On Schematic Orthogonal Arrays of Strength Two

Yong Lin Zhang

Statistics Department, Beijing Information Science and Technology University

Haidian District, 100192, Beijing,

E-mail: ylzhang9@126.com

November 10, 2008

Abstract

In the theory of orthogonal arrays, an orthogonal array is called schematic if its rows form an association scheme with respect to Hamming distances. Which orthogonal arrays are schematic orthogonal arrays and how to classify them is an open problem proposed by Hedayat et al.[12]. In this paper we study the Hamming distances of the rows in orthogonal arrays and construct association schemes according to the distances. The paper gives the partial solution of the problem by Hedayat et. al. for symmetric and some asymmetric orthogonal arrays of strength two.

key words: orthogonal array, Hamming distance, association scheme, schematic orthogonal array.

1 Introduction

In the past decades, people mainly focused on the properties of columns of orthogonal array because of the application. Only a few people considered the relations of the rows in the orthogonal arrays. Delsarte [9] gave the relations of rows in some orthogonal arrays first, Atsumi [4] also dealt with relations of rows in the orthogonal array.

Some authors studied schematic orthogonal arrays. Yoshizawa [25] shown schematic orthogonal arrays in some cases. Hedayat etc. [12] studied the orthogonal arrays in detail and proposed many open problems. One of these problems is that in which orthogonal array the rows form an association scheme and how to classify them. The paper deals with the schematic orthogonal arrays with strength 2 and gives the construction of association schemes with respect to Hamming distances of the rows in the orthogonal arrays.

2 The Hamming distances in symmetric orthogonal arrays of strength two

In the definition of orthogonal array by C. R. Rao [19], if $t_1 = \cdots = t_m = t$, an orthogonal array can be denoted simply by $OA(n, m, t, s)$. Such orthogonal arrays are called symmetric orthogonal arrays of strength s. The orthogonal array $OA(n, m, t, s)$ has the properties:

1. In each column, all the symbols occur equally often;
2. In any r columns ($r \leq s$), all the possible r-tuples of symbols occur equally often.

In practice, we usually take $s = 2$. In that case, we take Taguchi's notation $L_n(t^m)$ for $OA(n, m, t, s)$, here the strength $s = 2$ is omitted.

There are many ways to construct orthogonal arrays. The orthogonal arrays constructed by Latin squares are expressed by $L_{t^2}(t^{t+1})$, those constructed by vector spaces over Galois fields are expressed by $L_{t^n}(t^m)$(where $m = \frac{t^n-1}{t-1}$) and those constructed by Hadamard matrices H_{4n} are expressed by $L_{4n}(2^{4n-1})$. The different methods make different expressions. But they are all saturated in the sense that they leave no degree of freedom for error estimation in statistics. In $L_n(t^m)$, it means

$$m(t-1) = n - 1. \tag{2.1}$$

In asymmetric orthogonal array $OA(N, t_1^{m_1} t_2^{m_2} \cdots t_r^{m_r}, 2)$ it means

$$(t_1 - 1)m_1 + (t_2 - 1)m_2 + \cdots + (t_r - 1)m_r = N - 1. \tag{2.2}$$

In [15], there is a lemma describing the relations of rows in saturated asymmetric orthogonal arrays, it is stated as follows.

Lemma 2.1 *(Mukerjee and Jeff Wu) Consider two distinct rows of a saturated orthogonal array $OA(n, t_1^{m_1} t_2^{m_2}, 2)$. For $i = 1, 2$, let Δ_i be the number of coincidences between these two rows arising from the t_i-symbol columns. Then Δ_1 and Δ_2 are nonnegative integers satisfying $\Delta_1 \leq m_1, \Delta_2 \leq m_2$,*

$$t_1\Delta_1 + t_2\Delta_2 = m_1 + m_2 - 1 \tag{2.3}$$

In this lemma, let $m_2 = 0$, it becomes symmetric case.

If Δ denotes the number of coincidences between any two rows with t symbols, then Hamming distance should be $D_h = m - \Delta$. From 2.1 and 2.3, we have

$$D_H = m - \Delta = m - \frac{m-1}{t} = \frac{n-1}{t-1} - \frac{1}{t}\left[\frac{n-1}{t-1} - 1\right] = \frac{n}{t}.$$

There is one Hamming distance, so the saturated symmetric orthogonal array is equidistant with respect to Hamming distance.

In $L_{t^n}(t^m)$ where $m = \frac{t^n-1}{t-1}, n \geq 2$, $D_H = t^{n-1}$. In $L_{4n}(2^{4n-1}), D_H = 2n$.

But for unsaturated symmetric orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$ ($\lambda \geq 2$), Mukerjee and Jeff Wu do not give the expression of Hamming distances of the rows in the array.

The orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$ is derived from difference matrices based on the method of Bose and Bush [5]. The method is stated as follows.

Theorem 2.1 *(Bose and Bush) Let $D(\lambda t, \lambda t, t)$ be a difference matrix over an additive group G of order t, if every element in D is replaced by its correspondence in the additive table of G, we obtain a $\lambda t^2 \times \lambda t$ matrix, it is the orthogonal array $L_{\lambda t^2}(t^{\lambda t})$, and its λt^2 rows can be divided into λt groups with t rows each, the t elements in each group are just all the elements in G. By juxtaposing another column*

$$(0,0,\cdots 0,1,1,\cdots,1,\cdots,t-1,t-1,\cdots,t-1)'$$

we obtain the orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$.

For difference matrices of $\lambda = 2$, many authors gave the construction methods (Masuyama [17], for example Liu [14], Jungnickel[13] and Xiang [24]).

For $\lambda > 2$, there is no general construction method so far, some difference matrices are found case by case. Jungnickel [13] gave the following lemma.

