On Kirkman Packing Designs $KPD(\{3, 4^*, 5^*\}, v)$s

Renwang Su
College of Statistics and Computing Science
Zhejiang Gongshang University
Hangzhou 310018, P. R. China
srwang123@sina.com

Jinhua Wang*
School of Sciences
Nantong University
Nantong 226007, P. R. China
jhwang@ntu.edu.cn

Abstract

A Kirkman packing design $KPD(\{w, s^*, t^*\}, v)$ is a Kirkman packing with maximum possible number of parallel classes, such that each parallel class contains one block of size s, one block of size t and all other blocks of size w. A (k, w)-threshold scheme is a way of distributing partial information (shadows) to w participants, so that any k of them can determine a key easily, but no subset of fewer than k participants can calculate the key. In this paper, the existence of a $KPD(\{3, 4^*, 5^*\}, v)$ is established for every $v \equiv 3 \pmod 6$ with $v \geq 51$. As its consequence, some new $(2, w)$-threshold schemes have been obtained.

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*corresponding author

1 Introduction

Let X be a v-set of points. A packing of X of order v is a set of subsets (called blocks) of X such that any pair of distinct points from X occur together in at most one block in the set. A packing is called a Kirkman packing (KP) of a v-set X if its blocks set admits a partition into parallel classes, each parallel class being a partition of the point set X.

A Kikman packing design (KPD), denoted by $KPD(K,v)$, is a Kirkman packing of a v-set by the maximum possible number $m(v)$ of parallel classes, each class containing the same number of blocks of each size in K. If in every parallel class of a $KPD(K,v)$ there is only one block of size s, and all others have blocks of size w, then we denote this KPD by $KPD(\{w,s^*\},v)$. If $K=\{3,s\}$ with $s\in\{2,4\}$, then a $KPD(K,v)$ is also called a Kirkman school project design in [5, 7]. When $K=\{3\}$ and $v\equiv 3 \pmod 6$, a $KPD(K,v)$ is called a Kirkman triple system and denoted by $KTS(v)$. If $K=\{3\}$ and $v\equiv 0 \pmod 6$, then a $KPD(K,v)$ is called a nearly Kirkman triple system and denoted by $NKTS(v)$.

Let X be a set of v elements (shadows), and K be a set of m elements (keys). A (k,w)-threshold scheme is a pair $(\mathfrak{B},\emptyset)$, where $\mathfrak{B}$ is a set of b distinct w-subsets of X (blocks), and $\emptyset:\mathfrak{B}\to K$, such that

(i) any k shadows determine at most one key (i.e., for every k-subset S of X, $|\{\emptyset(B): S\subseteq B\in\mathfrak{B}\}|=0$ or 1),

(ii) any set of fewer than k shadows that occur in a block do not determine a unique key (i.e., for every k'-subset S' of X, where $k'<k$, $|\emptyset(B): S\subseteq B\in\mathfrak{B}|>1$).

It is shown in [3] that a $KPD(\{w,s^*\},v)$ can be used to construct a $(2,w)$-threshold scheme when $s\geq w$. In this scheme, the number of keys is the number of parallel classes in the KPD. Moreover, it has already been pointed out in [4] that any $KPD(K,v)$ can be used to construct a $(2,w)$-threshold scheme if $k\geq w$ for any $k\in K$. In this article, we shall focus our attention on the problem of the existence of KPDs. As to the relationship between KPDs and threshold schemes, we refer the reader to [3, 7].

The known results concerning a $KPD(\{3,s\},v)$ for $s\equiv 0,1 \pmod 3$ are as follows.

