The adjacent vertex-distinguishing total chromatic number of 1-tree

Haiying Wang[*†]
The School of Information Engineering
China University of Geosciences(Beijing)
Beijing 100083, P.R.China

Abstract Let $G = (V(G), E(G))$ be a simple graph and $T(G)$ be the set of vertices and edges of G. Let C be a $k-$color set. A (proper) total $k-$coloring f of G is a function f: $T(G) \longrightarrow C$ such that no adjacent or incident elements of $T(G)$ receive the same color. For any $u \in V(G)$, denote $C(u) = \{f(u)\} \cup \{f(uv)|uv \in E(G)\}$. The total $k-$coloring f of G is called the adjacent vertex-distinguishing if $C(u) \neq C(v)$ for any edge $uv \in E(G)$. And the smallest number of colors is called the adjacent vertex-distinguishing total chromatic number $\chi_{at}(G)$ of G. Let G be a connected graph. If there exists an vertex $v \in V(G)$ such that $G - v$ is a tree then G is a $1-$tree. In this paper, we will determine the adjacent vertex-distinguishing total chromatic number of 1-trees.

MSC: 05C15

Keywords The adjacent vertex-distinguishing total coloring; The adjacent vertex-distinguishing total chromatic number; 1-tree.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph and $T(G) = V(G) \cup E(G)$ be the set of vertices and edges of G. $\Delta(G)$, $\delta(G)$ and V_Δ denote the maximum degree, the minimum degree and the set of the vertices with degree $\Delta(G)$ respectively. For $v \in V(G)$, we use $N_G(v)$ to denote the neighbor set of v in $V(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S.

*E-mail: whycht@126.com.
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Let G be a connected graph. If there exists a vertex $v \in V(G)$ such that $G - v$ is a tree, then G is a $1-$tree.

Let C be a $k-$color set. A (proper) total $k-$coloring f of G is a function f: $T(G) \longrightarrow C$ such that no adjacent or incident elements of $T(G)$ receive the same color. For any $u \in V(G)$, denote $C(u) = \{f(u)\} \cup \{f(uv) | uv \in E(G)\}$. The total $k-$coloring f of G is called the adjacent vertex-distinguishing if $C(u) \neq C(v)$ for any edge $uv \in E(G)$. And the smallest integer k is called the adjacent vertex-distinguishing total chromatic number $\chi_{at}(G)$ of G. It is obvious that $\chi_{at}(G) \geq \Delta(G) + 1$.

Suppose that f is an adjacent vertex-distinguishing total coloring of G. For $w \in V(G)$, $f(N_G(w))$ denotes the color set of the edges incident to w in G and $f_G[w] = f(N_G(w)) \cup \{f(w)\}$. If an element t is colored α, then we denote $\alpha \Rightarrow t$.

After B.Bollóbas, A.C.Burris, R.H.Schelp, C.Bazgan and P.N.Balister discussed the vertex-distinguishing coloring in [1-3], Zhongfu Zhang, Linzhong Lin, Jianfang Wang and Xiangen Chen introduced the adjacent vertex-distinguishing edge coloring and the adjacent vertex-distinguishing total coloring in [6] and [7]. Some results on the subject have been obtained in [4-7].

Lemma 1([6]) If G has two vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G) + 2$.

Lemma 2([6]) If G has m components G_i $(i = 1, 2, \cdots, m)$ and $|V(G_i)| \geq 2$, $i = 1, 2, \cdots, m$, then $\chi_{at}(G) = \max\{\chi_{at}(G_1), \chi_{at}(G_2), \cdots, \chi_{at}(G_m)\}$.

Lemma 3([6]) Let C_n be a cycle with $n \geq 4$. Then $\chi_{at}(C_n) = 4$.

Lemma 4([6]) Let S_n be a star with $n \geq 3$. Then $\chi_{at}(S_n) = n + 1$.

Lemma 5([6]) Let F_n be a fan with $n \geq 4$. Then $\chi_{at}(F_n) = n + 1$.