Lemma 2.2 *The transpose of difference matrix $D(\lambda t, \lambda t, t)$ is also a difference matrix $D(\lambda t, \lambda t, t)$.*

According to Lemma 2.2, we have

Theorem 2.2 *In orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$, there are three Hamming distances between rows: $D_{H_1} = \lambda t, D_{H_2} = \lambda(t-1), D_{H_3} = \lambda(t-1)+1$.*

Proof. Lemma 2.2 means that the rows of the matrix form a difference matrix as well, so in the difference of any two rows of $D(\lambda t, \lambda t, t)$, there are λ 0's. That means there are λ corresponding symbols equal in the two rows, so the Hamming distance of the two rows is

$$\lambda t - \lambda = \lambda(t-1).$$

By the constructing method of Theorem 2.1, after replacing symbols in $D(\lambda t, \lambda t, t)$ by their corresponding columns in the additive table, juxtapose a column

$$(0,\cdots,0,1,\cdots,1,\cdots,(t-1),\cdots,(t-1))'.$$

There are three conditions for the Hamming distance of the array after juxtaposition.

(1). For any two rows corresponding to the same row in $D(\lambda t, \lambda t, t)$, all the symbols are different except the juxtaposed column, so the Hamming distance of them is

$$D_{H_1} = \lambda t,$$

(2). Take any two rows corresponding to different rows in $D(\lambda t, \lambda t, t)$ and corresponding to the same symbol in the juxtaposed column. One row can be obtained by permutation of the other, the Hamming distance of the original rows is $\lambda(t-1)$, after permutation, it keeps unchanged. The Hamming distance of them is

$$D_{H_2} = \lambda(t-1).$$

(3). For any two rows corresponding to the different rows in $D(\lambda t, \lambda t, t)$ and different symbols in the juxtaposed column, the Hamming distance is that in $D(\lambda t, \lambda t, t)$ plus one, so

$$D_{H_3} = \lambda(t-1)+1.$$

Hence the theorem is proved. ∎

3 The Construction of Association Schemes

We define associate relations by Hamming distances of the rows. When the relations satisfy some conditions, an association scheme is constructed.

For convenience in the construction, we give the definition of the association scheme first.

Definition 1 *Given v symbols $1, 2, \cdots, v$, a relation satisfying the following conditions is said to be an association scheme with d classes:*

(1) Any two symbols are either 1st, 2nd, ..., or d-th associate, the relation is symmetrical, that is, if the symbol x is the i-th associate of the symbol y, then y is the i-th associate of x.

(2) Each symbol x has n_i i-th associates, the number n_i is independent of the symbol x.

(3) If any two symbols x and y are i-th associates, then the number of symbols that are both j-th associate of x and k-th associate of y is p^i_{jk}, and it is independent of the pair of i-th associates x and y.

The numbers $v, n_i, (i = 1, 2, \cdots, d)$ and $p^i_{jk} (i, j, k = 1, 2, \cdots, d)$. are called the parameters of the association scheme. It is easy to show that $p^i_{jk} = p^i_{kj}$ and they form d symmetric $d \times d$ matrices:

$$P_i = (p^i_{jk}), (i = 1, 2, \cdots, d) \quad .$$

The two-associate-class association schemes related to statistics are summarized by Raghavarao [18].

3.1 For Saturated Symmetric Orthogonal Arrays

The orthogonal arrays constructed from Latin squares, Hadamard matrices and vector spaces over Galois fields are all saturated. There is only one Hamming distance of the rows in these arrays. If we delete some columns in the orthogonal array, the array becomes unsaturated, hence Hamming distance has two or more values.

If we delete one column, the Hamming distance has two different values, the rows are grouped according to the symbols in the deleted column. In the same group any two rows have one distance and in the different groups, any two rows have another distance. So the group divisible association scheme is constructed, and it is not related to the strength s of the array.

But if we delete two or more columns from the array, the case depends on the strength of the array. We give a lemma first.

Lemma 3.1 *Let there be a d-class association scheme with parameters:*

$$v, \ n_1, n_2, \cdots, n_d, p^i_{jk} \ (i, j, k = 1, 2, \cdots, d).$$

If we repeat each symbol r times and define the new associate relations of the repeated symbols as $(d+1)$-th associates, then a $(d+1)$-class association scheme is obtained with parameters:

$$v' = rv, n'_1 = rn_1, n'_2 = rn_2, \cdots, n'_{d+1} = r - 1,$$

and

$$p'^{i}_{jk} = \begin{cases} rp^{i}_{jk} & if\ i,j,k \neq d+1 \\ 0 & if\ i=j=d+1 \\ r-1 & if\ i=j\neq d+1\ and\ k=d+1 \\ r-2 & if\ i=j=k=d+1 \end{cases}$$

Proof. It is easy to see that if an associate relation is not newly defined in the new scheme, the related parameters are multiplied by r because each symbol repeats r times. For a pair of $(d+1)$-th associates x, y, there will not be a symbol that is both $(d+1)$-th associate of x and k-th associate of y. If $i = k$, $j = d+1$, the $r-1$ other symbols will be the $(d+1)$-th associates of x and i-th associates of y. If $i, j, k = d+1$, the other $r-2$ symbols will be the $(d+1)$-th associates of both x and y. The lemma is proved. ∎

Let there be an orthogonal array $OA(n, m, t, s)$ with strength s. Each r-tuple is ordered and can be looked as a point in $\mathbf{R}^r$. Then the t^r points can be arranged into an r-dimensional supercube according to their coordinates and the Hamming distances of them can be found. According to Brouwer, Cohen and Neumaie ([7]PP.261), the t^r points form a distance-regular graph, named Hamming graph (lattice graph) and it is an association scheme with respect to Hamming distance. So equivalently, all the t^r points form an r-class association scheme with respect to Hamming distance.

In any r columns, each r-tuples appears equally often, so by Lemma 3.1, these tuples form an association scheme. Since

$$D_{H(afterdeleting)} = D_{H(array)} - D_{H(deleted)}.$$

In case $s = 2$, we have:

Theorem 3.1 *In a saturated symmetric orthogonal array $OA(n, m, t, 2)$, if we delete any 2 columns, the rows of the array after deletion form an association scheme with respect to Hamming distances.*

Notice that the construction method is independent of the deleted column, so we can obtain the association scheme by deleting any 2 columns. The construction is shown in the following example.

