Maximum Packing of $D_t - P$ and $D_t \cup P$ with Mendelsohn Triples

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Abstract

In this paper, we extend the study on packing and covering of complete directed graph D_t with Mendelsohn triples [6]. Mainly, the maximum packing of $D_t - P$ and $D_t \cup P$ with Mendelsohn triples are obtained respectively where P is a vertex-disjoint union of directed cycles in D_t.

Keywords: Mendelsohn triples, Complete digraph, Maximum Packing.

1 Introduction

A Steiner triple system of order t denoted $STS(t)$, is a pair $(V, \mathcal{B})$ where V is a t-set and $\mathcal{B}$ is a collection of 3-element subsets (called triples) of V such that each pair of elements occurs in a unique triple of $\mathcal{B}$. It is well known that an $STS(t)$ exists if and only if $t \equiv 1$ or $3 \pmod 6$. In terms of graph decomposition, an $STS(t)$ can also be viewed as a partition of the edges of K_t, each element of which induces a triangle C_3; we denote such a decomposition by $C_3|K_t$.

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A packing of a graph G with triangles is a partition of the edge set of a subgraph H of G, each element of which induces a triangle; the remainder graph of this packing, also known as the leave, is the subgraph $G - H$ formed from G by removing the edges in H. If the remainder graph is minimum in size (that is, has the least number of edges among all possible leaves of G), then the packing is called a maximum packing. The following result is well-known.

Theorem 1.1 [5] The remainder graph P for the maximum packings of K_t with triangles are as follows:

t(mod 6)	0	1	2	3	4	5
P	F	$\emptyset$	F	$\emptyset$	F_1	C_4

F is a 1-factor, F_1 is an odd spanning forest with $\frac{t}{2}+1$ edges (tripole), and C_4 is a cycle of length 4.

It is natural to ask for which subgraphs H of K_t, $C_3 |\ K_t - H$. When t is odd and H is a 2-regular graph, the following result has been obtained by Colbourn and Rosa [3].

Theorem 1.2. [3] Let t be an odd positive integer. Let H be a 2-regular subgraph of K_t. If $t = 9$, then suppose that $H \neq C_4 \cup C_5$. Then $C_3 |\ K_t - H$ if and only if the number of edges in $K_t - H$ is a multiple of 3.

A covering of a graph G with triangles is a collection of triangles, $\mathcal{P}$, such that each edge of G occurs in at least one triangle in $\mathcal{P}$. So, if $G(\mathcal{P})$ is the multigraph formed by joining each pair of vertices u and v with x edges if and only if $\mathcal{P}$ contains x triples that contain both u and v, then clearly $C_3 |\ G(\mathcal{P})$. The multigraph $G(\mathcal{P}) - G$ is called the excess graph of G; it is also known as the padding of the covering $\mathcal{P}$ of G. A covering with smallest excess graph (in size) is called a minimum covering. The following result which is the companion of Theorem 1.1. considers minimum covering.

Theorem 1.3. [5] The excess graph P for the minimum coverings of K_t with triangles are as follows:

t(mod 6)	0	1	2	3	4	5
P	F	$\emptyset$	F_1	$\emptyset$	F_1	C_2

F is a 1-factor, F_1 is an odd spanning forest with $\frac{t}{2}+1$ edges (tripole), and C_2 is a cycle of length 2.

Colbourn and Rosa got the following result about covering and the result is also proved by C. M. Fu, H. L. Fu and C. A. Rodger using a different method [4].

Theorem 1.4. [2] Let H be a 2-regular (not necessarily spanning) subgraph of K_t. Then $C_3|\ K_t \cup H$ if and only if the number of edges in $K_t \cup H$ is a multiple of 3.

Similarly, corresponding problem about packing and covering of directed graphs can be considered.

A Mendelsohn triple system of order t, denoted $MTS(t)$, is a pair $(V, \mathcal{B})$ where V is a t-set, $\mathcal{B}$ is a collection of cyclically ordered 3-subsets of V(called Mendelsohn triples) such that each ordered pair of V appears in exactly one Mendelsohn triple of $\mathcal{B}$. In terms of graph decomposition, the existence of an $MTS(t)$ is equivalent to partition the directed edges (or edges in short) of the complete directed graph of order t, D_t into a collection of directed 3-cycles (directed $C_3's$). We denote such a decomposition by $\vec{C_3}\ |D_t$. The following is well-known. When $t \equiv 0,1 \pmod 3$, $t \neq 6$, there exists an $MTS(t)$ [1] .

Quite recently, L. Pu, H. L. Fu and H. Shen extend the work of Theorem 1.2 and Theorem 1.4. and prove the following result.

Theorem 1.5 [6]. Let t be an integer, $t \geq 13$. Let D_t be a complete directed graph without loops and its order is t. Let P be a vertex-disjoint union of directed cycles in D_t and $|E(P)|$ be the number of edges of P. Then $\vec{C_3}\ |D_t - P$ (or $\vec{C_3}\ |D_t \cup P$) if and only if $t(t-1) - |E(P)| \equiv 0 \pmod 3$ (or $t(t-1) + |E(P)| \equiv 0 \pmod 3$).

In this paper, we further extend the work of Theorem 1.1, Theorem 1.3 and Theorem 1.5. Mainly, we obtain the maximum packings of $D_t - P$ and $D_t \cup P$ with Mendelsohn triples respectively.

2. Preliminaries

Let C_{l_i} denote a cycle of length l_i and $V(C_{l_j})$ be the set of vertices of C_{l_j}. Also, let $P_n = \cup_{i=1}^{k} C_{l_i}$ whenever $n = \sum_{i=1}^{k} l_i$ and $V(C_{l_i}) \cap V(C_{l_j}) = \emptyset$ $(i \neq j)$. For convenience, we use $k \vec{C_t}$ to denote k vertex-disjoint directed t-cycles. If P_n is oriented, we get P_n^+. And P_n^- is obtained by reversing the direction of all edges in P_n^+.

Let A be an m-set, B be an n-set and $A \cap B = \emptyset$. A complete bipartite directed graph $D_{A,B}$ $(D_{m,n})$ with two partite sets A and B contains $2mn$ directed edges. Therefore, $D_{m+n} = D_m + D_n + D_{m,n}$ and $D_{m+1} = D_m + D_{m,1}$.

The join of two directed graphs G_1 and G_2 is denoted by $G_1 \vee G_2$. Then A($G_1 \vee G_2$)= A(G_1) $\cup$ A(G_2) $\cup$ $A(D_{|V(G_1)|,|V(G_2)|})$ where A(G) denotes the set of edges of G.

Definition 2.1. Let H be a 2-factor of G. If H is oriented such that for each vertex w in H, $\deg^+(w) = \deg^-(w) = 1$, then H is a directed 1-factor of G.

Lemma 2.1. Let C be a directed 1-factor. Then $\vec{C_3} \mid \vec{C} \vee K_1$ where $V(\vec{C}) \cap V(K_1) = \emptyset$.

The lemma can be deduced from the following example immediately.

Example 1. Let A $= Z_5$, B $= \{\infty\}$ and $\vec{C_5} = (0,1,2,3,4)$. The graph T contains a single point ∞. Then $T \vee \vec{C_5} = D_{A,B} + \vec{C_5} = \{(0,1,\infty),$ $(1,2,\infty)$, $(2,3,\infty)$, $(3,4,\infty)$, $(4,0,\infty)\}$. The following Lemma is useful and we adopt a difference method approach. Here we assume that the readers are familiar with the method of construction triple systems via difference triples [7, 8].

Throughout the rest of paper all packings are obtained by using Mendelsohn triples and thus we will not mention this again and again. For convenience, we use (a,b) to denote a directed cycle of length 2 and a directed edge from a to b is denoted by ab for simplicity.

Lemma 2.2. $D_4 - \vec{C_4}$, $D_7 - P_7$, $D_{10} - P_{10}$ and $D_{13} - P_{13}$ can be packed with leave $\vec{C_2}$.