Lemma 6([6]) Let T_n be a tree with $n \geq 4$. Then

$$\chi_{at}(T_n) = \begin{cases} \Delta(T_n) + 1, & \text{if} \quad E(T_n[V_\Delta]) = \emptyset, \\ \Delta(T_n) + 2, & \text{if} \quad E(T_n[V_\Delta]) \neq \emptyset. \end{cases}$$

Lemma 7([6]) Let $S_{n,m}$ be a double-star with $n \geq 4$ and $m \geq 4$. Then

$$\chi_{at}(S_{n,m}) = \begin{cases} \Delta(S_{n,m}) + 1, & \text{if} \quad n \neq m, \\ \Delta(S_{n,m}) + 2, & \text{if} \quad n = m. \end{cases}$$

In this paper, we will determine the adjacent vertex-distinguishing total chromatic number of 1-trees.

2. Main results

The adjacent vertex-distinguishing total chromatic number of the graphs G with $\Delta(G) \leq 2$ has been determined in [6] and [7]. So we only consider the graph G with $\Delta(G) \geq 3$. Then $|V(G)| \geq 4$.

Theorem 2.1 If G is a 1-tree, then

$$\chi_{at}(G) = \begin{cases} \Delta(G)+1, & \text{if} \quad E(G[V_\Delta]) = \emptyset, \\ \Delta(G)+2, & \text{if} \quad E(G[V_\Delta]) \neq \emptyset. \end{cases}$$

Proof: Suppose that G is a 1−tree. According to the definition of the adjacent vertex-distinguishing total chromatic number, $\chi_{at}(G) \geq \Delta(G) + 1$. By Lemma 1, if $E(G[V_\Delta]) \neq \emptyset$ then $\chi_{at}(G) \geq \Delta(G) + 2$. Thus,

$$\chi_{at}(G) \geq \begin{cases} \Delta(G)+1, & \text{if} \quad E(G[V_\Delta]) = \emptyset, \\ \Delta(G)+2, & \text{if} \quad E(G[V_\Delta]) \neq \emptyset. \end{cases}$$

So we will only prove that

$$\chi_{at}(G) \leq \begin{cases} \Delta(G)+1, & \text{if} \quad E(G[V_\Delta]) = \emptyset, \\ \Delta(G)+2, & \text{if} \quad E(G[V_\Delta]) \neq \emptyset. \end{cases}$$

According to the definition of 1−tree, there exists $v \in V(G)$ such that $G - v$ is a tree. Suppose that $T = G - v$.

If T is a star, then it is easy to prove the conclusion. So we assume that T is not a star. Then there exists $w \in V(T)$ such that $d_T(w) \neq 1$ and it has only one adjacent vertex u with $d_T(u) \geq 2$. Let $W(T)$ denote the set of such $w \in V(T)$ above. We will prove the conclusion by induction on $|V(G)|$.

It is obvious that Theorem 2.1 holds for $|V(G)| = 4$. Now assume that $|V(G)| \geq 5$. Suppose that w is a vertex with the smallest degree in $W(T)$. Let $N_T(w) = \{u, v_1, \cdots, v_s\}$ with $d_T(u) \geq 2$ and $d_T(v_i) = 1$ for all $i \in \{1, 2 \cdots, s\}$ with $s \geq 1$.

Case 1 $wv \in E(G)$.

Subcase 1.1 $d_G(v_i) = 2$ and $d_G(v_j) = 1$ for all $i \in \{1, 2, \cdots, k-1\}$ and $j \in \{k, \cdots, s\}$ with $k \geq 2$ and $s \geq k$.

In this subcase, we consider the graph $G_0 = G - \{v_k, \cdots, v_s\}$. It is easy to see that $|V(G_0)| < |V(G)|$ and G_0 is also a 1-tree with $\Delta(G_0) = \Delta(G) = \Delta$. By the induction assumption G_0 has an adjacent vertex-distinguishing total coloring f' with the color set C_0 with $|C_0| \in \{\Delta+1, \Delta+2\}$. In the following, we will extend f' to an adjacent vertex-distinguishing total coloring f of G with the color set C with $C_0 \subseteq C$ and $|C| \in \{\Delta+1, \Delta+2\}$, respectively.

There are four subcases in all, denoted (a)(b)(c)(d) below.

(a) $d_G(w) \neq d_G(v)$ and $d_G(w) \neq d_G(u)$.
(b) $d_G(w) \neq d_G(v)$ and $d_G(w) = d_G(u)$.
(c) $d_G(w) = d_G(v)$ and $d_G(w) \neq d_G(u)$.
(d) $d_G(w) = d_G(v) = d_G(u)$.