Example

The following table is orthogonal array $L_{27}(3^{13})$ (transposed):

1	1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3 3
2	1 1 1 2 2 2 3 3 3 1 1 1 2 2 2 3 3 3 1 1 1 2 2 2 3 3 3
3	1 1 1 2 2 2 3 3 3 2 2 2 3 3 3 1 1 1 3 3 3 1 1 1 2 2 2
4	1 1 1 2 2 2 3 3 3 3 3 3 1 1 1 2 2 2 2 2 2 3 3 3 1 1 1
5	1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3
6	1 2 3 1 2 3 1 2 3 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1
7	1 2 3 1 2 3 1 2 3 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2
8	1 2 3 2 3 1 3 1 2 1 2 3 2 3 1 3 1 2 1 2 3 2 3 1 3 1 2
9	1 2 3 2 3 1 3 1 2 2 3 1 3 1 2 1 2 3 3 1 2 1 2 3 2 3 1
10	1 2 3 2 3 1 3 1 2 3 1 2 1 2 3 2 3 1 2 3 1 3 1 2 1 2 3
11	1 2 3 3 1 2 2 3 1 1 2 3 3 1 2 2 3 1 1 2 3 3 1 2 2 3 1
12	1 2 3 3 1 2 2 3 1 2 3 1 1 2 3 3 1 2 3 1 2 2 3 1 1 2 3
13	1 2 3 3 1 2 2 3 1 3 1 2 2 3 1 1 2 3 2 3 1 1 2 3 3 1 2

Deleting the first two columns, by the discussion above, a 3-class association scheme is obtained with parameters:

$$v = 27,\ n_1 = 2,\ n_2 = 12,\ n_3 = 12,$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 12 & 0 \\ & & 12 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 2 & 0 \\ & 3 & 6 \\ & & 6 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & 2 \\ & 6 & 6 \\ & & 3 \end{pmatrix}.$$

The association scheme table is Table 1:

Table 1: The association scheme obtained from $L_{27}(3^{13})$ after deleting two columns

	1st	2nd asso.	3rd asso.
1	2 3	4 5 6 7 8 9 10 11 12 19 20 21	13 14 15 16 17 18 22 23 25 25 26 27
2	1 3	4 5 6 7 8 9 10 11 12 19 20 21	13 14 15 16 17 18 22 23 25 25 26 27
3	1 2	4 5 6 7 8 9 10 11 12 19 20 21	13 14 15 16 17 18 22 23 25 25 26 27
4	5 6	1 2 3 7 8 9 13 14 15 22 23 24	10 11 12 16 17 18 19 20 21 25 26 27
5	4 6	1 2 3 7 8 9 13 14 15 22 23 24	10 11 12 16 17 18 19 20 21 25 26 27
6	4 5	1 2 3 7 8 9 13 14 15 22 23 24	10 11 12 16 17 18 19 20 21 25 26 27
7	8 9	1 2 3 4 5 6 16 17 18 25 26 27	10 11 12 13 14 15 19 20 21 22 23 24
8	7 9	1 2 3 4 5 6 16 17 18 25 26 27	10 11 12 13 14 15 19 20 21 22 23 24
9	7 8	1 2 3 4 5 6 16 17 18 25 26 27	10 11 12 13 14 15 19 20 21 22 23 24
10	11 12	1 2 3 13 14 15 16 17 18 19 20 21	4 5 6 7 8 9 22 23 24 25 26 27
11	10 12	1 2 3 13 14 15 16 17 18 19 20 21	4 5 6 7 8 9 22 23 24 25 26 27
12	10 11	1 2 3 13 14 15 16 17 18 19 20 21	4 5 6 7 8 9 22 23 24 25 26 27
13	14 15	4 5 6 10 11 12 16 17 18 22 23 24	1 2 3 7 8 9 19 20 21 25 26 27
14	13 15	4 5 6 10 11 12 16 17 18 22 23 24	1 2 3 7 8 9 19 20 21 25 26 27
15	13 14	4 5 6 10 11 12 16 17 18 22 23 24	1 2 3 7 8 9 19 20 21 25 26 27
16	17 18	7 8 9 10 11 12 13 14 15 25 26 27	1 2 3 4 5 6 19 20 21 22 23 24
17	16 18	7 8 9 10 11 12 13 14 15 25 26 27	1 2 3 4 5 6 19 20 21 22 23 24
18	16 17	7 8 9 10 11 12 13 14 15 25 26 27	1 2 3 4 5 6 19 20 21 22 23 24
19	20 21	1 2 3 10 11 12 22 23 24 25 26 27	4 5 6 7 8 9 13 14 15 16 17 18
20	21 22	1 2 3 10 11 12 22 23 24 25 26 27	4 5 6 7 8 9 13 14 15 16 17 18
21	19 20	1 2 3 10 11 12 22 23 24 25 26 27	4 5 6 7 8 9 13 14 15 16 17 18
22	23 24	4 5 6 13 14 15 19 20 21 25 26 27	1 2 3 7 8 9 10 11 12 16 17 18
23	22 24	4 5 6 13 14 15 19 20 21 25 26 27	1 2 3 7 8 9 10 11 12 16 17 18
24	22 23	4 5 6 13 14 15 19 20 21 25 26 27	1 2 3 7 8 9 10 11 12 16 17 18
25	26 27	7 8 9 16 17 18 19 20 21 22 23 24	1 2 3 4 5 6 10 11 12 13 14 15
26	25 27	7 8 9 16 17 18 19 20 21 22 23 24	1 2 3 4 5 6 10 11 12 13 14 15
27	25 26	7 8 9 16 17 18 19 20 21 22 23 24	1 2 3 4 5 6 10 11 12 13 14 15

It is the No. 384 association scheme for 27 at the website:
http://kissme.shinshu-u.ac.jp/as/.

