Some new monoid and group constructions under semidirect products

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Abstract

In this paper we mainly define semidirect product version of the Schützenberger product and also a new two-sided semidirect product construction for arbitrary two monoids. Then, as main results, we present a generating and a relator set for these two products. Additionally, to explain why these products have been defined, we investigate the *regularity* for the semidirect product version of Schützenberger products and the *subgroup separability* for this new two-sided semidirect product.

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1 Introduction

In [1, 6], it has been defined a generator and a relator set for semidirect and the Schützenberger products of arbitrary monoids. Moreover, in [8], it has been given regularity of the semidirect product and, in [7], it has been proved some basic theorems for subgroup separability of HNN-extensions with abelian base group. By using similar methods as in these above papers, we propose to define a semidirect product version of the Schützenberger product and to give a related standard presentation of it. Besides that we give necessary and sufficient conditions for this product to be regular. We also propose to construct a new two-sided semidirect product and to give a related presentation of this new product. Then we will investigate its subgroup separability.

Let A and B be monoids with associated presentations $\mathcal{P}_A = [X; R]$ and $\mathcal{P}_B = [Y; S]$, respectively. Let $M = A \rtimes_\theta B$ be the corresponding semidirect product of these two monoids, where θ is a monoid homomorphism from B to $End(A)$ such that, for every $a \in A$, $b_1, b_2 \in B$,

$$(a)\theta_{b_1 b_2} = ((a)\theta_{b_2})\theta_{b_1}. \tag{1}$$

We recall that the elements of M can be regarded as ordered pairs (a, b), where $a \in A$, $b \in B$ with the multiplication given by

$$(a_1, b_1)(a_2, b_2) = (a_1(a_2)\theta_{b_1}, b_1 b_2),$$

and the monoids A and B are identified with the submonoids of M having elements $(a, 1_B)$ and $(1_A, b)$. For every $x \in X$ and $y \in Y$, choose a word, denoted by $(x)\theta_y$, on X such that $[(x)\theta_y] = [x]\theta_{[y]}$ as an element of A. To establish notation, let us denote the relation $yx = (x)\theta_y y$ on $X \cup Y$ by T_{yx} and write T for the set of relations T_{yx}. Then, for any choice of the words $(x)\theta_y$,

$$\mathcal{P}_M = [X, Y \; ; \; R, S, T]$$

is a standard monoid presentation for the semidirect product M.

Now for a subset P of $A \times B$ and $a \in A$, $b \in B$, we define

$$Pb = \{(c, db) \; ; \; (c, d) \in P\} \quad \text{and} \quad aP = \{(ac, d) \; ; \; (c, d) \in P\}.$$

Then the Schützenberger product of A and B, denoted by $A \Diamond B$, is the set $A \times P(A \times B) \times B$ with the multiplication

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 a_2, P_1 b_2 \cup a_1 P_2, b_1 b_2).$$

Clearly $A \Diamond B$ is a monoid with the identity $(1_A, \emptyset, 1_B)$.

We note that definitions of the above two products will be needed to construct Section 2 and 3 below.

2 A semidirect product version of the Schützenberger product

Let A and B be monoids. For $P \subseteq A \times B$ and $b \in B$, we define

$$Pb = \{(a, db) \; ; \; (a, d) \in P\}.$$

The semidirect product version of the Schützenberger product of A by B, denoted by $A \Diamond_{sv} B$, is the set $A \times P(A \times B) \times B$ with the multiplication

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 (a_2)\theta_{b_1}, P_1 b_2 \cup P_2, b_1 b_2).$$

Then $A \Diamond_{sv} B$ is a monoid with the identity $(1_A, \emptyset, 1_B)$ where θ is defined as in (1). Let us show the associative property:

Let $(a_1, P_1, b_1), (a_2, P_2, b_2)$ and (a_3, P_3, b_3) be the elements of $A \Diamond_{sv} B$. Thus we get

$$\begin{aligned} &((a_1, P_1, b_1)(a_2, P_2, b_2))(a_3, P_3, b_3) \\ &= (a_1 (a_2)\theta_{b_1}, P_1 b_2 \cup P_2, b_1 b_2)(a_3, P_3, b_3) \\ &= (a_1 (a_2)\theta_{b_1} (a_3)\theta_{b_1 b_2}, P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3) \end{aligned}$$

and

$$
\begin{aligned}
&(a_1, P_1, b_1)((a_2, P_2, b_2)(a_3, P_3, b_3)) \\
&= (a_1, P_1, b_1)(a_2(a_3)\theta_{b_2}, P_2 b_3 \cup P_3, b_2 b_3) \\
&= (a_1(a_2(a_3)\theta_{b_2})\theta_{b_1}, P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3) \\
&= (a_1(a_2)\theta_{b_1}(a_3)\theta_{b_1 b_2}, P_1 b_2 b_3 \cup P_2 b_3 \cup P_3, b_1 b_2 b_3).
\end{aligned}
$$

In the following lemma and theorem, we give a generating set and a presentation of this product, as a first main result of this paper.

Lemma 2.1 *Let us suppose that the monoids A and B are generated by the sets X and Y, respectively. Then the product $A \lozenge_{sv} B$ is generated by the union of the sets $\{(x, \emptyset, 1_B); x \in X\}$,$\{(1_A, \emptyset, y); y \in Y\}$ and $\{(1_A, \{(a,b)\}, 1_B); a \in A, b \in B\}$.*

Proof. For $x \in X$, $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $P_1, P_2 \subseteq A \times B$, we can easily show that the proof follows from

$$
\begin{aligned}
(a_1, \emptyset, 1_B)(a_2, \emptyset, 1_B) &= (a_1 a_2, \emptyset, 1_B), && (2)\\
(1_A, \emptyset, b_1)(1_A, \emptyset, b_2) &= (1_A, \emptyset, b_1 b_2), && (3)\\
(1_A, P_1, 1_B)(1_A, P_2, 1_B) &= (1_A, P_1 \cup P_2, 1_B), && (4)\\
(a, \emptyset, 1_B)(1_A, \emptyset, b)(1_A, P, 1_B) &= (a, P, b),
\end{aligned}
$$

as required. □

Theorem 2.2 *Let us suppose that the monoids A and B are defined by presentations $[X; R]$ and $[Y; S]$, respectively. Then the semidirect product version of the Schützenberger product of A by B is defined by generators*

$$Z = X \cup Y \cup \{z_{a,b} \ ; \ a \in A, b \in B\}$$

and the relations

$$R, \ S, \qquad (5)$$

$$yx = ((x)\theta_y)y \quad (x \in X,\ y \in Y), \tag{6}$$

$$z_{a,b}^2 = z_{a,b},\ z_{a,b}z_{c,d} = z_{c,d}z_{a,b}\,(a, c \in A,\ b, d \in B), \tag{7}$$

$$z_{a,b}y = yz_{a,by},\ xz_{a,b} = z_{a,b}x\,(x \in X,\ y \in Y,\ a \in A,\ b \in B). \tag{8}$$

Proof. Let us denote the set of all words in Z by Z^*. Also let

$$\psi : Z^* \longrightarrow A\Diamond_{sv}B$$

be a homomorphism defined by $(x)\psi = (x, \emptyset, 1_B)$,$(y)\psi = (1_A, \emptyset, y)$ and $(z_{a,b})\psi = (1_A, \{(a, b)\}, 1_B)$ where $x \in X$, $y \in Y$, $a \in A$ and $b \in B$. Then, by Lemma 2.1, we say that θ is onto. Now let us check whether $A\Diamond_{sv}B$ satisfies relations (5)-(8). In fact relations (5) and (7) follow from (2), (3) and (4). For relations (8), we have

$$\begin{aligned}(1_A, \{(a, b)\}, 1_B)(1_A, \emptyset, y) &= (1, \{(a, by)\}, y)\\ &= (1_A, \emptyset, y)(1_A, \{(a, by)\}, 1_B),\\ (x, \emptyset, 1_B)(1_A, \{(a, b)\}, 1_B) &= (x, \{(a, b)\}, 1_B)\\ &= (1_A, \{(a, b)\}, 1_B)(x, \emptyset, 1_B).\end{aligned}$$

Now let us show that relations (6) hold.

$$(1_A, \emptyset, y)(x, \emptyset, 1_B) = ((x)\theta_y, \emptyset, y) = ((x)\theta_y, \emptyset, 1_B)(1_A, \emptyset, y),$$

for all $x \in X$, $y \in Y$, $a \in A$, $b \in B$. Therefore ψ induces an epimorphism $\overline{\psi}$ from the monoid M defined by (5)-(8) onto $A\Diamond_{sv}B$.