Now we only deal with the case of $d_G(w) = d_G(v) = d_G(u) = \Delta = s + 2$ with $s \geq 2$ and $k \geq 2$ (see Figure 1) (A similar arguments work for other cases). In this case, $E(G_0[V_\Delta]) \neq \emptyset$. Then $|C_0| = \Delta + 2$.

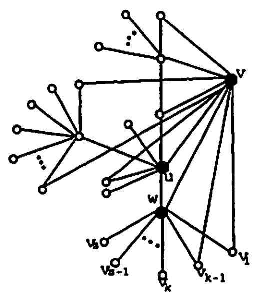

Figure 1

Firstly, let $C_0 = C$. Since $|f'_{G_0}[u]| = \Delta + 1$, there exists at least one color $c_1 \in C_0$ such that $c_1 \notin f'_{G_0}[u]$. Similarly, there exists at least one color $c_0 \in C$ such that $c_0 \notin f'_{G_0}[v]$. Secondly, we do it step by step below.

Note: the elements which have not yet colored, retain the colors given by f' when we extend f' to f of G respectively. In the proof below, we will not mention it again.

Step 1
If $c_1 \notin f'_{G_0}[w]$ and $c_0 \notin f'_{G_0}[w] \cup \{c_1\}$ with $d_G(v) = 2$ then $d_G(w) \geq 3$. Let $c_1 \Rightarrow wv_k$.
If $c_1 \notin f'_{G_0}[w]$ and $c_0 \notin f'_{G_0}[w] \cup \{c_1\}$ with $d_G(v) \neq 2$ then firstly select any color $c \in \{c_0, c_1\} - f'(vv_{k-1})$ and let $c \Rightarrow wv_{k-1}$; secondly, select any color $\alpha \in C - \{f'(w), c\} \cup \{f'(v), f'(vv_{k-1})\}$ and let $\alpha \Rightarrow v_{k-1}$.
If $c_1 \notin f'_{G_0}[w]$ and $c_0 = c_1$ then let $c_0 (= c_1) \Rightarrow wv_k$.
If $c_1 \in f'_{G_0}[w]$ and $c_0 \notin f'_{G_0}[w]$ then let $c_0 \Rightarrow wv_k$.
If $c_1 \in f'_{G_0}[w]$ and $c_0 \in f'_{G_0}[w]$ then do Step 2 directly.

Step 2 Let $\{f(wv_k), \cdots, f(wv_s)\} \subseteq C - \{f'(w), f'(wu), f'(wv), f'(wv_1), \cdots, f'(wv_{k-2}), f(wv_{k-1})\}$.

Step 3 Select $\alpha_i \in C - \{f(wv_i), f'(w)\}$ and let $\alpha_i \Rightarrow v_i$ for all $i \in \{k, \cdots, s\}$.

Subcase 1.2 $d_G(v_i) = 2$ for all $i \in \{1, 2, \cdots, s\}$ with $s \geq 1$.

(I) There exists one vertex $x \in V(G)$ such that $d_G(x) = 1$ (see Figure 2).

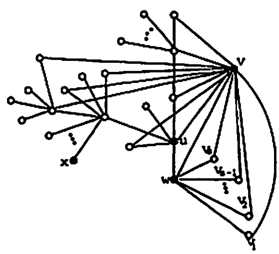

Figure 2

We consider the graph $G_0 = G - v_1 + vx$. It is easy to see that $|V(G_0)| < |V(G)|$ and G_0 is also a 1-tree with $\Delta(G_0) = \Delta(G) = \Delta$. By the induction assumption G_0 has an adjacent vertex-distinguishing total coloring f' with the color set C_0 with $|C_0| \in \{\Delta + 1, \Delta + 2\}$. In the following, we will extend f' to an adjacent vertex-distinguishing total coloring f of G with the color set C with $C_0 \subseteq C$ and $|C| \in \{\Delta + 1, \Delta + 2\}$, respectively.