3.2 For Unsaturated Symmetric Orthogonal Arrays

There are some orthogonal arrays $L_{\lambda t^2}(t^{\lambda t+1})$ ($\lambda \geq 2$) constructed from difference matrices $D(\lambda t, \lambda t, t)$ by Bose and Bush. From Theorem 2.2, it is seen that in orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$, there are three Hamming distances. Therefore we have:

Theorem 3.2 *From orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$, a three-class association scheme can be constructed with parameters*

$$n_1 = t-1,\ n_2 = (\lambda-1)t,\ n_3 = \lambda t(t-1),$$

$$P_1 = \begin{pmatrix} t-2 & 0 & 0 \\ & (\lambda-1)t & 0 \\ & & \lambda t(t-1) \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & t-1 & 0 \\ & (\lambda-2)t & 0 \\ & & \lambda t(t-1) \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & t-1 \\ & 0 & (\lambda-1)t \\ & & \lambda t(t-2) \end{pmatrix}.$$

Proof. By Theorem 2.2, there are three Hamming distances in the orthogonal array. Define that two rows are first associates if their Hamming distance is λt, second associates if their distance is $\lambda(t-1)$ and third associates if their distance is $\lambda(t-1)+1$.

All the rows with distance λt to each other come from the same row in the difference matrix, there are t such rows, so $n_1 = t-1$.

For each row, its second associates are those rows corresponding to different rows in the difference matrix and same symbol in the juxtaposed column. There are $(\lambda-t)t$ such rows, so $n_2 = (\lambda-t)t$.

For each row, the third associates of it are those rows corresponding to different symbols in the juxtaposed column. There are $\lambda t(t-1)$ such rows in number, so $n_3 = \lambda t(t-1)$.

Let's find the intersection matrices.

(1). Taking any pair (x,y) of first associates, they correspond to the same row in the original difference matrix and same symbol in the juxtaposed column. But there are no common symbol in the pair of the corresponding entries of the two rows, and the t rows are self-closed, so $p^1_{11} = t-2$, $p^1_{12} = 0$ and $p^1_{13} = 0$.

For this pair, their common second associates are those corresponding to the same symbol in the juxtaposed column. There are $(\lambda-1)t$ such rows, so $p^1_{22} = (\lambda-1)t$.

Because x's second associates are those corresponding to the same symbol in the juxtaposed column, but y's third associates are those rows corresponding to the different symbols in the juxtaposed column. There are no common rows satisfying the condition, so $p^1_{23} = 0$.

Their common third associates are those corresponding to the different symbols of that corresponding to both x and y. There are $(t-1)$ such symbols and each symbol corresponds to λt rows, so $p^1_{33} = \lambda t(t-1)$.

(2). Taking any second associate pair (x,y). x and y correspond to different two rows in the original difference matrix and the same symbol in the juxtaposed column, so they have no common first associate, hence $p^2_{11} = 0$. The $(t-1)$ first associates of x all correspond to the same row in the original difference matrix but different symbols in the juxtaposed column, so $p^2_{12} = t-1$ and the similar reason leads to $p^2_{13} = 0$.

(x,y)'s common second associates should be those corresponding to different rows in the original difference matrix with respect to x and y. There are $(\lambda-2)t$ such rows, so $p^2_{22} = (\lambda-2)t$. Because that x's second associates correspond to the same symbol in the juxtaposed column with y, and y's third associates correspond to different symbols in the juxtaposed column with x, there are no common rows among them, so $p^2_{23} = 0$.

It's easy to see that $p^2_{33} = (t-1)\lambda t$.

(3). Taking any third associate pair (x, y). They correspond to different rows in difference matrix and different symbols in the juxtaposed column. So obviously, $p^3_{11} = 0$, $p^3_{12} = 0$ and $p^3_{13} = (t-1)$. Their common second associates should be those rows corresponding to the same symbol in juxtaposed column with x and y respectively. But x and y correspond different symbols in the juxtaposed column and there are no such rows, so $p^3_{22} = 0$.

Meanwhile, there are $(\lambda - 1)t$ rows corresponding to the same symbol with x in the juxtaposed column and different symbol with y in the juxtaposed column, so $p^3_{23} = (\lambda - 1)t$.

At last, the common third associates of (x, y) should be those rows corresponding to different symbols with both x and y in the juxtaposed column. Since x and y correspond λt columns respectively, so $p^3_{33} = \lambda t^2 - 2\lambda t = \lambda t(t-2)$.

Hence the rows of the orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$ form an association scheme with respect to Hamming distance. ∎

From $L_{\lambda t^2}(t^{\lambda t+1})$, we may construct another association scheme as well.

In the construction of $L_{\lambda t^2}(t^{\lambda t+1})$, when the entries of $D(\lambda t, \lambda t, t)$ are replaced by the entries of additive table, we obtain $L_{\lambda t^2}(t^{\lambda t})$. There are two Hamming distances. For any row in $D(\lambda t, \lambda t, t)$, after replacement, it produce t new rows and there is no coincidence between any two such rows. If we define that such two rows are 1st associates each other, we have $n_1 = t - 1$.

Any two rows in $D(\lambda t, \lambda t, t)$ produce two groups of t rows each after replacement. Each new row has distance λt with those who are at the same position in the different groups. For a chosen row, there are $\lambda t - 1$ such rows, so $n_2 = \lambda - 1$. Other relations are defined as the 3rd associates and we have $n_3 = (\lambda t - 1)(t - 1)$.

In fact, for a given $L_{\lambda t^2}(t^{\lambda t+1})$, we need only to delete the juxtaposed column and any other one, so that there are three Hamming distances in the array. By the similar proof in 3.2, we have:

Theorem 3.3 *Let there be an orthogonal array $L_{\lambda t^2}(t^{\lambda t+1})$. By deleting the juxtaposed column in the construction and any other one, a 3-class association scheme is obtained with parameters:*

$$v = \lambda t^2,\ n_1 = t-1,\ n_2 = \lambda - 1,\ n_3 = (\lambda t - 1)(t-1),$$

$$P_1 = \begin{pmatrix} t-2 & 0 & 0 \\ & 0 & \lambda t - 1 \\ & & (t-2)(\lambda t - 1) \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & t-1 \\ & \lambda t - 2 & 0 \\ & & (\lambda t - 2)(t-1) \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 1 & t-2 \\ & 0 & \lambda t - 2 \\ & & (\lambda t - 2)(t-2) \end{pmatrix}.$$

It is seen that from both saturated and unsaturated symmetric orthogonal arrays, an association scheme with respect to Hamming distances can be obtained. As summary, we have the following theorem.