Theorem 1.1 (Ray-Chaudhuri and Wilson [12]). *There exists a $KST(v)$ containing $(v-1)/2$ parallel classes if and only if $v\equiv 3$ (mod 6).*

Theorem 1.2 (Kotzig and Rosa [10], Baker [1], Brouwer [2], Rees and Stinson [13]). *There exists an $NKTS(v)$ containing $(v-2)/2$ parallel classes if and only if $v\equiv 0$ (mod 6) and $v\geq 18$.*

Theorem 1.3 (Cerny et al. [6], Phillps et al. [11], Colbourn and Ling [9], Cao and Du [3]). *There is a $KPD(\{3,4^*\},v)$ containing $\lfloor(v-3)/2\rfloor$ parallel classes for every $v \equiv 1 \pmod 3$ with $v \geq 25$.*

The known results concerning a $KPD(\{3,s\},v)$ for $s \equiv 2 \pmod 3$ are as follows.

Theorem 1.4 (Cao and Zhu [5], Cao and Du [3]). *For every $v \equiv 2 \pmod 3$ and $v \notin \{23,26,29,83,107,155,173,179,197\}$, there exists a $KPD(\{3,5^*\},v)$ containing $\lfloor(v-7)/2\rfloor$ parallel classes.*

Let $KPD(\{3,4^{**}\},v)$ denote a KPD in which each parallel class consists of two blocks of size 4 and $(v-8)/3$ blocks of size 3, where $v \equiv 2 \pmod 3$. We have the following results for the existence of a $KPD(\{3,4^{**}\},v)$.

Theorem 1.5 (Cao and Tang [4]). *There exists a $KPD(\{3,4^{**}\},v)$ containing $\lfloor(v-5)/2\rfloor$ parallel classes for every $v \equiv 2 \pmod 3$ with $v \geq 32$.*

For $v \geq 9$, now let $KP(\{3,4^*,5^*\},v)$ denote a KP in which each parallel class consists of one block of size 4, one block of size 5 and $(v-9)/3$ blocks of size 3. Let $KPD(\{3,4^*,5^*\}, v)$ denote a $KP(\{3,4^*,5^*\}, v)$ which has maximum possible number $m(v)$ of parallel classes. Clearly, we have the following Lemma.

Lemma 1.6 *If there exists a $KPD(\{3,4^*,5^*\},v)$ for $v \geq 9$, then $v \equiv 0 \pmod 3$ and $m(v) \leq \lfloor(v-8)/2+28/(v+7)\rfloor$.*

Suppose that $v \equiv 3 \pmod 6$, $v \geq 51$ and there is a $KPD(\{3,4^*,5^*\}, v)$ with $m(v)$ parallel classes, then it is easy to see that $m(v) \leq (v-9)/2$ from Lemma 1.6. The main purpose of this paper is to establish the existence of a $KPD(\{3,4^*,5^*\},v)$ containing $(v-9)/2$ parallel classes for every $v \equiv 3 \pmod 6$ with $v \geq 51$.

2 Basic construction techniques

In this section, we will introduce some basic techniques for constructing $KPD(\{3,4^*,5^*\},v)$s, and generalize the idea of [3-6, 12].

Firstly, we need the following some definitions. We refer the reader to [8] for more information on design theory if necessary.

A group-divisible design (GDD) is a triple $(X,\mathcal{G},\mathcal{B})$ which satisfies the following properties: (i) X is a finite set of points, (ii) $\mathcal{G}$ is a partition of X into subsets called groups, (iii) $\mathcal{B}$ is a set of subsets (called blocks) of X, such that a group and a block contain at most one common point, and every pair of points from distinct groups occur in exactly one block.

The type of a GDD is the multset $\{|G|, G \in \mathcal{G}\}$. We denote the type by $1^{u_1}2^{u_2}\cdots$ where there are precisely u_i occurrences of i for any $i \geq 1$. The set of block sizes is denoted by K.

A $GDD(X, \mathcal{G}, \mathcal{B})$ is called frame resolvable if its block set $\mathcal{B}$ can be partitioned into frame parallel classes, each class being a partition of $X - G_j$ for some $G_j \in \mathcal{G}$. A Kirkman frame is a frame resolvable GDD in which all the blocks have size three. It is well known that to each G_j there are exactly $|G_j|/2$ frame parallel classes of triples so that each class is a partition of $X - G_j$. The groups in a Kirkman frame are often referred to as holes.