Proof. The proof is by direct constructions. Let D_4 be defined on Z_4 where $\vec{C_4} = (0,1,2,3)$ is the missing cycle. Then $D_4 - \vec{C_4} = \{(0,3,2),(0,2,1)\} \cup \{(1,3)\}$. Let D_7 be defined on Z_7 where $\vec{C_7} = (0,1,2,3,4,5,6)$ is the missing cycle. Then $D_7 - \vec{C_7} = \{(2,0,6),\ (3,0,4),\ (4,0,5),\ (1,0,3),\ (4,1,6),\ (5,0,2),\ (3,5,1),\ (4,6,1),\ (3,2,6),\ (5,2,1),\ (5,3,6)\} \cup \{(4,2)\}$. Let D_7 be defined on Z_7 where $(0,1),(2,3)$ and $(4,5,6)$ are the missing cycles. Then $D_7 - 2\vec{C_2} \cup \vec{C_3} = \{(2,0,6),\ (2,6,5),\ (4,0,5),\ (2,1,4),\ (3,6,0),\ (5,0,2),\ (3,0,4),\ (3,5,1),\ (4,1,2),\ (1,5,3),\ (4,6,3)\} \cup \{(1,6)\}$. Let D_7 be defined on $Z_5 \cup \{\infty_0,\infty_1\}$ and $\vec{C_5} \cup \vec{C_2} = (0,4,3,2,1) \cup (\infty_0,\infty_1)$. Then $D_7 - \vec{C_5} \cup \vec{C_2} = \{(3,2,\infty_0),\ (2,1,\infty_0),\ (1,0,\infty_0),\ (0,3,\infty_0),\ (0,2,\infty_1),\ (2,4,\infty_1),\ (4,1,\infty_1),\ (1,3,\infty_1),\ (3,0,\infty_1),\ (3,1,4),\ (4,2,0)\} \cup \{(4,\infty_0)\}$. Let D_7 be defined on Z_7 and $\vec{C_4} \cup \vec{C_3} = (0,1,2,3) \cup (4,5,6)$. Then $D_7 - \vec{C_4} \cup \vec{C_3} = \{(2,0,6),\ (2,6,5),\ (4,0,5),\ (2,1,4),\ (3,6,0),\ (0,4,1),\ (5,0,2),\ (1,5,3),\ (6,3,4),\ (4,3,2),\ (3,5,1)\} \cup \{(1,6)\}$. As for $D_{10} - P_{10}$ and $D_{13} - P_{13}$, the packings can be found in Appendix A and Appendix B respectively . □

Theorem 2.2. For $t \equiv 1,4 \pmod 6$ and $t \geq 13$, $D_t - P_t$ can be packed with leave $\vec{C_2}$ if and only if $t(t-1) - |E(P)| \equiv 2 \pmod 3$.

Proof. The necessity is obvious. We prove the sufficiency by induction on t. Lemma 2.2 shows that it is true for $t = 4,7,10,13$ except when $t=4$ and $D_4 - 2\vec{C_2}$. Assume the case is true for smaller t, we shall prove it is true for t. First we consider the case $t = 6k+1$. Then $|E(P)| \equiv 1$ (mod 6). Let $|E(P)| = 6h+1$, $0 < h < k$ or $|E(P)| = 6k+1$. For the former case, we can add Mendelsohn triples T to P where $V(T) \cap V(P) = \emptyset$. Therefore, it suffices to consider the case $|E(P)| = 6k+1$. We divide the proof into four cases when k is even.

Case 1. $P^+_{6k+1} = P^+_{3k+6} + P^+_{3k-5}$.

Then $D_{6k+1} - P^+_{6k+1} = [(D_{3k+6} - P^+_{3k+6} - P^-_{3k+6}) + D_{3k+6,3k-6}] + (D_{3k+6,1} + P^-_{3k+6}) + (D_{3k-5} - P^+_{3k-5}) = \text{(I)} + \text{(II)} + \text{(III)}$. By the hypothesis, $\vec{C_3} \mid \text{(III)}$. By Lemma 2.1, $\vec{C_3} \mid \text{(II)}$. In order to prove $\vec{C_3} \mid \text{(I)}$, write $2v = 3k+6$. Denote the $(2v)$-set as V and $V = \{i_j \mid i \in Z_v, j = 0,1\}$. We first map the directed cycles of total length $2r$ onto the $2r$ vertices $\{i_j\}$ as follows. Every even cycle of length $2m$ is mapped onto the vertices in $\{i_0,\ (i+1)_0,\ \dots,\ (i+m-1)_0,\ (i+m-1)_1,\ (i+m-2)_1,\ \dots,\ i_1\}$. These vertices can form a directed cycle $(i_0,\ (i+1)_0,\ \ldots,\ (i+m-1)_0,\ (i+m-1)_1,\ (i+m-2)_1,\ \ldots,\ i_1)$. There exist even number of odd directed cycles and they are mapped in pairs onto the vertex set $\{i_j\}$ as illustrated in the following example: $\vec{C}_3 \cup \vec{C}_5 = (i_0,\ (i+1)_0,\ i_1) \cup ((i+2)_0,\ (i+3)_0,\ (i+3)_1,\ (i+2)_1,\ (i+1)_1)$.

We choose a 1-factor for ∞_0 by including all edges of the form $\{i_0 i_1\}$ and $\{i_1 i_0\}$ which are not in P_{2r}, and completing the 1-factor with edges of the form $\{i_j(i+1)_j\}$ and $\{(i+1)_j i_j\}$, here the first subscript is taken modulo r. A 1-factor for ∞_1 is then chosen by taking all remaining edges of the form $\{i_j(i+1)_j\}$, $\{(i+1)_j i_j\}$, and all edges of the form $\{i_1(i+1)_0\}$ and $\{(i+1)_0 i_1\}$ which are not in P_{2r}.

The edges on vertex set $\{i_j\}$ remained unused by either P_{2r} or the factors for ∞_0 and ∞_1 can be partitioned into three groups: those on $\{i_0\}$, those on $\{i_1\}$, and those between the two classes. We partition these edges into Mendelsohn triples using a standard method of "pure and mixed differences", after noting that whenever $\{i_j k_l\}$ is a remaining edge, so is $\{(i+a)_j(k+a)_l\}$. The differences used in P_{2r}^+, P_{2r}^- and the two directed 1-factors for ∞_0 and ∞_1 are $1_{00}, (v-1)_{00}, 1_{11}, (v-1)_{11}, 0_{01}, 0_{10}, 1_{01}, (v-1)_{10}$. The remaining differences are: $2_{00}, 3_{00}, \ldots, (v-2)_{00}$; $2_{11}, 3_{11}, \ldots, (v-2)_{11}$; $2_{01}, 3_{01}, \ldots, (v-2)_{01}, (v-1)_{01}$; $1_{10}, 2_{10}, \ldots, (v-2)_{10}$. We can get six difference triples from these differences as follows: $(2_{01}, 3_{10}, (v-5)_{00})$,$(4_{01}, 5_{10}, (v-9)_{00})$, $(2_{10}, 3_{01}, (v-5)_{11})$, $(4_{10}, 5_{01}, (v-9)_{11})$, $(6_{01}, 7_{10}, (v-13)_{00})$, $(6_{10}, 7_{01}, (v-13)_{11})$. The remaining differences can form $3k-8$ directed 1-factors. By Lemma 2.1, we have $\vec{C}_3 \mid$(I). Thus we finish the case.

Case 2. $P_{6k+1}^+ = P_{3k+7}^+ + P_{3k-6}^+$.