Let $w \in Z^*$ be any non-empty word. By using relations (6) and (8), there exist words w_x in X^*, $w_y \in Y^*$ and $w_{a,b} \in \{z_{a,b} : a \in A, b \in B\}^*$ such that $w = w_x w_y w_{a,b}$ in M. Moreover it can be noted that relations (7) can be used to prove there exists a set $P(w) \subseteq A \times B$ such that $w_{a,b} = \prod_{(a,b)\in P(w)} z_{a,b}$. Therefore, for any word $w \in Z^*$, we have

$$\begin{aligned}(w)\psi &= (w_x w_y w_{a,b})\psi = (w_x)\psi(w_y)\psi(w_{a,b})\psi\\ &= (w_x, \emptyset, 1_B)(1_A, \emptyset, w_y)(1_A, P(w), 1_B)\\ &= (w_x, P(w), w_y).\end{aligned}$$

Now, if $(w')\psi = (w'')\psi$ for some $w', w'' \in Z^*$ then, by the equality of these components, we deduce that $w'_x = w''_x$ in A, $w'_y = w''_y$ in B and $P(w') = P(w'')$. Relations (5) imply that $w'_x = w''_x$ and $w'_y = w''_y$ hold in M. So that $w' = w''$ holds. Thus $\overline{\psi}$ is injective. □

As an application of Theorem 2.2, we can give the following corollary while A and B are finite cyclic monoids. We note that some of the fundamental facts about cyclic monoids can be found in [1, 5].

Corollary 2.3 *Let A and B be finite cyclic monoids with presentations*

$$\wp_A = [x; x^k = x^l (k > l)] \text{ and } \wp_B = [y; y^s = y^t (s > t)],$$

respectively. Suppose $x^{i^s} = x^{i^t}$ $(0 \leq i < k)$ holds. Then the product $A \Diamond_{sv} B$ has a presentation

$$\begin{aligned} \wp_{A\Diamond_{sv}B} = [x, y, \ z_{x^m,y^n} \ ; \ & x^k = x^l, \ x z_{x^m,y^n} = z_{x^m,y^n} x, \\ & yx = x^i y, \ z^2_{x^m y^n} = z_{x^m,y^n}, \\ & z_{x^m,y^n} z_{x^p,y^q} = z_{x^p,y^q} z_{x^m,y^n} \ , \\ & y^s = y^t \ , z_{x^m,y^n} y = y z_{x^m,y^{n+1}}] \end{aligned}$$

where $0 \leq m, p \leq k-1$, $0 \leq n, q \leq s-1$.

Proof. Let δ_i $(0 \leq i < k)$ be an endomorphism of A. Then we have a mapping

$$y \rightarrow End(A), \quad y \longmapsto \delta_i.$$

By [4], this induces a homomorphism

$$\theta : B \longrightarrow End(A), \quad y \longmapsto \delta_i$$

if and only if

$$\delta_i^s = \delta_i^t.$$

Since δ_i^s and δ_i^t are equal if and only if they agree on the generator x of A then we must have $x^{i^s} = x^{i^t}$. This implies that $yx = x^i y$. The rest of the relations can be easily seen by using relations (5), (7) and (8). Hence the result. □

By the following note, one can explain the reason why this semidirect product version of the Schützenberger product has been defined in this paper.

Remark 2.4 1) *In [2, 4], it has been investigated the p-Cockcroft property of some extensions with finite generating sets. Besides, in literature, we have not reached any study about the p-Cockcroft property of Schützenberger products of monoids. So one can examine whether this property holds under this new version. Because this new version can be thought as a homomorphic image of the semidirect product. So one can give some new efficient (equivalently, p-Cockcroft) presentation examples of monoids by using this type of homomorphic image of the semidirect product.*

2) *One can easily see that if A (or B) is infinite, then the product $A\Diamond_{sv}B$ has infinite generating set. So we can transfer some other algebraic properties from the semidirect product with finite generating set to this new version with infinite (or finite) generating set. For instance, in the following subsection, we give necessary and sufficient conditions for this new version to be regular by using homomorphic image of the semidirect product.*

2.1 Regularity of $A\Diamond_{sv}B$

In this section we aim to give the necessary and sufficient conditions for $A\Diamond_{sv}B$ to be regular while both A and B are arbitrary monoids. To do that, as depicted in Remark 2.4-2, we will work on not only finite generating set. In fact the generating set of $A\Diamond_{sv}B$ can be infinite while A and B have finite generating set.