For the existence of Kirkman frames, we require the following results.

Lemma 2.1 (Stinson [14]). *There exists a Kirkman frame of type g^u if and only if $u \geq 4$, g is even and $g(u-1) \equiv 0$ (mod 3).*

Lemma 2.2 (Cao and Tang [4]). *For each positive integer v with $v \equiv 0$ (mod 6) and $v \geq 234$, there is a Kirkman frame of type $42^a36^b30^c$, where $v = 42a + 36b + 30c$, $a \geq 4$ and $b, c \geq 0$ or $a = 0$, $b \geq 4$ and $c \geq 0$.*

Lemma 2.3 (Cao and Tang [4]). *There exists a Kirkman frame of type $(2g)^4(2m)^1$ with $m > 0$ if and only if $g \equiv m \equiv 0$ (mod 3) and $0 < m \leq 3g/2$.*

For given positive integers v and h with $v \equiv h \equiv 3$ (mod 6) and $h \geq 9$, an incomplete Kirkman packing design, denoted by $IKPD(\{3, 4^*, 5^*\}, v, h)$, is a triple $(V, H, \mathcal{B})$ which satisfies the following properties:

(1) V is a v-set of points, H(called a hole) is a h-subset of V and $\mathcal{B}$ is a set of subsets (called blocks) of V, each block having size of 3, 4 or 5;

(2) $|H \cap B| \leq 1$ for each $B \in \mathcal{B}$;

(3) any two points of V appear either in H or in at most one block of $\mathcal{B}$, but not both;

(4) $\mathcal{B}$ admits a partition into $(v-h)/2$ parallel classes, each consisting of one block of size 4, one block of size 5 and $(v-9)/3$ triples on V, and $(h-9)/2$ auxiliary parallel classes, each consists of $(v-h)/3$ triples on $V \backslash H$.

The following "filling in holes" construction is analogous to [Cao and Zhu [5], Lemma 4.1]. It provides a very useful tool for the existence of an incomplete Kirkman packing design.

Theorem 2.4 *Suppose that there exists a Kirkman frame of type $g_1g_2\cdots g_u$. If $g_i \equiv 0$ (mod 6) and there is an $IKPD(\{3, 4^*, 5^*\}, g_i + h, h)$ for any $1 \leq i \leq u$, then there is an $IKPD(\{3, 4^*, 5^*\}, h + \sum_{i=1}^{u} g_i, h)$. Further, if $h = 9$ and $\sum_{i=1}^{u} g_i \geq 42$, then the $IKPD$ is also a KPD.*

Proof: Suppose that $(X', \mathcal{G}, \mathcal{B}')$ is a Kirkman frame of type $g_1g_2\cdots g_u$. Let $t_i = g_i/2$ for $1 \leq i \leq u$ and $t = (h-9)/2$. For each $1 \leq i \leq u$, there are exactly t_i frame parallel classes $P'_{i1}, P'_{i2}, \cdots, P_{it_i}{}'$ each missing the group G_i of size g_i. Let H be a h-set and $H \cap X' = \Phi$. For each $1 \le i \le u$, let $(G_i \cup H, H, \mathcal{B}_i)$ be an $IKPD(\{3, 4^*, 5^*\}, h + g_i, h)$ with t_i parallel classes $P''_{i1}, P''_{i2}, \cdots, P_{it_i}{}''$ and t auxiliary parallel classes $AP_{i1}, AP_{i2}, \cdots, AP_{it}$. Now let

$$X = X' \cup H, and$$

$$\mathcal{B} = (\bigcup_{i=1}^{u} \mathcal{B}_i) \cup \mathcal{B}',$$

then it is easy to check that the $(X, \mathcal{B})$ is an $IKPD(\{3, 4^*, 5^*\}, h + \sum_{i=1}^{u} g_i, h)$ with $\sum_{i=1}^{u} t_i$ parallel classes $P_{ij} = P'_{ij} \cup P''_{ij}$, $1 \le i \le u$, $1 \le j \le t_i$ and t auxiliary parallel classes $AP_j = \bigcup_{i=1}^{u} AP_{ij}$, $1 \le j \le t$.