Suppose P_{3k+7}^+ contains an l-cycle $\vec{C}_l = (x_0, x_1, \ldots, x_{l-1})$. Let $V(P_{3k+7}^+) = V_1$ and $V = V_1 \setminus \{x_0\}$. By adding a Mendelsohn triple (x_0, x_{l-1}, x_1) to P_{3k+7}^+, we have

$$
\begin{aligned}
&D_{6k+1} - P_{6k+1}^+ \\
&= D_{6k+1} - (P_{3k-6}^+ + C_l + P_{3k+7-l}^+) \\
&= D_{6k+1} - [P_{3k-6}^+ + C_l + P_{3k+7-l}^+ + (x_0, x_{l-1}, x_1)] + (x_0, x_{l-1}, x_1) \\
&= D_{6k+1} - [P_{3k-6}^+ + P_{3k+7-l}^+ + (x_1, x_2, \ldots, x_{l-1}) + (x_0, x_1) + (x_0, x_{l-1})] \\
&\quad + (x_0, x_{l-1}, x_1) \\
&= D_{6k+1} - [P_{3k-6}^+ + P_{3k+6}^+ + (x_0, x_1) + (x_0, x_{l-1})] + (x_0, x_{l-1}, x_1) \\
&= [(D_{3k+6} - P_{3k+6}^+ - P_{3k+6}^-) + D_{3k-6,3k+6}] + (D_{3k-5} - P_{3k-6}^+)
\end{aligned}
$$

$+ [(D_{x_0,V-\{x_1,x_{l-1}\}} + P^-_{3k+6} + (x_0,x_{l-1},x_1)]$
$= (\mathrm{I}_a+\mathrm{I}_b) + (\mathrm{II}) + [D_{x_0,V-\{x_1,\dots,x_{l-1}\}} + (P^-_{3k+6} - (x_{l-1},x_{l-2},\dots,x_2,x_1))] +$
$(x_1,x_{l-1}) + \{(x_0,x_{2+i},x_{1+i})|0 \le i \le l-3\}] + (x_1,x_{l-1})$
$= (\mathrm{I}_a+\mathrm{I}_b) + (\mathrm{II}) + (\mathrm{III}) + \{(x_0,x_{2+i},x_{1+i})|0 \le i \le l-3\} + (x_1,x_{l-1}).$

Similarly, $\vec{C}_3 \mid (\mathrm{I}_a+\mathrm{I}_b) + (\mathrm{II}) + (\mathrm{III})$.

Case 3. $P_{6k+1} = \vec{C}_{6k+1}$.

Let $V_{6k+1} = V_{3k+6} \cup V_{3k-5} = \{\infty_i | i \in Z_{3k-5}\} \cup \{i | i \in Z_{3k+6}\}$ and $\vec{C}_{6k+1} = (\infty_0, \infty_1, \dots, \infty_{3k-6}, 0, 1, \dots, 3k+5)$. Add $A = \{(k+2, \infty_{3k-6}, \infty_0)$, $(\infty_0, 3k+5, k+2)$, $(\infty_{3k-6}, k+2, 0)$, $(0, k+2, 3k+5)\}$ to $\vec{C}_{6k+1}$ and divide $\vec{C}_{6k+1}$ into $\vec{C}_{3k+6}$, $\vec{C}_{3k-5}$ and six 2-cycles. The triples in A should be chosen carefully so that the two 2-cycles $(k+2, 3k+5)$ and $(0, k+2)$ are in two directed 1-factors f_1 and f_2 from $D_{3k+6} - \vec{C}_{3k+6}$ respectively. Then we have:

$D_{6k+1} - \vec{C}_{6k+1}$
$= D_{6k+1} - (C_{6k+1}+A)+A$
$= (D_{3k-5} - \vec{C}_{3k-5}) + (D_{3k+6} - \vec{C}_{3k+6}) + D_{(3k-7),(3k+6)}$
$+ D_{\{\infty_0\},V_{3k+6}-\{k+2,3k+5\}} + D_{\{\infty_{3k+1}\},V_{3k+6}-\{k+2,0\}} - (0,k+2) - (k+2,3k+5)+A$
$= (\mathrm{I}) + [(D_{3k+6} - \vec{C}_{3k+6} - f_1 - f_2) + D_{(3k-7),(3k+6)}] +$
$[D_{\{\infty_0\},V_{3k+6}-\{k+2,3k+5\}} + f_1 - (k+2,3k+5)] + [D_{\{\infty_{3k-6}\},V_{3k+6}-\{(k+2,0)\}} +$
$f_2 - (k+2,0)] + A$
$= (\mathrm{I}) + (\mathrm{II}_a+\mathrm{II}_b) + (\mathrm{III}) + (\mathrm{IV}) + A.$

By the hypothesis, (I) has a maximum packing with a leave $\vec{C}_2$. The differences from $D_{3k+6} - \vec{C}_{3k+6}$ can form difference triples $(2,3,3k+1)$, $(4,5,3k-3)$,$(6,7,3k-7)$ and $(3k-7)$ directed 1-factors. By Lemma 2.1, $\vec{C}_3 \mid (\mathrm{II}_a+\mathrm{II}_b)$ and $\vec{C}_3 \mid (\mathrm{III})+(\mathrm{IV})$. Thus we have the proof of this case.

Case 4. $P^+_{6k+1} = P^+_{3k+6+h} + P^+_{3k-5-h}$, $h \ge 2$.

We use a technique called "fitting" to solve it. The details can be found as follows.

Suppose P^+_{3k+6+h} contains at least two cycles $\vec{C}_j = (a_0, a_1, a_2, a_3, \dots, a_{j-1})$ $(j \ge h+2)$ and $\vec{C}_l = (b_1, b_2, \dots, b_l)$. Let $V_1 = V(P^+_{3k+6+h})$, $V_0 = \{a_i | 0 \le i \le h-1\}$ and $V = V_1 \backslash V_0$. We can add $A = \{(b_1, a_h, a_{h-1})$, (a_0, b_1, a_{h-1}), $(a_{j-1}, b_1, a_0)\}$ to P^+_{3k+6+h}. These triples should be chosen carefully so that the 2-cycle (b_1, a_{j-1}) is in a directed 1-factor f_1 and the 2-cycle (a_h, b_1) is in a directed 1-factor f_2. By combining edges in D_{6k+1} and P^+_{3k+6+h}, we can get $\vec{C}_h = (a_0, a_1, a_2, ..., a_{h-1})$, P^+_{3k+6}, and a Mendelsohn triple (a_{j-1}, a_h, b_1). Then we can use the result of Case 1. The details can be found as follows:

$D_{6k+1} - P^+_{6k+1}$
$= D_{6k+1} - (P^+_{3k-5-h} + \vec{C}_j + \vec{C}_l + P^+_{3k+6+h-j-l})$
$= D_{6k+1} - (P^+_{3k-5-h} + \vec{C}_j + \vec{C}_l + P^+_{3k+6+h-j-l} + A) + A$
$= D_{6k+1} - [P^+_{3k-5-h} + P^+_{3k+6+h-j-l} + (a_0, a_1, ..., a_{h-2}, a_{h-1}) + (b_1, b_2, ..., b_l)$
$+ (b_1, a_h, ..., a_{j-1}) + (b_1, a_0) + (a_{j-1}, a_0) + (b_1, a_{h-1}) + (a_h, a_{h-1})] + A$
$= D_{6k+1} - [P^+_{3k-5-h} + P^+_{3k+6+h-j-l} + C_h + (b_1, b_2, ..., b_l) + ((a_h, ..., a_{j-1}) +$
$\{b_1 a_h\} + \{a_{j-1} b_1\} - \{a_{j-1} a_h\}) + (b_1, a_0) + (a_{j-1}, a_0) + (b_1, a_{h-1}) + (a_h, a_{h-1})]$
$+ A$
$= D_{6k+1} - [P^+_{3k-5} + P^+_{3k+6} + \{b_1 a_h\} + \{a_{j-1} b_1\} - \{a_{j-1} a_h\} + (b_1, a_0) +$
$(a_{j-1}, a_0) + (b_1, a_{h-1}) + (a_h, a_{h-1})] + A$
$= (D_{3k+6} - P^+_{3k+6}) + (D_{3k-5} - P^+_{3k-5}) + [D_{3k-7,3k+6} + D_{\{a_0\}, V-\{b_1,\ a_{j-1}\}} +$
$D_{\{a_{h-1}\}, V-\{b_1,\ a_h\}} - \{b_1 a_h\} - \{a_{j-1} b_1\} + \{a_{j-1} a_h\}] + A$
$= [(D_{3k+6} - P^+_{3k+6} - f_1 - f_2) + D_{3k-7,3k+6}] + (D_{3k-5} - P^+_{3k-5}) + [D_{\{a_0\}, V-\{b_1,\ a_{j-1}\}}$
$+ (f_1 - (b_1,\ a_{j-1}))] + [D_{\{a_{h-1}\}, V-\{b_1,\ a_h\}} + (f_2 - (b_1,\ a_h))] + (b_1,\ a_{j-1}) +$
$(b_1,\ a_h) - \{b_1 a_h\} - \{a_{j-1} b_1\} + \{a_{j-1} a_h\} + A$
$= [(\mathrm{I}_a - f_1 - f_2) + \mathrm{I}_b] + (\mathrm{II}) + (\mathrm{III}_a) + (\mathrm{III}_b) + (a_{j-1}, a_h, b_1) + A.$

The same as Case 1, I_a can be decomposed into difference triples and $3k-5$ directed 1-factors which contain f_1 and f_2. The $3k-5$ directed 1-factors minus two directed 1-factors f_1 and f_2 is $3k-7$. By Lemma 2.1, $\vec{C}_3 \mid (\mathrm{I}_a - f_1 - f_2) + \mathrm{I}_b$ and $\vec{C}_3 \mid (\mathrm{III}_a) + (\mathrm{III}_b)$. By the hypothesis, (II) has a maximum packing with a leave $\vec{C}_2$.