(A) If $E(G_0[V_\Delta]) \neq \emptyset$ then $|C_0| = \Delta(G) + 2$ and let $C = C_0$. Since $|f'_{G_0}[v]| = d_G(v) + 1 \leq \Delta(G) + 1$, there exists at least one color $c_0 \in C$ such that $c_0 \notin f'_{G_0}[v]$. Similarly, there exists at least one color $c_1 \in C$ such that $c_1 \notin f'_{G_0}[u]$. So $c_0 \neq f'(vx)$ and $c_1 \neq f'(uw)$. Assume that $f'(v_i) = c_1$ for $i \in S \subseteq \{2, \cdots, s\}$ below.

(A.1) $d_G(v) \geq 3$.

Step 1 Let $f'(vx) \Rightarrow vv_1$.
Step 2

If $c_0 \notin f'_{G_0}[w] \cup \{c_1\}$ and $c_1 \notin f'_{G_0}[w] \cup \{f'(vx)\}$ then firstly let $c_1 \Rightarrow w$ and $c_0 \Rightarrow wv_1$; secondly, select any color $a_i \in C - (\{f'(wv_i), f'(vv_i)\} \cup \{f'(w), f'(v)\})$ and let $a_i \Rightarrow v_i$ for all $i \in S$.

If $c_0 = c_1 \notin f'_{G_0}[w]$ then $c_0 = c_1 \neq f'(vx)$. Let $c_0 = c_1 \Rightarrow wv_1$.

If $c_0 \notin f'_{G_0}[w]$ and $c_1 = f'(vx) \notin f'_{G_0}[w]$ then $c_0 \neq c_1$. Firstly, let $c_0 \Rightarrow wv_1$ and

$c_1 \Rightarrow w$; secondly, select any color $a_i \in C - (\{f'(wv_i), f'(vv_i)\} \cup \{f'(w), f'(v)\})$ and let $a_i \Rightarrow v_i$ for all $i \in S$.

If $c_0 \notin f'_{G_0}[w]$ and $c_1 \in f'_{G_0}[w]$ then let $c_0 \Rightarrow wv_1$.

If $c_0 \in f'_{G_0}[w]$ and $c_1 \notin f'_{G_0}[w] \cup \{f'(vx)\}$ then let $c_1 \Rightarrow wv_1$.

If $c_0 = f'(w) \in f'_{G_0}[w]$ and $c_1 = f'(vx) \notin f'_{G_0}[w]$, then firstly let $c_1 \Rightarrow w$ and $f'(w)(= c_0) \Rightarrow wv_1$; secondly, select any color $a_i \in C - (\{f'(wv_i), f'(vv_i)\} \cup \{f'(w), f'(v)\})$ and let $a_i \Rightarrow v_i$ for all $i \in S$.

If $c_0 \in f'_{G_0}[w]$ and $c_1 = f'(vx) \notin f'_{G_0}[w]$, but $f'(w) \neq c_0$, then firstly let $c_1 \Rightarrow w$ and $f'(w) \Rightarrow wv_1$; secondly, select any color $a_i \in C - (\{f'(wv_i), f'(vv_i)\} \cup \{f'(w), f'(v)\})$ and let $a_i \Rightarrow v_i$ for all $i \in S$.

If $c_0 \in f'_{G_0}[w]$ and $c_1 \in f'_{G_0}[w]$ then select any color $\alpha \in C - (f'_{G_0}[w] \cup \{f'(vx)\})$ and let $\alpha \Rightarrow wv_1$.

Step 3 Select any $\alpha_1 \in C - (\{f(w), f(wv_1)\} \cup \{f'(vx), f'(v)\})$ and let $\alpha_1 \Rightarrow v_1$.

(A.2) $d_G(v) = 2$.

In this subcase, $s = 1$. We reconsider the graph $G_0 = G - v + wx$. It is easy to see that $|V(G_0)| < |V(G)|$ and G_0 is also a 1-tree with $\Delta(G_0) = \Delta(G) = \Delta$. By the induction assumption G_0 has an adjacent vertex-distinguishing total coloring f' with the color set C_0 with $|C_0| = \Delta + 2$. Let $C = C_0$. Firstly, select any color $c \in C - f'_{G_0}[w]$ and let $c \Rightarrow v_1$. Secondly, $f'(wu) \Rightarrow v$ and $f'(w) \Rightarrow vv_1$.