Theorem 3.4 *(1). Let there be a saturated symmetric orthogonal array $OA(n,m,t,s)$, all the orthogonal arrays $OA(n,m-r,t,s)$ obtained by deleting r ($r \leq s$) columns from it are schematic.*
(2). All the symmetric orthogonal arrays in form of $L_{\lambda t^2}(t^{\lambda t+1})$ are schematic.

Notice that Delsarte [9] gave a sufficient condition for an orthogonal array to be schematic.

Theorem 3.5 *(Delsarte) Let σ be the number of distinct nonzero Hamming distances between the runs of an $OA(n,m,t,s)$. If $s \geq 2\sigma - 2$ then the orthogonal array is schematic.*

According to the Theorem 3.5, the orthogonal arrays constructed from Latin squares, Hadamard matrices and vector spaces over Galois fields are all schematic.

The condition of Theorem 3.5 is not necessary. An orthogonal array constructed from a difference matrix does not satisfy the condition in 3.5. But we can still construct association scheme from it, so it is schematic as well.

So far, we have given the answer of Hedayat's open problem (see [12])for symmetric orthogonal arrays with strength two.

Example

There is the orthogonal array $L_{18}(3^7)$ as follows. (transposed):

1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	2	3	1	3	1	2	2	3	1	3	1	2
1	2	3	2	3	1	1	2	3	3	1	2	3	1	2	2	3	1
1	2	3	2	3	1	3	1	2	2	3	1	1	2	3	3	1	2
1	2	3	3	1	2	2	3	1	2	3	1	3	1	2	1	2	3
1	2	3	3	1	2	3	1	2	1	2	3	2	3	1	2	3	1

By the method of Theorem 3.2, we obtain the association scheme as follows.

$$v = 18, n_1 = 2, n_2 = 3, n_3 = 12$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 3 & 0 \\ & & 12 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 2 & 0 \\ & 0 & 0 \\ & & 12 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & 2 \\ & 0 & 3 \\ & & 6 \end{pmatrix}.$$

The scheme is listed in Table 2.

By the method of Theorem 3.3, by deleting the juxtaposed column and second columns of $L_{18}(3^7)$, we obtain a 3-class association scheme with parameters:

$$v = 18,\ n_1 = 2,\ n_2 = 5,\ n_3 = 10$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 5 \\ & & 5 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 2 \\ & 4 & 0 \\ & & 8 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 4 \\ & & 4 \end{pmatrix}.$$

Table 2: Table of first association scheme obtained from $L_{18}(3^7)$

run	1st assoc.	2nd assoc.	3rd assoc.
1	2 3	10 11 12	4 5 6 7 8 9 13 14 15 16 17 18
2	1 3	10 11 12	4 5 6 7 8 9 13 14 15 16 17 18
3	1 2	10 11 12	4 5 6 7 8 9 13 14 15 16 17 18
4	5 6	13 14 15	1 2 3 7 8 9 10 11 12 16 17 18
5	4 6	13 14 15	1 2 3 7 8 9 10 11 12 16 17 18
6	4 5	13 14 15	1 2 3 7 8 9 10 11 12 16 17 18
7	8 9	16 17 18	1 2 3 4 5 6 10 11 12 13 14 15
8	7 9	16 17 18	1 2 3 4 5 6 10 11 12 13 14 15
9	7 8	16 17 18	1 2 3 4 5 6 10 11 12 13 14 15
10	11 12	1 2 3	4 5 6 7 8 9 13 14 15 16 17 18
11	10 12	1 2 3	4 5 6 7 8 9 13 14 15 16 17 18
12	10 11	1 2 3	4 5 6 7 8 9 13 14 15 16 17 18
13	14 15	4 5 6	1 2 3 7 8 9 10 11 12 16 17 18
14	13 15	4 5 6	1 2 3 7 8 9 10 11 12 16 17 18
15	13 14	4 5 6	1 2 3 7 8 9 10 11 12 16 17 18
16	17 18	7 8 9	1 2 3 4 5 6 10 11 12 13 14 15
17	16 18	7 8 9	1 2 3 4 5 6 10 11 12 13 14 15
18	16 17	7 8 9	1 2 3 4 5 6 10 11 12 13 14 15

The scheme is listed in Table 3.

There are some other methods to construct symmetric orthogonal arrays. Shrikhande [21] gave a method by the Kronecker sum of an orthogonal array and a difference matrix. It is stated as follows.

Theorem 3.6 *(Shrikhande) If there exists an orthogonal array $L_{\mu m}(m^n)$ and a difference matrix $D(\lambda m, r, m)$, then $D = L_{\mu m}(m^n) \oplus D(\lambda m, r, m)$ is an orthogonal array $L_{\lambda \mu m^2}(m^{rn})$.*

In the theorem, the Kronecker sum of two matrices $A \oplus B$ means adding every entry of B to the whole matrix A, then the order of the sum is the product of the orders of the two matrices. If the given difference matrix is symmetric, it is equidistant. After Kronecker sum, the Hamming distances of the rows depend on the distances of the orthogonal array, so we have:

Corollary 3.1 *By Shrikhande's construction, if $D(\lambda m, r, m)$ is symmetric and $L_{\mu m}(m^n)$ is schematic, then $D = L_{\mu m}(m^n) \oplus D(\lambda m, r, m)$ is a schematic orthogonal array.*

4 Some Asymmetric Orthogonal Arrays

In design of experiments, there is a demand for the factors to take different numbers of levels, so orthogonal arrays with different numbers of levels