For an element a in a monoid M, let us take a^{-1} for the set of inverses of a in M, that is, $a^{-1} = \{b \in B : aba = a$ and $bab = b\}$. Hence M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the emptyset.

Theorem 2.5 *Let A and B be any monoids. The product $A\Diamond_{sv}B$ is regular if and only if*

(i) A and B are regular,

(ii) for every $a \in A$ and $b \in B$, there exists an idempotent $e^2 = e \in B$ such that $bB = eB$ and $a \in A(a)\theta_e$, where $\theta : B \to End(A)$ is homomorphism as in (1).

(iii) for every $(a, P, b) \in A\Diamond_{sv}B$, either

$$P = P_1 b = \bigcup_{(a_1,b_1)\in P_1} \{(a_1, b_1 b)\}$$

or

$$P = P_1 bd = \bigcup_{(a_1,b_1)\in P_1} \{(a_1, b_1 bd)\},$$

where $P_1 \subseteq A \times B$ and $d \in b^{-1}$.

Proof. Let us suppose that $A\Diamond_{sv}B$ is regular. Thus, for $(a, \emptyset, 1_B) \in A\Diamond_{sv}B$, there exists (c, P, d) such that

$$\begin{aligned}(a,\ \emptyset, 1_B) &= (a,\ \emptyset, 1_B)(c, P, d)(a,\ \emptyset, 1_B) = (ac(a)\theta_d, P, d),\\ (c, P, d) &= (c, P, d)(a,\ \emptyset, 1_B)(c, P, d) = (c(a)\theta_d(c)\theta_d, P, dd).\end{aligned}$$

Therefore $d = 1_B$. This shows that $a = aca$ and $c = cac$. By using the similar argument, we can show that $b = bdb$ and $d = dbd$. This implies that $bd = e = e^2$ satisfies $bB = eB$ and $a \in A(a)\theta_e$. Hence both (i) and (ii) hold. By the assumption on the regularity of $A\Diamond_{sv}B$, for $(a, P, b) \in A\Diamond_{sv}B$, we have $(c, P_2, d) \in A\Diamond_{sv}B$ such that

$$\begin{aligned}(a, P, b) &= (a, P, b)(c, P_2, d)(a, P, b),\\ (c, P_2, d) &= (c, P_2, d)(a, P, b)(c, P_2, d).\end{aligned}$$

Hence $P = Pdb \cup P_2 b \cup P$ and $P_2 = P_2 bd \cup Pd \cup P_2$. Since $b = bdb$ and $d = dbd$, for every $(a, P, b) \in A\Diamond_{sv}B$, either $P = P_1 b$ or $P = P_1 bd$, where $P_1 \subseteq A \times B$ and $d \in b^{-1}$. Otherwise P would not be equal to $Pdb \cup P_2 b \cup P$, for all $P_2 \subseteq A \times B$, which gives a contradiction with the regularity of $A\Diamond_{sv}B$. Therefore condition (iii) must hold.

Suppose conversely A and B satisfy (i), (ii) and (iii). Let $(a, P, b) \in A\Diamond_{sv}B$, and let $e^2 = e \in B$ such that $bB = eB$ and $a \in A(a)\theta_e$. Then there are some $u \in A$ such that $a = u(a)\theta_e$ and some $d \in B$ such that $bd = e$ and $d \in b^{-1}$. Since A is regular, we can take $c = (v)\theta_d$ for some $v \in a^{-1}$. Now

$$\begin{aligned} a(c)\theta_b(a)\theta_{bd} &= u(a)\theta_e((v)\theta_d)\theta_b(a)\theta_{bd} = u(a)\theta_e(v)\theta_{bd}(a)\theta_{bd} \\ &= u(a)\theta_e(v)\theta_e(a)\theta_e = u(ava)\theta_e = u(a)\theta_e = a, \\ c(a)\theta_d(c)\theta_{db} &= (v)\theta_d(a)\theta_d((v)\theta_d)\theta_{db} = (v)\theta_d(a)\theta_d(v)\theta_{dbd} \\ &= (v)\theta_d(a)\theta_d(v)\theta_d = (vav)\theta_d = (v)\theta_d = c. \end{aligned}$$