(B) If $E(G_0[V_\Delta]) = \emptyset$ and $E(G[V_\Delta]) \neq \emptyset$ then $|C_0| = \Delta(G) + 1$. Let $C_0 \subset C$ and $|C| = \Delta(G) + 2$. Assume that $c \in C - C_0$.
Step 1 Let $f'(vx) \Rightarrow vv_1$.
Step 2 Let $c \Rightarrow wv_1$.
Step 3 Select any $\alpha_1 \in C - (\{f'(w), f(wv_1)\} \cup \{f'(vx), f'(v)\})$ and let $\alpha_1 \Rightarrow v_1$.

(C) If $E(G_0[V_\Delta]) = \emptyset$ and $E(G[V_\Delta]) = \emptyset$ then $|C_0| = \Delta(G) + 1$. Let $C = C_0$.

(C.1) $d_G(v) \geq 3$.

(C.1.1) If $d_G(v) = d_G(u) = \Delta(G)$ then $d_G(w) \neq \Delta(G) \geq 4$ and $uv \notin E(G)$.
Step 1 Let $f'(vx) \Rightarrow vv_1$.
Step 2 Select any $\alpha \in C - (f'_{G_0}[w] \cup \{f'(vx)\})$ and let $\alpha \Rightarrow wv_1$.
Step 3 Select any $\alpha_1 \in C - (\{f'(w), f(wv_1)\} \cup \{f'(vx), f'(v)\})$ and let $\alpha_1 \Rightarrow v_1$.

(C.1.2) If $d_G(v) = \Delta(G)$ and $d_G(u) \neq \Delta(G)$ then $d_G(w) \neq \Delta(G) \geq 4$. So there exists at least one color $c_1 \in C$ such that $c_1 \notin f'_{G_0}[u]$. Assume that $f'(v_i) = c_1$ for $i \in S \subseteq \{2, \cdots, s\}$ below.
Step 1 Let $f'(vx) \Rightarrow vv_1$.
Step 2
If $c_1 \notin f'_{G_0}[w]$ and $c_1 \neq f'(vx)$ then let $c_1 \Rightarrow wv_1$.
If $c_1 \notin f'_{G_0}[w]$ and $c_1 = f'(vx)$ then firstly let $c_1 \Rightarrow w$ and $f'(w) \Rightarrow wv_1$; secondly, select any color $a_i \in C - (\{f'(wv_i), f'(vv_i)\} \cup \{f'(w), f'(v)\})$ and let $a_i \Rightarrow v_i$ for all $i \in S$.
If $c_1 \in f'_{G_0}[w]$ then select any $\alpha \in C - f'_{G_0}[w]$ and let $\alpha \Rightarrow wv_1$.
Step 3 Select any $\alpha_1 \in C - (\{f'(w), f(wv_1)\} \cup \{f'(vx), f'(v)\})$ and let $\alpha_1 \Rightarrow v_1$.

(C.1.3) If $d_G(v) \neq \Delta(G)$ and $d_G(u) = \Delta(G)$ then $d_G(w) \neq \Delta(G)$. So there exists one color $c_0 \in C_0$ such that $c_0 \notin f'_{G_0}[v]$.
Step 1 Let $f'(vx) \Rightarrow vv_1$.
Step 2
If $c_0 \notin f'_{G_0}[w]$ then let $c_0 \Rightarrow wv_1$.
If $c_0 \in f'_{G_0}[w]$ then select any $\alpha \in C - f'_{G_0}[w]$ and let $\alpha \Rightarrow wv_1$.
Step 3 Select any $\alpha_1 \in C - (\{f'(w), f(wv_1)\} \cup \{f'(vx), f'(v)\})$ and let $\alpha_1 \Rightarrow v_1$.

(C.1.4) If $d_G(v) \neq \Delta(G)$ and $d_G(u) \neq \Delta(G)$ then there exists at least one color $c_0 \in C$ such that $c_0 \notin f'_{G_0}[v]$. Similarly, there exists at least one color $c_1 \in C$ such that $c_1 \notin f'_{G_0}[u]$ with $c_1 \neq f'(uw)$. Its steps are similar to (A).

(C.2) $d_G(v) = 2$.