Table 3: Table of second association scheme obtained from $L_{18}(3^7)$

run	1st assoc.	2nd assoc.	3rd assoc.
1	2 3	4 7 10 13 16	5 6 8 9 11 12 14 15 17 18
2	1 3	5 8 11 14 17	4 6 7 9 10 12 13 15 16 18
3	1 2	6 9 12 15 18	4 5 7 8 10 11 13 14 16 17
4	5 6	1 7 10 13 16	2 3 8 9 11 12 14 15 17 18
5	4 6	2 8 11 14 17	1 3 7 9 10 12 13 15 16 18
6	4 5	3 9 12 15 18	1 2 7 8 10 11 13 14 16 17
7	8 9	1 4 10 13 16	2 3 5 6 11 12 14 15 17 18
8	7 9	2 5 11 14 17	1 3 4 6 10 12 13 15 16 18
9	7 8	3 6 12 15 18	1 2 4 5 10 11 13 14 16 17
10	11 12	1 4 7 13 16	2 3 5 6 8 9 14 15 17 18
11	10 12	2 5 8 14 17	1 3 4 6 7 9 13 15 16 18
12	10 11	3 6 9 15 18	1 2 4 5 7 8 13 14 16 17
13	14 15	1 4 7 10 16	2 3 5 6 8 9 11 12 17 18
14	13 15	2 5 8 11 17	1 3 4 6 7 9 10 12 16 18
15	13 14	3 6 9 12 18	1 2 4 5 7 8 10 11 16 17
16	17 18	1 4 7 10 13	2 3 5 6 8 9 11 12 14 15
17	16 18	2 5 8 11 14	1 3 4 6 7 9 10 12 13 15
18	16 17	3 6 9 12 15	1 2 4 5 7 8 10 11 13 14

are needed. Such arrays are called asymmetric (or mixed-level) orthogonal arrays.

There is a method to construct asymmetric orthogonal array by Addelman [1]: suppose that an orthogonal array of strength 2 has a column involving m symbols, and let m' be a positive integer such that $m = 0 \pmod{m'}$. Then the m-symbol column can be collapsed into an m'-symbol column by first grouping the m symbols into m' sets of m/m' symbols each and then replacing the symbols belonging to the same set by a common symbol. It is an orthogonal array of strength 2. For example, the array $OA(16,5,4,2)$ can be used in the way to construct asymmetric orthogonal array $OA(16,5,2 \times 4^4,2)$.

If the given array is produced from Latin square or vector space over Galois field, there is one Hamming distance. After substituting by m' sets of m/m' symbols, the rows are grouped into m' groups according to the sets. There is one Hamming distance in the same group while in different groups there are two Hamming distances. So the rows of the new orthogonal array form a group divisible association scheme. It can be shown as following theorem.

Theorem 4.1 *The derived asymmetric orthogonal array by Addelman's method from the saturated orthogonal array is a schematic orthogonal array.*

In an Hadamard matrix $H_n (n \geq 4)$, a set of three distinct columns of H_n is said to have the Hadamard property if the Hadamard product of any two columns in the set equals the third (Hadamard product of two vectors $\mathbf{a} = (a_1, a_2, \cdots, a_t)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_t)$ is defined as $\mathbf{a} * \mathbf{b} = (a_1 b_1, a_2 b_2, \cdots, a_t b_t)$).

The following result belong to Wang [22].

Lemma 4.1 *(Wang's Lemma) Let H_n be an Hadamard matrix and suppose that there is a set of three columns of H_n having the Hadamard property. Denote these columns by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then in any $n \times 4$ submatrix of H_n made up of the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{c}$, where $\mathbf{c}$ is any column of H_n other than $\mathbf{1}_n$, and the three with the Hadamard property, each of the eight vectors*

$$(1,1,1,\pm 1), (1,-1,-1,\pm 1), (-1,1,-1,\pm 1), (-1,-1,1,\pm 1)$$

occurs equally often as a row.

According to Cheng [8] and others, several asymmetric orthogonal arrays of strength two can be constructed by Hadamard matrices, as summarized in [10].

If an Hadamard matrix contains a set of three columns with Hadamard property, replace the 4 rows $(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$ in the three columns with the Hadamard property by the symbols 0,1,2 and 3 respectively and delete the initial column $\mathbf{1}_n$ of H_n by Wang's lemma, then an $n \times (n-3)$ matrix is obtained. It is an asymmetric orthogonal array $OA(n, n-3, 4 \times 2^{n-4}, 2)$. We shall show that the asymmetric OA constructed in such way is a schematic orthogonal array.

Corollary 4.1 *The asymmetric orthogonal array $OA(n, n-3, 4 \times 2^{n-4}, 2)$ constructed in the above way is a schematic orthogonal array.*

Proof. It can be seen that in the chosen three columns, the rows

$$(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$$

are equidistant. In the replacement, they are repeated for the same times.

In H_n, the rows fall into two cases: two rows corresponding to the same symbol in the new column have one distance, those two corresponding to different symbols in the new column have another distance. So they are grouped. The rows in the asymmetric $OA(n, n-3, 4 \times 2^{n-4}, 2)$ form a group divisible association scheme according to Hamming distances, so the orthogonal array is schematic . ∎

For instance, we have an $OA(8, 5, 4 \times 2^4, 2)$ from $H_8 = H_2 \otimes H_4$ by replacement. It is a schematic orthogonal array ([10]pp.52).

The column of 1 in H_8 is deleted and the rows of the three columns that have Hadamard property are replaced by 0,1,2 and 3 respectively. Then the asymmetric orthogonal array $OA(8, 5, 4 \times 2^4, 2)$ is obtained. It is shown as follows.

$$H_8 = \begin{pmatrix} +\vdots & + & + & +\vdots & + & + & + & + \\ +\vdots & - & + & -\vdots & + & - & + & - \\ +\vdots & + & + & +\vdots & - & - & - & - \\ +\vdots & - & + & -\vdots & - & + & - & + \\ +\vdots & + & - & -\vdots & + & + & - & - \\ +\vdots & - & - & +\vdots & + & - & - & + \\ +\vdots & + & - & -\vdots & - & - & + & + \\ +\vdots & - & - & +\vdots & - & + & + & - \end{pmatrix} \rightarrow \begin{pmatrix} 0 & + & + & + & + \\ 2 & + & - & + & - \\ 0 & - & - & - & - \\ 2 & - & + & - & + \\ 1 & + & + & - & - \\ 3 & + & - & - & + \\ 1 & - & - & + & + \\ 3 & - & + & + & - \end{pmatrix}.$$

In $OA(8, 5, 4 \times 2^4, 2)$, the first row and the third row have one distance, while the first row and the second have another distance and so on. All the rows form an association scheme, hence the obtained array is a schematic orthogonal array.