Now we need to consider the cases that all cycles in P have length less that $h+2$. If P contains at least one cycle $\vec{C}_{h+1}$, it must have another cycle $\vec{C}_l$. Thus by fitting, we can manage to get the form as that in Case 1. Now it is left the cases in which all cycles in P^+_{3k+6+h} are less than h. If P^+_{3k+6+h} contains at least one cycle $\vec{C}_{h-1}$, it can be deduced to Case 2. Now it is left the cases in which all cycles in P^+_{3k+6+h} are less than $h-1$. If P^+_{3k+6+h} contains at least one $\vec{C}_{h-2}$, then we can get $P^+_{6k+1} = P^+_{3k+8} + P^+_{3k-7}$. If there exist a j-cycle $\vec{C}_j$ $(j \geq 4)$ in P^+_{3k+8} and we can settle this cases by fitting. If all cycles in P^+_{3k+8} have length less than 4, then we can find a $\vec{C}_2$. Now, it is left the cases in which all cycles in P^+_{3k+6+h} are less than $h-2$. Repeat the process until all cycles in P^+_{3k+6+h} are less than $\lceil \frac{h}{2} \rceil$. Now, we write $P^+_{6k+1} = P^+_{3k+6+\lceil\frac{h}{2}\rceil} + P^+_{3k-5-\lceil\frac{h}{2}\rceil}$ and by repeating the process as above we solve all cases.

When k is odd, we first consider $P^+_{6k+1} = P^+_{3k+3} + P^+_{3k-2}$ and proceed similarly as above.

Case 5. $P^+_{6k+1} = C^+_{l_1} + C^+_{l_2}$, $3k-4 < l_2 < l_1 < 3k+6$.

We need to consider the following cases: (i) $l_1 = 3k+5$, $l_2 = 3k-4$; (ii) $l_1 = 3k+4$, $l_2 = 3k-3$; (iii) $l_1 = 3k+3$, $l_2 = 3k-2$; (iv) $l_1 = 3k+2$, $l_2 = 3k-1$; (v) $l_1 = 3k+1$, $l_2 = 3k$. For any of the cases, we can first divide $C^+_{l_2}$ into two cycles using the method of Case 3. Then we can change all the cases into the form of Case 2 and Case 4. Thus we finish Case 5.

As for $t = 6k+4$, we first consider the cases $P^+_{6k+4} = P^+_{3k+6} + P^+_{3k-2}$ when k is even and $P^+_{6k+4} = P^+_{3k+3} + P^+_{3k+1}$ when k is odd, then we can proceed similarly as above. □

3. Maximum Packing of $D_t - P$

With above preparations, we can prove the following theorem.

Theorem 3.1. For each directed 2-regular subgraph P of D_t and an integer t, $t \geq 13$, $D_t - P$ can be packed with leave L_i if and only if $t(t-1) - |E(P)| \equiv i \pmod 3$ where $i = 0, 1, 2$. Here, $L_0 = \emptyset$, $L_1 = \vec{C}_4$ (or $2\vec{C}_2$) and $L_2 = \vec{C}_2$.

Proof. The necessity is obvious. We only need to prove the sufficiency. Note that Theorem 1.5 is a special case where $i = 0$. Now we consider the cases $i = 1, 2$.

Case 1. $t(t-1) - |E(P)| \equiv 2 \pmod 3$.

The same as that in Theorem 2.1, we only need to consider the following cases: (i) $t = 6k$, $|E(P)| = 6k-2$; (ii) $t = 6k+2$, $|E(P)| = 6k$; (iii) $t = 6k+3$, $|E(P)| = 6k+1$; (iv) $t = 6k+5$, $|E(P)| = 6k+3$; (v) $t = 6k+1$, $|E(P)| = 6k+1$; (vi) $t = 6k+4$, $|E(P)| = 6k+4$. For cases (i) $\sim$ (iv), let $P^* = P \cup \vec{C}_2$ where $V(P) \cap V(\vec{C}_2) = \emptyset$. By Theorem 1.5, $\vec{C}_3 \mid P^*$ and then we can pack $D_t - P$ with leave $\vec{C}_2$. As for cases (v) and (vi), they have been proved in Theorem 2.1.

Case 2. $t(t-1) - |E(P)| \equiv 1 \pmod 3$.

The same as before, we only consider the following cases. (i) $t = 6k$, $|E(P)| = 6k-1$; (ii) $t = 6k+2$, $|E(P)| = 6k+1$; (iii) $t = 6k+3$, $|E(P)| = 6k+2$; (iv) $t = 6k+5$, $|E(P)| = 6k+4$. (v) $t = 6k+1$, $|E(P)| = 6k-1$; (vi) $t = 6k+4$, $|E(P)| = 6k+2$. Let $P = \vec{C}_n \cup H$ where $\vec{C}_n = (x_1, x_2, ..., x_n)$. Since $|V(P)| \leq t-1$, let v be a vertex not in P and let $\vec{C}_{n+1} = (x_1, x_2, ..., x_n, v)$. By Theorem 1.5, $D_t - \vec{C}_{n+1} \cup H$ has $\vec{C}_3$-decompositions and the edge $\{x_1 x_n\}$ is in a Mendelsohn triple (x_1, x_n, y). By deleting this triple, we have a packing of $D_t - P$ with a 4-cycle (x_1, y, x_n, v) leave. □

4. Maximum Packing of $D_t \cup P$

We are now in a position to obtain the packing of $D_t \cup P$.

Theorem 4.1. For each directed 2-regular subgraph P of D_t and an integer t, $t \geq 13$, $D_t \cup P$ can be packed with leave L_i if and only if $t(t-1) + |E(P)| \equiv i \pmod 3$ where $i = 0, 1, 2$. Here, $L_0 = \emptyset$, $L_1 = \vec{C}_4$ (or $2\vec{C}_2$) and $L_2 = \vec{C}_2$.

Proof. The necessity is obvious. We only need to prove the sufficiency. Theorem 1.5 is a special case of $i = 0$. We divide the proof into two cases.

Case 1. $t(t-1) + |E(P)| \equiv 1 \pmod 3$

For $t = 3k$, $|E(P)| = 3l + 1 < 3k - 2$, let $P_{3k-1} = P_{3l+1} + P_{3k-3l-2}$ where $V(P_{3l+1}) \cap V(P_{3k-3l-2}) = \emptyset$ and $x \in V(D_{3k}) \setminus V(P_{3k-1})$. Then $D_{3k} + P_{3l+1} = D_{3k} + P_{3l+1} + P_{3k-3l-2} - P_{3k-3l-2} = (D_{3k-1} - P_{3k-3l-2}) + (D_{\{x\},3k-1} + P_{3k-1}) = \text{(I)} + \text{(II)}$. By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $2\vec{C}_2$ or $\vec{C}_4$. If $|E(P)| = 3k-2$ and $x, y \notin V(P)$, we have $D_{3k} + P_{3k-2} = D_2 + D_{3k-2} + D_{2,3k-2} + P_{3k-2} = D_2 + (D_{3k-2} - P_{3k-2}) + (D_{2,3k-2} + 2P_{3k-2}) = D_2 + \text{(I)} + \text{(II)}$ where D_2 is defined on $\{x, y\}$. By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $\vec{C}_2$.