In this subcase, $s = 1$. We reconsider the graph $G_0 = G - v + wx$. It is easy to see that $|V(G_0)| < |V(G)|$ and G_0 is also a 1-tree with $\Delta(G_0) = \Delta(G) = \Delta$. By the induction assumption G_0 has an adjacent vertex-distinguishing total coloring f' with the color set C_0 with $|C_0| = \Delta + 1$. Let $C = C_0$. Firstly, let $f'(wx) \Rightarrow wv$ and $f'(wx) \Rightarrow v_1$. Secondly, $f'(wu) \Rightarrow v$ and $f'(w) \Rightarrow vv_1$.

(II) $d_G(x) \geq 2$ for all $x \in V(G)$ (see Figure 3).

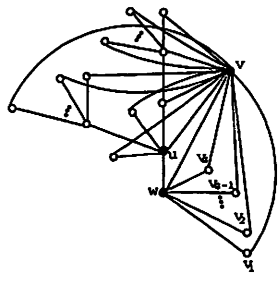

Figure 3

In this subcase, v is the unique vertex with maximum degree in G. We consider the graph $G_0 = G - v_1$. It is easy to see that $|V(G_0)| < |V(G)|$ and G_0 is also a 1−tree with $\Delta(G) = \Delta(G_0) + 1$. By the induction assumption G_0 has an adjacent vertex-distinguishing total coloring f' with the color set C_0 with $|C_0| = d_{G_0}(v) + 1$. In the following, we can extend f' to an adjacent vertex-distinguishing total coloring f of G with the color set C with $C_0 \subseteq C$ step by step.

Since $d_{G_0}(x) \geq 2$ for all $x \in V(G_0)$, v is also the unique vertex with maximum degree in G_0. So $d_G(v) = d_{G_0}(v) + 1$ and $\Delta(G) = \Delta(G_0) + 1$. Firstly, let $|C| = |C_0| + 1 (= d_G(v) + 1)$. Assume that $\alpha \in C - C_0$.

Step 1 Let $\alpha \Rightarrow vv_1$.
Step 2 Select any $\beta \in C - (f'_{G_0}[w] \cup \{\alpha\})$ and let $\beta \Rightarrow wv_1$.
Step 3 Select any $\gamma \in C - (\{\alpha, \beta\} \cup \{f'(w), f'(v)\})$ and let $\gamma \Rightarrow v_1$.

Subcase 1.3 $d_G(v_i) = 1$ for all $i = 1, 2, \cdots, s$ with $s \geq 1$ (see Figure 4).

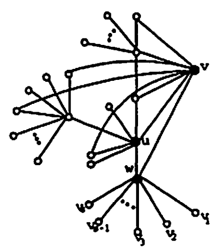

Figure 4

In this subcase, we can consider the graph $G_0 = G - \{v_1, \cdots, v_s\}$. Then $\Delta(G) = \Delta(G_0)$ or $\Delta(G) = \Delta(G_0) + 1$. It is easy to see that $|V(G_0)| < |V(G)|$ and G_0 is also a 1-tree with $\Delta(G_0) \leq \Delta(G)$. By the induction assumption G_0 has an adjacent vertex-distinguishing total coloring f' with the color set C_0 with $|C_0| \in \{\Delta(G_0) + 1, \Delta(G_0) + 2\}$. In the following, we can extend f' to an adjacent vertex-distinguishing total coloring f of G with the color set C with $|C| \in \{\Delta(G) + 1, \Delta(G) + 2\}$ and $C_0 \subseteq C$.

Step 1

(A) If $\Delta(G) = \Delta(G_0) + 1$ then w is the unique vertex with maximum degree in G. Let $C_0 \subseteq C$ and $|C| = \Delta(G) + 1 (= \Delta(G_0) + 2)$. Firstly, let

$\{f(wv_1), \cdots, f(wv_s)\} \subseteq C - f'_{G_0}[w]$. Secondly, do Step 3 directly.

(B) If $\Delta(G) = \Delta(G_0) = \Delta$ then we may assume that $wv \in E(G)$ and $uv \in E(G)$.

(B.1) If $E(G_0[V_\Delta]) \neq \emptyset$ then $E(G[V_\Delta]) \neq \emptyset$ and $|C_0| = \Delta(G) + 2$. Let $C_0 = C$. Assume that $d_G(w) = d_G(v) = d_G(u) = \Delta$ and $\{uv, wv\} \subset E(G)$. Assume that $c_1 \in C_0 - f'_{G_0}[u]$ and $c_0 \in C_0 - f'_{G_0}[v]$. Then $c_0 \neq c_1$.