In the Hadamard matrix satisfying the condition of Wang's lemma, there are three Hamming distances of the chosen eight rows. It is not difficult to see that the given eight rows form a 3-class association scheme with parameters

$$v = 8, n_1 = 1,\ n_2 = 3,\ n_3 = 3,$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 3 \\ & & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 1 \\ & 2 & 0 \\ & & 2 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 0 \end{pmatrix}.$$

Suppose there is an $H_n, (n \geq 16)$ with three columns having Hadamard property. If we replace the rows of the three columns and one other column together with symbols 1,2,...,8, then we obtain an asymmetric orthogonal array $OA(n, n-4, 8 \times 2^{n-3}, 2)$. In the submatrix with the eight rows, there are 4 Hamming distances (one distance is zero because of repeating). Since the original matrix is equidistant, after replacement, there are 4 distances of the rows in the array according to the distances of the chosen 8 rows in the submatrix. But the rows in the submatrix form an association scheme. If the rows in the submatrix are repeated, according to Lemma 3.1, they also form an association scheme with 4 classes. So equivalently, the rows in the asymmetric array form an association scheme as well, it is a schematic orthogonal array.

We may generalize the discussion and obtain the following result.

Theorem 4.2 *If the asymmetric orthogonal array is obtained by replacing the rows of a submatrix in an Hadamard matrix H_n by a set of finite number and the rows of the submatrix form an association scheme, then the asymmetric orthogonal array is a schematic orthogonal array.*

The idea of Theorem 4.2 leads to another case of schematic orthogonal array.

Theorem 4.3 *If the orthogonal array A_1 is equidistant and A_2 is schematic. Juxtapose the two arrays, if the result is an asymmetric orthogonal array, then it must be a schematic orthogonal array.*

The discussion above deals with adding or subtracting columns of the orthogonal array. But what if we add some rows to an orthogonal array?

In design of experiments, there is a method of foldover to obtain a higher resolution of the 2-level factorial design by orthogonal arrays [6].

By the method of foldover, we can obtain schematic orthogonal arrays as well.

Lemma 4.2 *Let there be an association scheme with parameters*

$$v, n_1, n_2, \cdots n_d, p^i_{jk}(i, j, k = 1, 2, \cdots, m).$$

If we duplicate the scheme and define one more associate class between any symbols in the different schemes, then we obtain a $(d+1)$-class association scheme with parameters

$$\hat{v} = 2v, \hat{n}_1 = n_1, \hat{n}_2 = n_2, \cdots \hat{n}_d = n_d, \hat{n}_{d+1} = v,$$

$$\hat{p}^i_{jk} = \begin{cases} p^i_{jk} & if\ i,j,k \leq d+1 \\ 0 & if\ i \leq d,\ j = d+1, \\ v & if\ i \leq d\ j = k = d+1 \end{cases} \quad \hat{P}_{d+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & n_1 \\ & 0 & \cdots & 0 & n_2 \\ & & \cdots & \cdots & \cdots \\ & & & 0 & n_d \\ & & & & 0 \end{pmatrix}.$$

Proof. For the two schemes with the same parameters, we define the original associate relations unchanged within the schemes and define two symbols in different schemes to be the $(d+1)$-th associates. It is easy to see that for $i, j, k \leq d$, p^i_{jk} is unchanged.

When $i \leq d$ and $j = d+1$, for a pair of i-th associate (x, y), if $k \leq d$, the k-th associates of x are those that are in the same scheme with x. Meanwhile, they are $(d+1)$-th associates of y. But there are no such symbols because x, y are in the same scheme, so $\hat{p}^i_{jk} = 0$ if $i \leq d, j = d+1$. If $j = k = d+1$, then $\hat{p}^i_{jk} = v$.

For a given pair of $(d+1)$-th associates x, y, they are in the different schemes, so all n_j-th associates of x must be $(d+1)$-th associates of y. So in $\hat{P}_{d+1}$, only the last column has nonzero elements but the last one. It is easy to see that for $i, j, k = d+1$, $\hat{p}^i_{jk} = 0$. So the last column of $\hat{P}_{d+1}$ is

$$(n_1, n_2, \cdots, n_d, 0)'.$$

∎

By Shrikhande's Theorem 3.6, Wang and Wu [23] gave a method of constructing asymmetric orthogonal array, we state it in a simple form (for the case of $i = 2$).

Theorem 4.4 *Let A be an orthogonal array $OA(n, n_1 + n_2, m_1^{n_1} m_2^{n_2}, 2)$ and suppose that there exist two difference matrices $D(M, k_i, m_i)(i = 1, 2)$,* *where n and M are both multiples of the m_i's. Partition A as $A = [C_1 \vdots C_2]$, where C_i is a symmetric orthogonal array $OA(n, n_i, m_i, 2)$ with symbols from the Abelian group over which $D(M, k_i, m_i)$ is defined, then by the generalized Kronecker sum*

$$[C_1 \oplus D(M, k_1, m_1) \vdots C_2 \oplus D(M, k_2, m_2)] \quad ,$$

the matrix is an orthogonal array $OA(Mn, k_1 n_1 + k_2 n_2, m_1^{k_1 n_1} m_2^{k_2 n_2}, 2)$.

By the Corollary 3.1, in the generalized sum of an orthogonal array and a difference matrix, if the orthogonal array is schematic and others are equidistant, then the rows of the new array depend on that of the schematic orthogonal array. Since a difference matrix is equidistant, we have:

Corollary 4.2 *In the construction of Theorem 4.4, if one of the orthogonal array is schematic and the other orthogonal array and difference matrices are equidistant, then the orthogonal array generated is a schematic orthogonal array.*

Many symmetric orthogonal arrays are constructed via difference matrices, there are some methods to do so for asymmetric orthogonal arrays as well. Wang and Wu [23] gave a method.