For $t = 3k+1$, $|E(P)| = 3l+1 < 3k+1$, let $P_{3k} = P_{3l+1} + P_{3k-3l-1}$ where $V(P_{3l+1}) \cap V(P_{3k-3l-1}) = \emptyset$ and $x \in V(D_{3k+1}) \setminus V(P_{3k})$. Then $D_{3k+1} + P_{3l+1} = D_{3k+1} + P_{3l+1} + P_{3k-3l-1} - P_{3k-3l-1} = (D_{3k} - P_{3k-3l-1}) + (D_{\{x\},3k} + P_{3k}) = \text{(I)} + \text{(II)}$. By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by

Theorem 3.1, (I) can be packed with leave 2 $\vec{C}_2$ or $\vec{C}_4$. If $|E(P)| = 3k+1$ and $P_{3k+1} = P_{3k+1-l} \cup \vec{C}_l$ where $\vec{C}_l = (x_0, x_1, ..., x_{l-1})$, we have $D_{3k+1} + P_{3k+1} = D_{3k+1} + P_{3k+1-l} + (x_0, x_1, x_{l-1}) + (x_1, ..., x_{l-1}) - (x_1, x_{l-1}) = (x_0, x_1, x_{l-1}) + (D_{3k} - (x_1, x_{l-1})) + (D_{\{x_0\},3k} + P_{3k}) = (x_0, x_1, x_{l-1})$ + (I) + (II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $\vec{C}_4$ or 2 $\vec{C}_2$.

For $t = 3k+2$, $|E(P)| = 3l+2 < 3k+2$, let $P_{3k+1} = P_{3l+2} + P_{3k-3l-1}$ where $V(P_{3l+2}) \cap V(P_{3k-3l-1}) = \emptyset$ and $x \in V(D_{3k+2}) \setminus V(P_{3k+1})$. Then $D_{3k+2} + P_{3l+2} = D_{3k+2} + P_{3l+2} + P_{3k-3l-1} - P_{3k-3l-1} = (D_{3k+1} - P_{3k-3l-1}) + (D_{\{x\},3k+1} + P_{3k+1})$ = (I) + (II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave 2 $\vec{C}_2$ or $\vec{C}_4$. If $|E(P)| = 3k+2$ and $P_{3k+2} = P_{3k+2-l} \cup \vec{C}_l$ where $\vec{C}_l = (x_0, x_1, ..., x_{l-1})$, then $D_{3k+2} + P_{3k+2} = D_{3k+2} + P_{3k+2-l} + (x_0, x_1, x_{l-1}) + (x_1, ..., x_{l-1}) - (x_1, x_{l-1}) = (x_0, x_1, x_{l-1}) + (D_{3k+1} - (x_1, x_{l-1})) + (D_{\{x_0\},3k+1} + P_{3k+1}) = (x_0, x_1, x_{l-1})$ + (I) + (II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $\vec{C}_4$ or 2 $\vec{C}_2$.

Case 2. $t(t-1) + |E(P)| \equiv 2 \pmod 3$

For $t = 3k$, $|E(P)| = 3l+2 < 3k-1$, let $P_{3k-1} = P_{3l+2} + P_{3k-3l-3}$ where $V(P_{3l+2}) \cap V(P_{3k-3l-3}) = \emptyset$ and $x \in V(D_{3k}) \setminus V(P_{3k-1})$. Then $D_{3k} + P_{3l+2} = D_{3k} + P_{3l+2} + P_{3k-3l-3} - P_{3k-3l-3} = (D_{3k-1} - P_{3k-3l-3}) + (D_{\{x\},3k-1} + P_{3k-1})$ = (I)+(II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $\vec{C}_2$. If $|E(P)| = 3k-1$ and $x \notin V(P)$, we have $D_{3k} + P_{3k-1} = D_{3k-1} + (D_{\{x\},3k-1} + P_{3k-1}) = \vec{C}_2 + (D_{3k-1} - \vec{C}_2) + (D_{\{x\},3k-1} + P_{3k-1}) = \vec{C}_2$+(I)+(II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 1.5, (I) can be packed with Mendelsohn triples.

For $t = 3k+1$, $|E(P)| = 3l+2 < 3k-1$, let $P_{3k} = P_{3l+2} + P_{3k-3l-2}$ where $V(P_{3l+2}) \cap V(P_{3k-3l-2}) = \emptyset$ and $x \in V(D_{3k+1}) \setminus V(P_{3k})$. Then $D_{3k+1} + P_{3l+2} = D_{3k+1} + P_{3l+2} + P_{3k-3l-2} - P_{3k-3l-2} = (D_{3k} - P_{3k-3l-2}) + (D_{\{x\},3k} + P_{3k})$ = (I)+(II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $\vec{C}_2$. If $|E(P)| = 3k-1$ and $x, y \notin V(P)$, we have $D_{3k+1} + P_{3k-1} = D_2 + (D_{3k-1} - P_{3k-1}) + [(D_{\{x\},3k-1} + P_{3k-1}) + (D_{\{y\},3k-1} + P_{3k-1})] = D_2$+(I)+(II) where D_2 is defined on $\{x, y\}$. By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 1.5, (I) can be packed with Mendelsohn triples. Moreover, D_2 is the leave. Here, D_2 can be considered as $\vec{C}_2$.

For $t = 3k + 2$, $|E(P)| = 3l < 3k$, let $P_{3k+1} = P_{3l} + P_{3k-3l+1}$ where $V(P_{3l}) \cap V(P_{3k-3l+1}) = \emptyset$ and $x \in V(D_{3k+2}) \setminus V(P_{3k+1})$. Then $D_{3k+2} + P_{3l} = D_{3k+2} + P_{3l} + P_{3k-3l+1} - P_{3k-3l+1} = (D_{3k+1} - P_{3k-3l+1}) + (D_{\{x\},3k+1} + P_{3k+1})$ = (I)+(II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 3.1, (I) can be packed with leave $\vec{C}_2$. If $|E(P)| = 3k$ and $x, y \notin V(P)$, we have $D_{3k+2} + P_{3k} = D_2 + (D_{3k} - P_{3k}) + [(D_{\{x\},3k} + P_{3k}) + (D_{\{y\},3k} + P_{3k})] = D_2$+(I)+(II). By Lemma 2.1, (II) has $\vec{C}_3$-decompositions while by Theorem 1.5, (I) can be packed with Mendelsohn triples. Moreover, D_2 is the leave. □

5. Concluding Remark

It can be seen that all the packings of $D_t - P$ and $D_t \cup P$ we obtained in this paper are in fact maximum packings with Mendelsohn triples. For the values $t < 13$, the maximum packings of $D_t - P$ and $D_t \cup P$ can also be obtained by direct constructions. Since it is a matter of routine work, we omit the details. Moreover, with the results obtained in this paper, we can extend the study to packing λD_t with Mendelsohn triples or packing $\lambda D_t - P$ ($\lambda D_t \cup P$, respectively) with Mendelsohn triples where P can be a larger subgraph of λD_t.

Appendices

Appendix A: $D_{10} - P_{10}$ can be packed leave $\vec{C_2}$.

Proof. Let D_{10} be defined on $\{i_j | i \in Z_3, j = 0, 1\} \cup \{\infty_i | i \in Z_4\}$ where $\vec{C_4}$ $= (\infty_0, \infty_1, \infty_2, \infty_3)$. Then $D_{10} - P_6 \cup \vec{C_4} = (D_6 - P_6) + D_{4,6} + (D_4 - \vec{C_4})$ where $D_6 - P_6$ can form four directed 1-factors and associate these with $\infty_i, i \in Z_4$, respectively. By lemma 2.2, $D_4 - \vec{C_4}$ can be packed with leave.

Let D_{10} be defined on $Z_8 \cup \{\infty_i | i \in Z_2\}$ where $\vec{C_8} = (0, 1, 2, 3, 4, 5, 6, 7)$ and (∞_0, ∞_1) are the missing cycles. Then $D_{10} - \vec{C_8} \cup \vec{C_2} = (D_8 - \vec{C_8}) + D_{2,8} + (D_2 - \vec{C_2}) = \{(1+i, i, \infty_0) | i \in Z_8\} \cup \{(1, 6, \infty_1), (6, 3, \infty_1), (3, 1, \infty_1), (5, 2, \infty_1), (2, 7, \infty_1), (7, 5, \infty_1), (0, 4, \infty_1), (4, 0, \infty_1), (7, 4, 1), (3, 0, 5), (2, 5, 0), (2, 0, 6), (6, 0, 3), (6, 4, 2), (3, 5, 7), (7, 2, 4), (1, 4, 6), (1, 3, 7), (1, 5)\}$.