Moreover, by condition (iii), we have $P = P_1 b$ where $P_1 \subseteq A \times B$. Then there exists $P_2 = P_1 bd \subseteq A \times B$ such that

$$\begin{aligned} Pdb \cup P_2 b \cup P = P_1 bdb \cup P_1 bdb \cup P_1 b &= P_1 b \cup P_1 b \cup P_1 b \\ &= P_1 b = P, \\ P_2 bd \cup Pd \cup P_2 = P_1 bdbd \cup P_1 bd \cup P_1 bd &= P_1 bd \cup P_1 bd \cup P_1 bd \\ &= P_1 bd = P_2. \end{aligned}$$

Consequently, for every $(a, P, b) \in A\Diamond_{sv}B$, there exists $(c, P_2, d) \in A\Diamond_{sv}B$ such that

$$\begin{aligned} (a, P, b)(c, P_2, d)(a, P, b) &= (a(c)\theta_b(a)\theta_{bd}, Pdb \cup P_2 b \cup P, bdb) \\ &= (a, P, b) \\ (c, P_2, d)(a, P, b)(c, P_2, d) &= (c(a)\theta_d(c)\theta_{db}, P_2 bd \cup Pd \cup P_2, dbd) \\ &= (c, P_2, d). \end{aligned}$$

Hence the result. □

3 A new two-sided semidirect product construction for monoids

Let A and B be arbitrary monoids. For $P_1 = (a_1, b_1), P_2 = (a_2, b_2) \in A \times B$, we define $P_1 P_2 = (a_1 a_2, b_1 b_2)$. A new two-sided semidirect product of A and B, denoted by $A\,{}_{\beta}\!\bowtie_{\alpha} B$, is the set $A \times (A \times B) \times B$ with the multiplication

$$(a, P_1, b)(c, P_2, d) = (a(c)\alpha_{b_1}, P_1P_2, (b)\beta_{a_2}d), \tag{9}$$

where $\alpha_b : B \to End(A)$ and $\beta_a : A \to End(B)$ for all $a \in A$ and $b \in B$ are monoid homomorphisms such that, for every $x, a_1, a_2 \in A$, $y, b_1, b_2 \in B$,

$$(x)\alpha_{b_1b_2} = ((x)\alpha_{b_2})\alpha_{b_1} \quad \text{and} \quad (y)\beta_{a_1a_2} = ((y)\beta_{a_1})\beta_{a_2}. \tag{10}$$

Then $A_{\beta} \bowtie_{\alpha} B$ is a monoid with the identity element $(1_A, P_e, 1_B)$, where $P_e = (1_A, 1_B)$. Let us show the associative property:

Let $(a, P_1, b), (c, P_2, d)$ and (e, P_3, f) be the elements of $A_{\beta} \bowtie_{\alpha} B$ where $P_1 = (a_1, b_1)$, $P_2 = (a_2, b_2)$ and $P_3 = (a_3, b_3)$. Therefore we have

$$\begin{aligned}
((a, P_1, b)(c, P_2, d))(e, P_3, f) &= (a(c)\alpha_{b_1}, P_1P_2, (b)\beta_{a_2}d)(e, P_3, f) \\
&= (a(c)\alpha_{b_1}(e)\alpha_{b_1b_2}, P_1P_2P_3, ((b)\beta_{a_2}d)\beta_{a_3}f) \\
&= (a(c)\alpha_{b_1}(e)\alpha_{b_1b_2}, P_1P_2P_3, (b)\beta_{a_2a_3}(d)\beta_{a_3}f) \\
(a, P_1, b)((c, P_2, d)(e, P_3, f)) &= (a, P_1, b)(c(e)\alpha_{b_2}, P_2P_3, (d)\beta_{a_3}f) \\
&= (a(c(e)\alpha_{b_2})\alpha_{b_1}, P_1P_2P_3, (b)\beta_{a_2a_3}(d)\beta_{a_3}f) \\
&= (a(c)\alpha_{b_1}(e)\alpha_{b_1b_2}, P_1P_2P_3, (b)\beta_{a_2a_3}(d)\beta_{a_3}f).
\end{aligned}$$

Now, in the following lemma and theorem, we give a generating set and a presentation of this two-sided semidirect product, as a main result of this section.