If Δ represents a difference matrix $D(\lambda m, n, m)$ and $\Delta_i = \Delta \oplus i (i = 0, 1, 2, \cdots, m-1)$, then develop Δ as $[\Delta'_0, \Delta'_1, \cdots, \Delta'_{m-1}]'$. Construct

$$\begin{pmatrix} \Delta'_0 & \Delta'_1 & \cdots & \Delta'_{m-1} \\ \mathbf{a}' & \mathbf{a}' & \cdots & \mathbf{a}' \end{pmatrix}',$$

where $\mathbf{a} = (1, 2, \cdots \lambda m)'$, the array is $OA(\lambda m^2, n+1, m^n \times (\lambda m), 2)$.

Then we have the lemma (Dey and Mukerjee [10], pp. 62).

Lemma 4.3 *The existence of a difference matrix $D(\lambda m, n, m)$ implies that of an orthogonal array $OA(\lambda m^2, n+1, m^n \times (\lambda m), 2)$.*

From the lemma above, we have the following theorem.

Theorem 4.5 *The asymmetric orthogonal array $OA(\lambda m^2, n+1, m^n \times (\lambda m), 2)$ constructed in Lemma 4.3 is schematic.*

Proof. In Lemma 4.3, Δ is the given difference matrix, $\Delta_i = \Delta \oplus i, i = 0, 1, \cdots, m-1$. The entries of a row in Δ'_i are totally different to their correspondences in Δ'_j. The Hamming distance of such two rows is n. But the rows within Δ'_i and out of Δ'_i except their correspondences are equidistant. So the rows in the array are grouped according to their positions in Δ'_i. The rows in the same position are in one class, the others are in another class. They form a group divisible association scheme, hence the array is a schematic orthogonal array. ■

The following example is also in [10], it is a schematic orthogonal array.

$$OA(18,7,3^6\times 6,2)=\begin{pmatrix} 022011 & 100122 & 211200 \\ 121020 & 202101 & 010212 \\ 112002 & 220110 & 001221 \\ 000000 & 111111 & 222222 \\ 201012 & 012120 & 120201 \\ 210021 & 021102 & 102210 \\ 123456 & 123456 & 123456 \end{pmatrix}'.$$

Acknowledgement The paper is supported by Beijing Information Science and Technology University.

References

[1] Addelman, S., Orthogonal main effect plans for asymmetrical factorial experiments, *Technometrics*, **4**(1962), 21-46.

[2] Addelman, S. and Kempthorne, O., Some main-effect plans and orthogonal arrays of strength two. *Ann Math. Statist.*, **32** (1961), 1167-1176.

[3] Dey, A., R. Mukerjee , *Fractional Factorial Plans*, John wiley & Sons, Inc, New York, (1999).

[4] Atsumi, T. , A study of orthogonal arrays from the point of view of design theory. *J. Combin. Theory.* A **35**(1983), 241-251.

[5] Bose, C. R., Bush, K. A., Orthogonal arrays of strength two and three. Ann. Math. Statist., **23**(1952), 508-524.

[6] Box, G. E. P. and Hunter, J. S. , The 2^{k-p} fractional factorial designs,*Technometrics*, **3**(1961), 311-351 and 449-458.

[7] Brouwer, A. E., Cohen, A. M. and Neumaier, A., *Distance-regular graphs*, Springer-Verlag,(1989).

[8] Cheng, C. S., Some orthogonal main effect plans for asymmetrical factorials, *Technometrics*,**31**, 475-477.

[9] Delsarte, P. , An algebraic approach to the association schemes of coding theory. *Philips Res. Reports Suppl.*, **10**(1973).

[10] Dey, Aloke and Mukerjee, Rahul , *Fractional factorial plans*, John Wiley & Sons, Inc, (1999).

[11] Hedayat, A. S.,K. Pu and J. Stufken , On the construction of asymmetrical orthogonal arrays, *Ann. Statist.*20, (1992)2142-2152.

[12] Hedayat, A. S., N. J. A., Sloane and John Stufken *Orthogonal arrays*, Springer-Verlag, New York,(1999).

[13] Jungnickel, D. , On difference matrices, resolvable transversal designs and generalized Hadamard matrices. *Math. Z.* **167**, (1979) 49-60.

[14] Liu, Z. W., The construction of difference matrix for orthogonal array $(2p^2, 2p+1, m, 2)$ with odd prome p. *Acta Mathematicae Applicatae Sinica*, vol1,3,(1977), 35-45.

[15] Mukerjee, R. and Wu, C. F. Jeff , On the existence of saturated and nearly saturated asymmetrical orthognonal arrays. *Ann. stat.*, **23**, 6, (1995),2102-2115.

[16] Mukerjee, R. and Kageyama, S. , On existence of two symbol complete orthogonal arrays. *J. Combin. Thorey Ser.A* **66**, (1994) 176-181.

[17] Masuyama, M., Construction of difference sets for $OA(2p^2, 2P+1, p, 2)$, p being an odd prime, Rep. Statist. Appl. Res., JUSE, **16** (1969) 1-9.

[18] Raghavarao, D. *Constructions and combinatorial problems in design of experiments*, Dover, New York,(1988).

[19] Rao, C. R. , Some combinatorial problems of arrays and applications to design of experiments, In *A Survey of Combinatorial Theory* (J. N. Srivastava. ed.), 349-359, Amsterdam: North-Holland,(1973).

[20] Sloane, N. J. A. , An introduction to association schemes and coding theory. Pages 225-260 of Alkey, R.A.(ed), *Theory of Applications of Special functions*, New York: Acdemic Press,(1975).

[21] Shrikhande, S. S., Generalized Hadamard matrices and orthogonal arrays of strength two. *Canadian J. Math.* **16** (1964) 736-740.

[22] Wang, P. C. , On the construction of some orthogonal main efffect plans, *Sankyā* **b52**, (1990)319-323.

[23] Wang, J. C. and Wu, C. F. Jeff , An approach to the construction of asymmetrical orthgonal arrays, *J. Amer. Statist. Assoc.* **86**, (1991)450-456.

[24] Xiang, K. F. , The difference matrix with $\lambda = 2$. *Acta Mathematicae Applicatae Sinica*, vol**6**, 2,(1983) 160-166.

[25] Yoshizawa, M. , On schematic orthogonal arrays $OA(q^t, k, q, t)$. *Graphs combin.*, **3**,(1987) 81-89.