Let D_{10} be defined on $Z_8 \cup \{\infty_i | i \in Z_2\}$ and $5\vec{C_2} = (0, 1) \cup (2, 3) \cup (4, 5) \cup (6, 7) \cup (\infty_0, \infty_1)$. Then $D_{10} - 5\vec{C_2} = (D_8 - 4\vec{C_2}) + D_{2,8} + (D_2 - \vec{C_2}) = \{(\infty_0, 1, 2), (\infty_0, 2, 1), (\infty_0, 3, 4), (\infty_0, 4, 3), (\infty_0, 5, 6), (\infty_0, 6, 5), (\infty_0, 0, 7), (\infty_0, 7, 0), (\infty_1, 1, 6), (\infty_1, 6, 3), (\infty_1, 3, 1), (\infty_1, 5, 2), (\infty_1, 2, 7), (\infty_1, 7, 5), (\infty_1, 0, 4), (\infty_1, 4, 0), (7, 4, 1), (2, 0, 6), (6, 4, 2), (1, 4, 6), (1, 3, 7), (3, 0, 5), (5, 7, 3), (7, 2, 4), (6, 0, 3), (5, 0, 2), (1, 5)\}$.

Let D_{10} be defined on $Z_8 \cup \{\infty_i | i \in Z_2\}$ and $2\vec{C_3} \cup 2\vec{C_2} = (1, 6, 3) \cup (5, 2, 7) \cup (0, 4) \cup (\infty_0, \infty_1)$. Then $D_{10} - 2\vec{C_3} \cup 2\vec{C_2} = (D_8 - 2\vec{C_3} \cup \vec{C_2}) + D_{2,8} + (D_2 - \vec{C_2}) = \{(1+i, i, \infty_0), (i, 1+i, \infty_1) | i \in Z_8\} \cup \{(7, 4, 1), (3, 0, 5), (2, 5, 0), (2, 0, 6), (6, 0, 3), (6, 4, 2), (3, 5, 7), (7, 2, 4), (1, 4, 6), (1, 3, 7)\} \cup \{(1, 5)\}$.

Let D_{10} be defined on $Z_8 \cup \{\infty_i | i \in Z_2\}$ and $\vec{C_6} \cup 2\vec{C_2} = (0, 1, 2, 3, 4, 5) \cup (6, 7) \cup (\infty_0, \infty_1)$. Then $D_{10} - \vec{C_6} \cup 2\vec{C_2} = D_8 - \vec{C_6} \cup \vec{C_2} + D_{2,8} + (D_2 - \vec{C_2}) = \{(7, 2, \infty_0), (2, 4, \infty_0), (4, 7, \infty_0), (6, 0, \infty_0), (0, 3, \infty_0), (3, 6, \infty_0), (1, 5, \infty_0), (5, 1, \infty_0), (1, 6, \infty_1), (6, 3, \infty_1), (3, 1, \infty_1), (5, 2, \infty_1), (2, 7, \infty_1), (7, 5, \infty_1), (0, 4, \infty_1), (4, 0, \infty_1), (7, 4, 1), (2, 0, 6), (6, 4, 2), (1, 4, 6), (1, 3, 7), (1, 0, 2), (5, 4, 3), (3, 2, 5), (7, 0, 5), (3, 0, 7)\} \cup \{(5, 6)\}$.

Let D_{10} be defined on $Z_7 \cup \{\infty_i | i \in Z_3\}$ and $\vec{C_7} \cup \vec{C_3} = (0, 1, 2, 3, 4, 5, 6) \cup (\infty_0, \infty_1, \infty_2)$. Then $D_{10} - \vec{C_7} \cup \vec{C_3} = D_7 - \vec{C_7} + D_{3,7} + (D_3 - \vec{C_3}) = \{(0, 2, \infty_0), (2, 4, \infty_0), (4, 6, \infty_0), (6, 1, \infty_0), (1, 3, \infty_0), (3, 5, \infty_0), (5, 0, \infty_0), (1, 5, \infty_1), (5, 1, \infty_1), (6, 3, \infty_1), (3, 6, \infty_1), (2, 0, \infty_1), (0, 4, \infty_1), (4, 2, \infty_1), (3, 0, \infty_2), (0, 6, \infty_2), (6, 2, \infty_2), (2, 1, \infty_2), (1, 4, \infty_2), (4, 3, \infty_2), (6, 4, 1), (6, 5, 2), (2, 5, 3), (5, 4, 0), (1, 0, 3), (\infty_0, 5)\}$.

Let D_{10} be defined on $Z_8 \cup \{\infty_i | i \in Z_2\}$ and $\vec{C_5} \cup \vec{C_3} \cup \vec{C_2} =$

$(0,1,2,3,4) \cup (5,6,7) \cup (\infty_0,\infty_1)$. Then $D_{10} - \vec{C_5} \cup \vec{C_3} \cup \vec{C_2} = (D_8 - \vec{C_5} \cup \vec{C_3}) + D_{2,8} + (D_2 - \vec{C_2}) = \{(1+i,i,\infty_0)|i \in Z_8\} \cup \{(3,0,\infty_1),$ $(0,5,\infty_1),(5,3,\infty_1),$ $(1,4,\infty_1),$ $(4,6,\infty_1),(6,1,\infty_1),$ $(2,7,\infty_1),(7,2,\infty_1),$ $(7,4,1),(2,0,6),(6,4,2),(1,3,7),(2,5,0),(6,0,3),(3,5,7),(4,5,2),(1,6,3),(7,0,4)\} \cup \{(1,5)\}$.

Let D_{10} be defined on Z_{10} and $\vec{C_{10}} = (0,1,2,3,4,5,6,7,8,9)$. Then $D_{10} - \vec{C_{10}} = \{(7,9,1),(6,8,0),(5,7,0),(8,1,9),(4,8,6),(5,9,6),(4,9,7),$ $(5,0,9),(3,0,8),(2,0,7),(6,0,3),(6,9,2),(4,3,9),(4,2,8),(4,0,2),(1,0,4),$ $(2,9,3),(1,4,6),(7,3,8),(7,5,4),(5,8,2),(1,8,5),(7,1,3),(6,2,7),(1,6,3),$ $(5,2,1),(5,3)\}$. □

Appendix B. $D_{13} - P_{13}$ can be packed with leave $\vec{C_2}$.

Proof. $D_{13} - \vec{C_9} \cup \vec{C_4} = [(D_9 - \vec{C_9}) + D_{4,9}] + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $\vec{C_9} = (0,1,...,7,8)$ is the missing cycle. Let D_4 be defined on $\{\infty_i|i \in Z_4\}$. We can get a difference triple $(2,3,4)$ and the remaining differences can associate with $\infty_i (i \in Z_4)$, respectively.

$D_{13} - 2\vec{C_4} \cup \vec{C_5} = (D_9 - \vec{C_4} \cup \vec{C_5}) + D_{4,9} + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $\vec{C_4} \cup \vec{C_5} = (0,1,2,3) \cup (4,5,6,7,8)$. Let D_4 be defined on $\{\infty_i|i \in Z_4\}$. We can get a difference triple $(2,3,4)$ and the differences 7 and 8 can associate with ∞_0 and ∞_1, respectively. Moreover, there are two directed 1-factors $(8,0,5,2,7,4,1,6,3)$ and $(8,5,1,7,3,4,0,6,2)$ for ∞_2 and ∞_3, respectively.

$D_{13} - \vec{C_7} \cup \vec{C_2} \cup \vec{C_4} = (D_9 - \vec{C_7} \cup \vec{C_2}) + D_{4,9} + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $\vec{C_2} \cup \vec{C_7} = (0,1) \cup (2,3,4,5,6,7,8)$. Let D_4 be defined on $\{\infty_i|i \in Z_4\}$. We can get a difference triple $(5,6,7)$ and the differences 2 and 4 can associate with ∞_0 and ∞_1, respectively. Moreover, there are two directed 1-factors $(8,0,3,2,1,4,7,6,5)$ and $(1,2,5,4,3,6,0,8,7)$ for ∞_2 and ∞_3.