If $c_1 \notin \{f'(wv), f'(v)\}$ and $f'(wu) = c_0$ then firstly let $c_1 \Rightarrow w$; secondly, do Step 2(1).
If $c_1 \notin \{f'(wv), f'(v)\}$ and $f'(wu) \neq c_0$ then firstly let $c_0 \Rightarrow wv_1$; secondly, do Step 2(2).

If $c_1 = f'(wv)$ then firstly let $c_0 \Rightarrow wv_1$; secondly, do Step 2(2).

If $c_1 = f'(v)$ and $c_0 \notin \{f'(u), f'(wu)\}$ then firstly let $c_0 \Rightarrow w$ and $c_1 \Rightarrow wv_1$; secondly, do Step 2(2).

If $c_1 = f'(v)$ and $c_0 = f'(wu)$ then firstly let $c_1 \Rightarrow wv_1$; secondly, do Step 2(2).

If $c_1 = f'(v)$ and $c_0 = f'(u)$ with $s \geq 2$, then firstly let $c_0 \Rightarrow wv_1$ and $c_1 \Rightarrow wv_2$; secondly, do Step 2(3).

If $c_1 = f'(v)$ and $c_0 = f'(u)$ with $s = 1$, then $\Delta = 3$. Assume that $v' \in N_G(v) - \{u, w\}$.
If $f'(uv) \neq f'(v')$ then firstly let $f'(uv) \Rightarrow v$ and $c_1 \Rightarrow uv$; secondly, let $c_1 \Rightarrow w$ and $c_0 \Rightarrow wv_1$; finally, do Step 3.
If $f'(uv) = f'(v')$ then $f'(wv) \neq f'(v')$. Firstly, let $f'(wv) \Rightarrow v$ and $c_1 \Rightarrow wv$; secondly, let $c_0 \Rightarrow wv_1$; thirdly, select any color $\alpha \in C - (\{c_0, c_1\} - \{f'(wu)\})$ and let $\alpha \Rightarrow w$; finally, do Step 3.

(B.2) If $E(G_0[V_\Delta]) = E(G[V_\Delta]) = \emptyset$ then $|C_0| = \Delta + 1$. Let $C = C_0$.

If $d_G(w) = \Delta$ then $d_G(u) \neq \Delta$ and $d_G(v) \neq \Delta$. Let $\{f(wv_1), \cdots, f(wv_s)\} \subseteq C - f'_{G_0}[w]$. Finally, do Step 3 directly.

If $d_G(w) \neq \Delta$ then we only deal with the case of $d_G(u) = d_G(v) = d_G(w) < \Delta$ with $wv \in E(G)$. Assume that $c_0 \in C_0 - f'_{G_0}[v]$ and $c_1 \in C_0 - f'_{G_0}[u]$. Its steps are similar to Subcase (B.1).

(B.3) If $E(G_0[V_\Delta]) = \emptyset$ and $E(G[V_\Delta]) \neq \emptyset$ then $|C_0| = \Delta(G) + 1$. Let $C_0 \subset C$ and $|C| = |C_0| + 1$. Assume that $\alpha \in C - C_0$. Firstly, let $\alpha \Rightarrow w$. Secondly, do Step 2(1).

Step 2
(1) Let $\{f(wv_1), \cdots, f(wv_s)\} \subseteq C - \{f'(uw), f'(wv), f(w)\}$.
(2) Let $\{f(wv_2), \cdots, f(wv_s)\} \subseteq C - (\{f'(uw), f'(wv), f(w)\} \cup \{ f(wv_1)\})$.
(3) Let $\{f(wv_3), \cdots, f(wv_s)\} \subseteq C - (f'_{G_0}[w] \cup \{ f(wv_1), f(wv_2)\})$.

Step 3 Select any $\alpha_i \in C - \{f(wv_i), f'(w)\}$ and let $\alpha_i \Rightarrow v_i$ for all $i = 1, \cdots, s$.

It is easy to verify that the coloring f above is an adjacent vertex-distinguishing total coloring of G in the cases respectively.

Case 2 If $wv \notin E(G)$ then its proof is very similar to Case 1 by the definition of the adjacent vertex-distinguishing total coloring.

Thus, Theorem 2.1 holds. □

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