Lemma 3.1 *Let us suppose that the monoids A and B are generated by the sets X and Y, respectively. Then the union of the sets $\{(x_1, P_e, 1_B); x_1 \in X\}$,$\{(1_A, (x_2, 1_B), 1_B); x_2 \in X\}$, $\{(1_A, P_e, y_1); y_1 \in Y\}$, and $\{(1_A, (1_A, y_2), 1_B); y_2 \in Y\}$ generates $A\,{}_{\beta}\!\bowtie_{\alpha} B$.*

Proof. For $a, a_1, a_2, c \in A$, $b, b_1, b_2, d \in B$, $P_1, P_2 \in A \times B$, we can easily show that the proof follows from

$$\begin{aligned}(a_1, P_e, 1_B)(a_2, P_e, 1_B) &= (a_1a_2, P_e, 1_B), & (11)\\ (1_A, P_e, b_1)(1_A, P_e, b_2) &= (1_A, P_e, b_1b_2), & (12)\\ (1_A, P_1, 1_B)(1_A, P_2, 1_B) &= (1_A, P_1P_2, 1_B) & (13)\end{aligned}$$

and

$$(a, P_e, 1_B)(1_A, (c, 1_B), 1_B)(1_A, (1_A, d), 1_B)(1_A, P_e, b) = (a, (c, d), b)$$

as required. □

Theorem 3.2 *Let us suppose that the monoids A and B are defined by presentations $[X; R]$ and $[Y; S]$, respectively. Then the two-sided semidirect product of A and B is defined by generators*

$$Z = X \cup Y \cup \{z_k : k \in X \text{ or } k \in Y\}$$

and relations

$$R,\ S, \tag{14}$$
$$xy = yx, \tag{15}$$
$$z_{k_1} z_{k_2} = z_{k_1 k_2} = z_{k_2} z_{k_1}, \tag{16}$$
$$z_{k_1} x = x z_{k_1}, \tag{17}$$
$$y z_{k_2} = z_{k_2} y, \tag{18}$$
$$z_{k_2} x = (x)\alpha_{k_2} z_{k_2} \tag{19}$$
$$y z_{k_1} = z_{k_1} (y)\beta_{k_1} \tag{20}$$

where $x, k_1 \in X$ and $y, k_2 \in Y$.

Before giving the proof, for a set Z, let us denote the set of all words in Z by Z^*.

Proof. Let

$$\psi : Z^* \longrightarrow A\ {}_{\beta}\!\bowtie_{\alpha} B$$

be a homomophism defined by $(x)\psi = (x, P_e, 1_A)$, $(y)\psi = (1_A, P_e, y)$, $(z_{k_1})\psi = (1_A, (k_1, 1_B), 1_B)$ and $(z_{k_2})\psi = (1_A, (1_A, k_2), 1_B)$ where $x, k_1 \in X$ and $y, k_2 \in Y$. Then, by Lemma 3.1, we say that ψ is onto. Now we need to check that whether $A\ {}_\beta \bowtie_\alpha B$ satisfies relations (14)-(20). In fact the relations (14) follow from (11), (12) and (13). For relations (15), (16), (17) and (18) we have

$$\begin{aligned}
(x, P_e, 1_B)(1_A, P_e, y) &= (x, P_e, y) \\
&= (1_A, P_e, y)(x, P_e, 1_B), \\
(1_A, (k_1, 1_B), 1_B)(1_A, (1_A, k_2), 1_B) &= (1_A, (k_1, k_2), 1_B) \\
&= (1_A, (1_A, k_2), 1_B)(1_A, (k_1, 1_B), 1_B), \\
(1_A, (k_1, 1_B), 1_B)(x, P_e, 1_B) &= (x, (k_1, 1_B), 1_B) \\
&= (x, P_e, 1_B)(1_A, (k_1, 1_B), 1_B), \\
(1_A, P_e, y)(1_A, (1_A, k_2), 1_B) &= (1_A, (1_A, k_2), y) \\
&= (1_A, (1_A, k_2), 1_B)(1_A, P_e, y).
\end{aligned}$$