$D_{13} - \vec{C_3} \cup \vec{C_6} \cup \vec{C_4} = (D_9 - \vec{C_3} \cup \vec{C_6}) + D_{4,9} + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $\vec{C_3} \cup \vec{C_6} = (0,1,2) \cup (3,4,5,6,7,8)$. Let D_4 be defined on $\{\infty_i|i \in Z_4\}$. $D_{13} - \vec{C_3} \cup \vec{C_6} \cup \vec{C_4} = \{(i,i+2,i+5)|i \in Z_9 \setminus \{3\}\} \cup \{(\infty_0,i,i+7)|i \in Z_9 \setminus \{2\}\} \cup \{(\infty_1,i,i+5)|i \in Z_9 \setminus \{0\}\} \cup \{(\infty_2,i,i+6)|i \in Z_9 \setminus \{5\}\} \cup \{(2,3,5),(5,8,0),(2,\infty_2,\infty_0),(5,\infty_1,\infty_2),(0,\infty_0,\infty_1),(\infty_2,\infty_3,\infty_0),(\infty_1,\infty_3)\}$.

$D_{13} - 3\vec{C_3} \cup \vec{C_4} = (D_9 - 3\vec{C_3}) + D_{4,9} + (D_4 - \vec{C_4})$. $D_9 = K_9^+ + K_9^-$,

K_9 can be decomposed into Kirkman triple system [3] and contains four parallel classes. If all these parallel classes are oriented properly, then one oriented parallel class can be chosen as $\cup_{i=1}^{3} \vec{C_3}$ and four oriented parallel classes associated with ∞_i ($i \in Z_4$), respectively.

$D_{13} - \vec{C_5} \cup 2\vec{C_2} \cup \vec{C_4} = (D_9 - \vec{C_5} \cup 2\vec{C_2}) + D_{4,9} + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $C_5 \cup C_2 \cup C_2 = (0,1,2,3,4) \cup (5,6) \cup (7,8)$. Let D_4 be defined on $\{\infty_i | i \in Z_4\}$. We can get a difference triple $(2,3,4)$ and the differences 7 and 6 can associate with ∞_0 and ∞_1, respectively. Moreover, there are two directed 1-factors $(0,5,1) \cup (2,7,6) \cup (3,8,4)$ and $(8,0) \cup (4,5) \cup (3,2,1,6,7)$ for ∞_2 and ∞_3, respectively.

$D_{13} - \vec{C_3} \cup 2\vec{C_4} \cup \vec{C_2} = (D_9 - \vec{C_3} \cup \vec{C_4} \cup \vec{C_2}) + D_{4,9} + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $\vec{C_3} \cup \vec{C_4} \cup \vec{C_2} = (0,1,2) \cup (3,4,5,6) \cup (7,8)$. Let D_4 be defined on $\{\infty_i | i \in Z_4\}$. We can get a difference triple $(2,3,4)$ and the differences 5 can associate with ∞_0. Moreover, there are three directed 1-factors $(2,8,6,5,3,1,0,7,4)$, $(0,8,5,2,1,7,6,4,3)$ and $(1,8,0,6,7,5,4) \cup (2,3)$ for ∞_1, ∞_2 and ∞_3, respectively.

$D_{13} - 3\vec{C_2} \cup \vec{C_3} \cup \vec{C_4} = (D_9 - 3\vec{C_2} \cup \vec{C_3}) + D_{4,9} + (D_4 - \vec{C_4})$. Let D_9 be defined on Z_9 where $3\vec{C_2} \cup \vec{C_3} = (0,1) \cup (2,3) \cup (4,5) \cup (6,7,8)$. Let D_4 be defined on $\{\infty_i | i \in Z_4\}$. We can get a difference triple $(2,3,4)$ and the differences 5 and 6 can associate with ∞_0 and ∞_1, respectively. Moreover, there are two directed 1-factors $(2,0,8,7,5,6,4,3,1)$ and $(8,0,7,6,5,3,4,2,1)$ for ∞_2 and ∞_3, respectively.

For $D_{13} - 2\vec{C_5} \cup \vec{C_3}$, let D_9 be defined on Z_9 where $2\vec{C_5} \cup \vec{C_3} = (0,1,2,3,4) \cup (5,6,7,8,\infty_0) \cup (\infty_1,\infty_2,\infty_3)$. Let D_4 be defined on $\{\infty_i | i \in Z_4\}$. Difference 8 can associate with ∞_2, directed 1-factor $(3,0,6,8,2,5) \cup (1,7,4)$ can associate with ∞_3. Moreover, there are Mendelsohn triples and a leave as follows. $\{(\infty_0,8,0), (4,5,\infty_0), (4,\infty_0,0), (\infty_0,1,6), (\infty_0,6,2), (\infty_0,2,7), (\infty_0,7,3), (\infty_0,3,1), (3,8,6), (8,4,2), (5,2,0), (0,7,5), (6,0,2), (1,3,6), (2,4,7), (3,5,8), (5,7,1), (8,1,4), (\infty_1,7,0), (\infty_1,0,3), (\infty_1,3,7), (\infty_1,5,1), (\infty_1,1,8), (\infty_1,8,5), (\infty_1,4,6), (\infty_1,6,4)\} \cup \{(\infty_1,2)\}$.

For $D_{13} - \vec{C_{11}} \cup \vec{C_2}$, Let D_{13} be defined on $Z_{12} \cup \{\infty_0\}$ where $\vec{C_{11}} \cup \vec{C_2} = (0,1,2,3,4,5,6,7,8,9,10) \cup (11,\infty_0)$. Then $D_{13} - \vec{C_{11}} \cup \vec{C_2} = \{(k,k+2,k+6) | k \in Z_{12} \setminus \{5,9,10\}\} \cup \{(k,k+3,k+1) | k \in Z_{12} \setminus \{3,5,6,8\}\} \cup \{(k,k+7,k+3) | k \in Z_{12} \setminus \{0,4,9\}\} \cup \{(\infty_0,k,k+5) | k \in Z_{12} \setminus \{6,11\}\} \cup \{(\infty_0,6,4), (0,4,3), (4,10,11), (3,6,11), (4,0,7), (11,0,9),$

$(9,4,11),(7,3,9),(6,9,8),(5,8,11),(5,7,6),(7,11)\}$.

For $D_{13}-\vec{C_6}\cup\vec{C_7}$, Let D_{13} be defined on $Z_{12}\cup\{\infty\}$ where $\vec{C_6}\cup\vec{C_7}=(0,1,2,3,4,5)\cup(6,7,8,9,10,11,\infty)$. Then $D_{13}-\vec{C_6}\cup\vec{C_7}=\{(k,k+4,k+1)|k\in Z_{12}\setminus\{9,1\}\}\cup\{(k,k+2,k+5)|k\in Z_{12}\setminus\{0,4\}\}\cup\{(k,k+6,k+2)|k\in Z_{12}\setminus\{0,8\}\}\cup\{(\infty_0,k,k+5)|k\in Z_{12}\setminus\{6,11\}\}\cup\{(\infty,11,0),(\infty,0,6),(5,6,2),(7,0,2),(6,11,4),(1,6,9),(8,1,10),(10,9,2),(2,0,5),(\infty,1,5),(\infty,5,10),(\infty,10,3),(\infty,3,8),(\infty,8,2),(\infty,2,1),(\infty,9,4),(\infty,4,9),(\infty,7)\}$.

Let D_{13} be defined on $Z_9\cup\{\infty_i|i\in Z_4\}$ where $\vec{C_{10}}\cup\vec{C_3}=(0,1,2,3,4,5,6,7,8,\infty_0)\cup(\infty_1,\infty_2,\infty_3)$. Then $D_{13}-\vec{C_{10}}\cup\vec{C_3}=\{(k,k+3,k+1)|k\in Z_9\setminus\{2\}\}\cup\{(\infty_1,k,k+2)|k\in Z_9\}\cup\{(\infty_2,,k,k+5)|k\in Z_9\}\cup\{(\infty_3,k,k+4)|k\in Z_9\}\cup\{(\infty_0,8,0),(2,8,5),(3,0,6),(\infty_0,4,1),(\infty_0,1,7),(\infty_0,7,4),(\infty_0,2,5),(\infty_0,5,3),(\infty_0,3,2),(\infty_0,6)\}$.

For $D_{13}-\vec{C_5}\cup\vec{C_8}$, let D_{13} be defined on $Z_{12}\cup\{\infty\}$ where $\vec{C_5}\cup\vec{C_8}=(0,1,2,3,4)\cup(5,6,7,8,9,10,11,\infty)$. Then $D_{13}-\vec{C_{11}}\cup\vec{C_2}=\{(k,k+4,k+3)|k\in Z_{12}\setminus\{7\}\}\cup\{(k,k+2,k+5)|k\in Z_{12}\setminus\{2\}\}\cup\{(k,k+6,k+2)|k\in Z_{12}\setminus\{10,4\}\}\cup\{(\infty,11,0),(5,\infty,0),(4,5,10),(7,0,10),(6,11,10),(7,11,4),(9,2,4),(\infty,7,2),(\infty,2,7),(\infty,3,8),(\infty,8,1),(\infty,1,6),(\infty,6,4),(\infty,4,10),(\infty,10,3),(\infty,9)\}$.

