then we write $G - \{v_1, \cdots, v_k\}$ for the subgraph of G obtained from G by removing the vertices $v_1, \cdots, v_k$ and all edges incident to any of them. If $e_i \in E(G)$

$(1 \leq i \leq \ell)$ then define $G - \{e_1, \cdots, e_\ell\}$ to be the subgraph of G obtained from G by removing the edges $e_1, \cdots, e_\ell$. A vertex of a graph is called a *pendant vertex* if its degree is 1.

Lemma 1. *[2] Suppose that v is a pendant vertex of G and u is adjacent to v. Then $\eta(G) = \eta(G - \{u, v\})$.*

The following result is trivial, but we will use it repeatedly.

Lemma 2. *Suppose $G = \bigcup_{i=1}^{\ell} G_i$, where $G_1, \cdots, G_\ell$ are connected components of G. Then $\eta(G) = \sum_{i=1}^{\ell} \eta(G_i)$.*

Suppose that G_0 is an n-vertex graph with a pendant vertex u_0 and that v_0 is adjacent to u_0. Let $G_{i+1} = G_i - \{u_i, v_i\}$, where u_i is a pendant vertex of G_i and v_i is adjacent to u_i $(i = 0, 1, \ldots)$. Then obviously there exists some integer ℓ such that $G_{\ell+1}$ has no pendant vertex. (Specifically, if G_0 is an edge, then $G_{\ell+1}$ will be an empty set.) The process of getting $G_{\ell+1}$ from G_0 is called a *pendant edge deleting* (PED for short) of G_0. By Lemma 1, $\eta(G_0) = \eta(G_{\ell+1})$. G_0 is a *PED-graph* if $G_{\ell+1}$ only contains isolated vertices. For instance, a tree is a PED-graph. Since v_i is always covered by any maximum matching of G_i, if G_0 is a PED-graph, then by Lemmas 1 and 2, $\eta(G_0) = n - 2m(G_0)$.

Lemma 3. *[2] A path with four vertices of degree 2 in a graph G can be replaced by an edge without changing $\eta(G)$.*

Let C_n be a cycle on n vertices. Then $\eta(C_n) = 2$ if $n \equiv 0(mod 4)$ and $\eta(C_n) = 0$ otherwise.

Lemma 4. *If G is either a b-graph or a θ-graph, then $\eta(G) \leq 3$.*

Proof. By lemma 3 we only need to observe the b_1-graphs and the θ-graphs on at most 11 vertices and b_2-graphs on at most 7 vertices (See Fig. 1). By MATLAB we can verify that the result holds. □

Suppose that H is a graph without any pendant vertex and that $u_i \in V(H)$ $(1 \leq i \leq k)$. Suppose that F is a forest with components $T_1, \ldots, T_k$ disjoint with H and that $v_i \in V(T_i)$ $(1 \leq i \leq k)$. Then we denote by $H \cdot F$ the graph obtained by identifying u_i and v_i $(1 \leq i \leq k)$, where $u_1, \ldots, u_k$

are called *overlapped vertices* of $H \cdot F$ and k is the *overlapped number*, denoted by $\Box(H \cdot F)$. A vertex in $\{u_1, \cdots, u_k\}$ is called *reserved* if it is undeleted after a PED of $H \cdot F$. In fact, a reserved vertex is such that is mismatched by some maximum matching from F. We denote by $\diamond(H \cdot F)$ the number of all reserved vertices of $H \cdot F$. Clearly, $\diamond(H \cdot F) \le \Box(H \cdot F)$.

Theorem 5. *Suppose that $H \cdot F$ is a bicyclic graph on n vertices, where H is either a b-graph or a θ-graph on ϑ vertices and F is a forest. If $\diamond(H \cdot F) = \Box(H \cdot F)$, then $\eta(H \cdot F) \le n - 2m(F) - \vartheta + 3$.*

Proof. Since $\diamond(H \cdot F) = \Box(H \cdot F)$, we obtain a subgraph which consists of H and $n - 2m(F) - \vartheta$ isolated vertices after a PED of $H \cdot F$. By Lemmas 1 and 2, $\eta(H \cdot F) = n - 2m(F) - \vartheta + \eta(H)$. It follows from Lemma 4 that the result is true. $\Box$

Theorem 6. *Suppose that $B_1 \cdot F$ is a bicyclic graph on n vertices, where B_1 is a b_1-graph with two cycles C^1 and C^2 of length ℓ_1 and ℓ_2 respectively and a path P (See Fig. 1: b_1-graph), and that F is a forest. Suppose that C^i contains r_i overlapped vertices and s_i reserved vertices ($i = 1, 2$) and that P contains r_3 overlapped vertices and s_3 reserved vertices. Let $m = m(B_1 \cdot F)$.*

(1.) *If $s_1 < r_1$ and $s_2 < r_2$, then $\eta(B_1 \cdot F) = n - 2m$;*

(2.) *If $s_1 < r_1$ and $s_2 = r_2$, then*

$$\eta(B_1 \cdot F) = \begin{cases} n - 2m + 2, & \text{if } \ell_2 \equiv 0 (mod\ 4); \\ n - 2m, & \text{if } \ell_2 \not\equiv 0 (mod\ 4) \text{ and } \ell_2 \text{ is even}; \\ n - 2m - 1, & \text{if } \ell_2 \text{ is odd}. \end{cases}$$