Now let us show that relations (19) and (20) hold. To do that we need to use equalities

$$\begin{aligned}
(1_A, (1_A, k_2), 1_B)(x, P_e, 1_B) &= ((x)\alpha_{k_2}, (1_A, k_2), 1_B) \\
&= ((x)\alpha_{k_2}, P_e, 1_B)(1_A, (1_A, k_2), 1_B), \\
(1_A, P_e, y)(1_A, (k_1, 1_B), 1_B) &= (1_A, (k_1, 1_B), (y)\beta_{k_1}) \\
&= (1_A, (k_1, 1_B), 1_B)(1_A, P_e, (y)\beta_{k_1})
\end{aligned}$$

for all $x, k_1 \in X$ and $y, k_2 \in Y$. Therefore ψ induces an epimorphism $\overline{\psi}$ from the monoid M defined by (14)-(20) onto $A_\beta \bowtie_\alpha B$.

Let $w \in Z^*$ be any non-empty word. By using relations (15)-(20), we can write that there exist words $w_x \in X^*$, $w_y \in Y^*$, $w_{k_1} \in \{z_{k_1} : k_1 \in X\}^*$ and $w_{k_2} \in \{z_{k_2} : k_2 \in X\}^*$ such that $w = w_x w_{k_1} w_{k_2} w_y$ in M. Here, let us suppose that $w_{k_1} = z_{x_1} z_{x_2} \cdots z_{x_m}$ and $w_{k_2} = z_{y_1} z_{y_2} \cdots z_{y_n}$ where $x_1, x_2, \cdots, x_m \in X$ and $y_1, y_2, \cdots, y_n \in Y$. Therefore, for any word $w \in Z^*$, we have

$$\begin{aligned}(w)\psi &= (w_x w_{k_1} w_{k_2} w_y)\psi = (w_x)\psi(w_{k_1})\psi(w_{k_2})\psi(w_y)\psi \\ &= (w_x, \emptyset, 1_B)(1_A, (x_1x_2\cdots x_m, y_1y_2\cdots y_n), 1_B)(1_A, \emptyset, w_y) \\ &= (w_x, P(w), w_y)\end{aligned}$$

where $P(w) = (x_1x_2\cdots x_m, y_1y_2\cdots y_n)$. Now, if $(w')\psi = (w'')\psi$, for some $w', w'' \in Z^*$ then, by the equality of these components, we deduce that $w'_x = w''_x$ in A, $w'_y = w''_y$ in B and $P(w') = P(w'')$. Relations (14) imply that $w'_x = w''_x$ and $w'_y = w''_y$ hold in M, so that $w' = w''$ holds as well, and $\overline{\psi}$ is injective. □

The following corollary guaranted that this two-sided semi-direct product construction is a group.

Corollary 3.3 *Let A and B be groups. Then the product $A_\beta \bowtie_\alpha B$ is a group under the multiplication in* (9), *where $\alpha : B \to Aut(A)$ and $\beta : A \to Aut(B)$ are homomorphism as in* (10).

Proof. For all $(a, (c, d), b) \in A \times (A \times B) \times B$, we have

$$((a^{-1})\alpha_{d^{-1}}, (c^{-1}, d^{-1}), (b^{-1})\beta_{c^{-1}}) \in A \times (A \times B) \times B$$

such that

$$(a, (c, d), b)((a^{-1})\alpha_{d^{-1}}, (c^{-1}, d^{-1}), (b^{-1})\beta_{c^{-1}}) = (1_A, P_e, 1_B)$$
$$((a^{-1})\alpha_{d^{-1}}, (c^{-1}, d^{-1}), (b^{-1})\beta_{c^{-1}})(a, (c, d), b) = (1_A, P_e, 1_B).$$

Hence the result. □

3.1 Subgroup separability

Let F and K be free groups presented by $\mathcal{P}_F = [x_1, x_2, \cdots, x_n;]$ and $\mathcal{P}_K = [y\,; \ ]$. In this section, by using the following lemma, we will investigate the subgroup separability of $F_\beta \bowtie_\alpha K$. First let us recall the definition of the subgroup separability. The group G is said to be *subgroup separable* if, for every finitely generated subgroup H of G, H is the intersection of finite index subgroup of G.

Lemma 3.4 ([7]) *Let K be a finitely generated abelian group and let A and B be subgroups of K such that G is the HNN-extension*

$$G = [t, K : t^{-1}At = B].$$

Then G is subgroup separable if and only if $A \cap B$ is a subgroup of finite index in both A and B, and there is a finitely generated normal subgroup of G, say H, such that H has finite index in $A \cap B$.