For $D_{13}-\vec{C_5}\cup\vec{C_6}\cup\vec{C_2}$, let D_{13} be defined on $Z_{12}\cup\{\infty\}$ where $\vec{C_5}\cup\vec{C_6}\cup\vec{C_2}=(0,1,2,3,4)\cup(5,6,7,8,9,10)\cup(11,\infty)$. Then $D_{13}-\vec{C_5}\cup\vec{C_6}\cup\vec{C_2}=\{(k,k+3,k+1)|k\in Z_{12}\setminus\{3\}\}\cup\{(k,k+7,k+3)|k\in Z_{12}\setminus\{9,10\}\}\cup\{(k,k+6,k+8)|k\in Z_{12}\setminus\{3,5\}\}\cup\{(\infty,0,5),(\infty,1,6),(\infty,2,7),(\infty,3,8),(\infty,4,9),(\infty,5,10),(\infty,7,0),(\infty,8,1),(\infty,9,2),(\infty,10,3),(5,11,4),(10,11,1),(\infty,6,4),(3,6,11),(4,3,9),(9,11,0),(1,5)\}$.

Let D_{13} be defined on $Z_{12}\cup\{\infty\}$ where $2\vec{C_2}\cup\vec{C_9}=(0,1)\cup(2,3,4,5,6,7,8,9,10)\cup(11,\infty)$. Then $D_{13}-2\vec{C_2}\cup\vec{C_9}=\{(k,k+7,k+3)|k\in Z_{12}\setminus\{5,6,8\}\}\cup\{(k,k+3,k+1)|k\in Z_{12}\setminus\{0,3,5\}\}\cup\{(1,7,9),(3,9,11),(4,10,0),(5,11,1),(7,1,3),(8,2,4),(10,4,6),(11,5,7),(\infty_0,0,5),(\infty_0,1,6),(\infty_0,2,7),(\infty_0,3,8),(\infty_0,4,9),(\infty_0,5,10),(\infty_0,7,0),(\infty_0,8,1),(\infty_0,6,4),(\infty_0,9,2),(\infty_0,10,3),(6,11,0),(11,4,3),(10,11,8),(3,6,8),(2,8,0),(1,2,6),(3,1,9),(0,3,5),(5,9,6),(8,6,0),(5,8)\}$.

For $D_{13}-\vec{C_2}\cup\vec{C_8}\cup\vec{C_3}$, let D_{13} be defined on $Z_{12}\cup\{\infty\}$ where $\vec{C_2}\cup\vec{C_8}\cup\vec{C_3}=(0,1,2)\cup(3,4,5,6,7,8,9,10)\cup(11,\infty)$. Then $D_{13}-\vec{C_2}\cup\vec{C_8}\cup\vec{C_3}=\{(k,k+7,k+3)|k\in Z_{12}\setminus\{3,6,7,8\}\}\cup\{(k,k+3,k+1)|k\in$

$Z_{12}\setminus\{0,2,9,11\}\} \cup \{(1,7,9),(3,9,11),(4,10,0),(5,11,1),(6,0,2),(8,2,4),(7,1,3),(11,5,7),(\infty,0,5),(\infty,1,6),(\infty,2,7),(\infty,3,8),(\infty,4,9),(\infty,5,10),(\infty,8,1),(\infty,6,3),(4,\infty,10),(\infty,9,0),(\infty,7,2),(2,3,10),(6,11,4),(9,6,10),(8,10,6),(3,1,9),(8,0,11),(2,8,3),(11,0,3),(10,11,2),(1,0,6),(2,5,9),(7,0,10),(3,5)\}$.

For $D_{13} - 3\vec{C_3} \cup 2\vec{C_2}$, Let D_{13} be defined on $Z_{12} \cup \{\infty\}$ where $3\vec{C_3} \cup 2\vec{C_2} = (0,1) \cup (2,3,4) \cup (5,6,7) \cup (8,9,10) \cup (11,\infty)$. Then $D_{13} - C_3 \cup C_3 \cup C_2 \cup C_2 \cup C_3 = \{(k,k+4,k+1)|k \in Z_{12}\setminus\{0,2,11\}\} \cup \{(k,k+2,k+5)|k \in Z_{12}\setminus\{2,3,9,11\}\} \cup \{(\infty,0,5),(\infty,3,8),(\infty,4,9),(\infty,5,10),(\infty,7,0),(\infty,8,1),(\infty,9,2),(\infty,10,3),(\infty,2,6),(\infty,1,7),(\infty,6,4),(11,4,10),(11,0,4),(4,1,2),(7,8,2),(6,11,1),(11,7,3),(5,11,9),(2,10,6),(1,4,7),(1,9,3),(0,6,3),(2,0,11),(9,7,2),(9,11,3),(3,2,8),(5,8,4),(5,3)\}$.

For $D_{13} - 2\vec{C_2} \cup \vec{C_6} \cup \vec{C_3}$, let D_{13} be defined on $Z_{12} \cup \{\infty\}$ where $2\vec{C_2} \cup \vec{C_6} \cup \vec{C_3} = (0,1) \cup (2,3,4,5,6,7) \cup (8,9,10) \cup (11,\infty)$. Then $D_{13} - 2\vec{C_2} \cup \vec{C_6} \cup \vec{C_3} = \{(k,k+2,k+5)|k \in Z_{12} \setminus \{2\}\} \cup \{(k,k+4,k+1)|k \in Z_{12} \setminus \{0\}\} \cup \{(k,k+6,k+2)|k \in Z_{12} \setminus \{0,4,8\}\} \cup \{(\infty,0,5),(\infty,1,6),(\infty,3,8),(\infty,4,9),(\infty,5,10),(\infty,8,1),(\infty,2,0),(\infty,9,2),(\infty,10,3),(\infty,6,4),(1,2,4),(10,11,4),(6,11,0),(10,6,2),(7,8,2),(0,4,7)\} \cup \{(\infty,7)\}$.

For $D_{13} - 5\vec{C_2} \cup \vec{C_3}$, let D_{13} be defined on $Z_{10} \cup \{\infty_i | i \in Z_3\}$ where $5\vec{C_2} \cup \vec{C_3} = (0,1) \cup (2,3,) \cup (4,5) \cup (6,7) \cup (8,9) \cup (\infty_0,\infty_1,\infty_2)$. Associate ∞_0 with difference 1 while $(9,3,2,5,1,7) \cup (6,8,0,4)$ can form 1-factor for ∞_1. Then the maximum packing is $\{(\infty_2,3,7),(\infty_2,7,1),(\infty_2,1,3),(\infty_2,9,2),(\infty_2,2,0),(\infty_2,0,9),(\infty_2,5,8),(\infty_2,8,6),(\infty_2,6,5),(0,3,1),(1,4,2),(3,6,4),(4,7,5),(6,9,7),(7,0,8),(8,1,9),(1,5,9),(3,5,7),(4,1,8),(8,5,2),(6,3,0),(0,7,4),(6,0,2),(4,8,2),(3,9,5),(6,2,9)\}$ with leave $(\infty_2,4)$.

For $D_{13} - \vec{C_{13}}$, let D_{13} be defined on $Z_{12} \cup \{\infty\}$ where $\vec{C_{13}} = (0,1,2,3,4,5,6,7,8,9,10,11,\infty)$. Then $D_{13} - \vec{C_{13}} = \{(k,k+2,k+6)|k \in Z_{12} \setminus \{11\}\} \cup \{(k,k+3,k+1)|k \in Z_{12} \setminus \{5\}\} \cup \{(k,k+7,k+3)|k \in Z_{12}\} \cup \{(\infty,2,7),(\infty,3,8),(\infty,4,9),(\infty,5,10),(\infty,7,0),(\infty,9,2),(\infty,10,3),(\infty,11,4),(\infty,6,5),(\infty,8,6),(11,0,5),(6,11,1),(1,5,8)\} \cup \{(\infty,1)\}$. □

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