(3.) *If $s_1 = r_1$, $s_2 = r_2$ and $s_3 < r_3$, then*

$$\eta(B_1 \cdot F) = \begin{cases} n - 2m + 4, & \text{if } \ell_1, \ell_2 \equiv 0 (mod\ 4); \\ n - 2m + 2, & \text{if } \ell_1 \equiv 0 (mod\ 4), \ell_2 \not\equiv 0 (mod\ 4) \text{ and } \ell_2 \text{ is even}; \\ n - 2m + 1, & \text{if } \ell_1 \equiv 0 (mod\ 4) \text{ and } \ell_2 \text{ is odd}; \\ n - 2m, & \text{if } \ell_1, \ell_2 \not\equiv 0 (mod\ 4) \text{ and } \ell_1, \ell_2 \text{ are even}; \\ n - 2m - 1, & \text{if } \ell_1 \not\equiv 0 (mod\ 4) \text{ and } \ell_2 \text{ is odd}; \\ n - 2m - 2, & \text{if } \ell_1, \ell_2 \text{ are odd}. \end{cases}$$

Proof. Suppose that $s_1 < r_1$ and $s_2 < r_2$. Then $B_1 \cdot F$ is a PED-graph, and so $\eta(B_1 \cdot F) = n - 2m$.

Suppose that $s_1 < r_1$ and $s_2 = r_2$. Because $s_2 = r_2$ implies that $s_3 < r_3$ and the paste vertex v can not be deleted after a PED of $B_1 \cdot F$, we get a subgraph consisting of the cycle C^2 and $n - \ell_2 - 2m(G')$ isolated vertices, where G' is the subgraph $B_1 \cdot F - E(C^2)$ of $B_1 \cdot F$. Hence $\eta(B_1 \cdot F) = n - \ell_2 - 2m(G') + \eta(C^2)$ by Lemmas 1 and 2. Note that if $\ell_2 \equiv 0(mod4)$ then $\eta(C^2) = 2$ and $\ell_2 = 2m(C^2)$; if ℓ_2 is even and $\ell_2 \not\equiv 0(mod4)$ then $\eta(C^2) = 0$ and $\ell_2 = 2m(C^2)$ and if ℓ_2 is odd then $\eta(C^2) = 0$ and $\ell_2 = 2m(C^2) + 1$. It follows that (2) is true.

Suppose that $s_1 = r_1$, $s_2 = r_2$ and $s_3 < r_3$. Because $s_1 = r_1$ and $s_2 = r_2$ imply that the paste vertices u and v can not be deleted after a PED of $B_1 \cdot F$, we get a subgraph which consists of C^1, C^2 and $n - \ell_1 - \ell_2 - 2m(G'')$ isolated vertices, where G'' is the subgraph $B_1 \cdot F - E(C^1 \cup C^2)$ of $B_1 \cdot F$. By Lemmas 1 and 2, $\eta(B_1 \cdot F) = n - \ell_1 - \ell_2 - 2m(G'') + \eta(C^1) + \eta(C^2)$. A simple case argument shows that (3) is true. □

Similar to the proof of Theorem 6, we can obtain

Theorem 7. *Suppose that $B_2 \cdot F$ is a bicyclic graph on n vertices, where B_2 is a b_2-graph with two cycles C^1 and C^2 of length ℓ_1 and ℓ_2 respectively (See Fig. 1: b_2-graph), and that F is a forest. Suppose that C^i contains r_i overlapped vertices and s_i reserved vertices ($i = 1, 2$). Let $m = m(B_2 \cdot F)$.*

(1.) *If $s_1 < r_1$ and $s_2 < r_2$, then $\eta(B_1 \cdot F) = n - 2m$;*

(2.) *If $s_1 < r_1$ and $s_2 = r_2$, then*

$$\eta(B_1 \cdot F) = \begin{cases} n - 2m + 2, & \text{if } \ell_2 \equiv 0(mod\ 4); \\ n - 2m, & \text{if } \ell_2 \not\equiv 0(mod\ 4) \text{ and } \ell_2 \text{ is even;} \\ n - 2m - 1, & \text{if } \ell_2 \text{ is odd.} \end{cases}$$

Theorem 8. *Suppose that $\Theta \cdot F$ is a bicyclic graph on n vertices, where Θ is a θ-graph with three paths P^1, P^2 and P^3 of length ℓ_1, ℓ_2 and ℓ_3, respectively (See Fig. 1: θ-graph), and that F is a forest. Suppose that P^i contains r_i overlapped vertices and s_i reserved vertices ($i = 1, 2, 3$). Let $m = m(\Theta \cdot F)$.*

(1.) *If $s_1 < r_1$ and $s_2 < r_2$, then $\eta(\Theta \cdot F) = n - 2m$.*

(2.) *If $s_1 < r_1$, $s_2 = r_2$ and $s_3 = r_3$, then*

$$\eta(\Theta \cdot F) = \begin{cases} n - 2m, & \text{if } \ell_2 + \ell_3 \not\equiv 0(mod\ 4); \\ n - 2m + 2, & \text{if } \ell_2 + \ell_3 \equiv 0(mod\ 4); \\ n - 2m - 1, & \text{if } \ell_2 + \ell_3 \text{ is odd.} \end{cases}$$

Proof. Suppose that $s_1 < r_1$ and $s_2 < r_2$. Because this implies that $s_3 < r_3$, $\Theta \cdot F$ is a PED-graph, and so $\eta(\Theta \cdot F) = n - 2m$. Suppose that $s_1 < r_1$, $s_2 = r_2$ and $s_3 = r_3$. Then the paste vertices u and v can not be deleted after a PED of $\Theta \cdot F$. As in the proof of Theorem 6 (2.), we can show that (2.) is true. □

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