Now, before giving our theorem, let us define the notion of the right layered basis which will be needed in our proof. Suppose that α is an automorphism of the free group F with base $X = \{x_1, x_2, \cdots, x_n\}$. Then X is a right layered basis for α if

$$\alpha(x_1) = x_1 \quad \text{and} \quad \alpha(x_i) = x_i w_i, \tag{21}$$

where $w_i \in \{x_1, \cdots, x_{i-1}\}^*$ for $2 \leq i \leq n$. For all the words w_i, the existence of a right layered basis for an automorphism of free groups with rank of Fix(f)=n has been shown by Collins and Turner [3].

Let $G = F\ {}_{\beta}\!\bowtie_{\alpha} K$ where α is an automorphism of the free group F with base X (such that $n > 1$) defined as in (21). We then have following theorem.

Theorem 3.5 *G is not subgroup separable.*

Proof. Let us suppose that $1 \neq \alpha \in Aut(F)$. Then, by (21), one can easily show that G has a presentation

$$\begin{aligned}\mathcal{P}_G = [\, x_i,\ y,\ z_{x_i},\ z_y \ \ ; \ \ & x_i y = y x_i,\ z_{x_i} z_y = z_y z_{x_i},\ z_{x_i} x_j z_{x_i}^{-1} = x_j, \\ & z_y x_1 z_y^{-1} = x_1,\ z_y x_i z_y^{-1} = x_i w_i \ (2 \leq i \leq n) \\ & y z_y y^{-1} = z_y,\ y z_{x_i} = z_{x_i}(y)\beta_{x_i}\,],\end{aligned}$$

where w_i are words as above and $1 \leq i, j \leq n$. Since $\alpha \neq 1$, at least one element of the w_i's is a non-trivial word. Now assume that w_s is the first such a word. Also let H_s be a subgroup of

G generated by $\{x_s,\ w_s,\ z_y\}$. Then H_s has a presentation of the form

$$\mathcal{P}_{H_s} = [\, x_s, w_s, z_y \; ; \; z_y w_s z_y^{-1} = w_s, \; z_y x_s z_y^{-1} = x_s w_s \,]$$

since $z_y x_k z_y^{-1} = x_k$, for all x_k $(k < s)$. In fact the above subgroup presentation can be rewritten as follows:

$$\mathcal{P}_{H_s} = [\, x_s, w_s, z_y \; ; \; z_y w_s z_y^{-1} = w_s, \; x_s^{-1} z_y x_s = z_y w_s \,].$$

So H_s is an HNN-extension with base, say T, a free abelian group of rank two generated by $\{w_s, z_y\}$ and stable letter x_s. Now assume that the isomorphic subgroups of T are generated by $\{z_y\}$ and $\{z_y w_s\}$, say C and D, respectively. It is obvious that $C \cap D = \{1\}$ and by Lemma 3.4, H_s is not subgroup separable. Therefore G cannot be subgroup separable since it contains H_s.

Let us suppose that $\alpha = 1$. Then the presentation of G becomes, for $1 \le i, j \le n$,

$$\begin{aligned}[\, x_i,\ y,\ z_{x_i},\ z_y \quad ; \quad & x_i y = y x_i,\ z_{x_i} z_y = z_y z_{x_i}, \\ & z_{x_i} x_j z_{x_i}^{-1} = x_j,\ z_y x_i z_y^{-1} = x_i \\ & y z_y y^{-1} = z_y,\ y z_{x_i} = z_{x_i}(y)\beta_{x_i} \,].\end{aligned}$$

Thus, for $1 \le i, j \le n$, if we consider the generators x_i, z_{x_i} and the relations $z_{x_i} x_j z_{x_i}^{-1} = x_j$, then we see that G contains a subgroup with finite index which is isomorphic to $F \times F$. In [9], Michailova showed that $F \times F$, where F is a free group of rank at least 2, is not subgroup separable. Thus, in our case, since the group G contains a subgroup as the form of $F \times F$, this gives us that G is not subgroup separable.

These all above procedure finish the proof, as required. □

Remark 3.6 *Let us take the rank of F is 1. Then, by considering the second part of the proof of Theorem 3.5, we see that G has a finite index subgroup which is isomorphic to K^4. In fact K^4 is subgroup separable. Therefore G is subgroup separable.*

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