Moreover, suppose that there is an $IKPD(\{3, 4^*, 5^*\}, 9 + \sum_{i=1}^{u} g_i, 9)$, then the $IKPD$ has only $((9 + \sum_{i=1}^{u} g_i) - 9)/2$ parallel classes. By Lemma 1.6, there are $(v-9)/2$ parallel classes if there exists a $KPD(\{3, 4^*, 5^*\}, v)$ for $v \equiv 3 \pmod 6$ with $v \ge 51$. Therefore, the $IKPD$ is also a KPD if $9 + \sum_{i=1}^{u} g_i \ge 51$., i.e., $\sum_{i=1}^{u} g_i \ge 42$. This completes the proof.

Lemma 2.5 *There is a* $KP(\{3, 4^*, 5^*\}, 27)$.

Proof: Take the point set Z_{27}. The blocks of the initial parallel class are listed below, where all base blocks are developed +3 modulo 27.

0 1 2 3 7	4 8 11 17	5 13 21	6 19 22
9 18 23	10 15 25	12 20 24	14 16 26

Lemma 2.6 *There is an* $IKPD(\{3, 4^*, 5^*\}, 39, 9)$.

Proof: Take the point set $Z_{15} \times \{1, 2\} \cup \{a_i, b_i, c_i | i \in Z_3\}$. The blocks of the initial parallel class are listed below, where the subscripts on a, b, c are developed modulo 3.

$1_1 8_1 10_2 13_2$	$0_1 2_1 0_2 1_2 6_2$	$7_1 10_1 2_2$	$3_2 11_2 a_0$	$3_1 14_2 a_1$
$5_1 9_1 a_2$	$11_1 4_2 b_0$	$4_1 14_1 b_1$	$5_2 7_2 b_2$	$6_1 9_2 c_0$
$8_2 12_2 c_1$	$12_1 13_1 c_2$			

Lemma 2.7 *There is an* $IKPD(\{3, 4^*, 5^*\}, 45, 9)$.

Proof: Take the point set $Z_{18} \times \{1, 2\} \cup \{\infty_i | 1 \le i \le 7\} \cup \{a_i | i \in Z_2\}$. The blocks of the initial parallel class are listed below, where the subscripts on a are developed modulo 2.

$2_1 2_2 5_2 6_2$	$11_1 17_1 1_1 13_2 0_2$	$5_1 10_1 16_2$	$9_2 15_2 17_2$	$12_1 15_1 16_1$
$0_1 7_1 a_0$	$1_2 8_2 a_1$	$6_1 11_2 \infty_1$	$14_1 4_2 \infty_2$	$3_1 12_2 \infty_3$
$4_1 14_2 \infty_4$	$8_1 3_2 \infty_5$	$13_1 10_2 \infty_6$	$9_1 7_2 \infty_7$	

Lemma 2.8 *There is an* $IKPD(\{3, 4^*, 5^*\}, 141, 27)$.

Proof: Take the point set $Z_{57} \times \{1, 2\} \cup \{\infty_i | 1 \le i \le 24\} \cup \{a_i | i \in Z_3\}$. Nine auxiliary parallel classes can be generated mod 57 from the following three initial classes S_1, S_2 and S_3, each generating three auxiliary parallel classes.

S_1: $0_1 1_1 5_1$ $0_2 1_2 5_2$
S_2: $0_1 11_1 28_1$ $0_2 11_2 28_2$
S_3: $8_1 10_1 9_2$ $0_1 5_2 7_2$

The blocks of the initial parallel class are listed below, where the subscripts on a are developed modulo 3.

$1_1 3_2 13_2 29_2$	$2_1 15_1 29_1 32_2 52_2$	$11_1 20_1 30_1$	$3_1 6_1 28_1$
$8_1 14_1 32_1$	$5_1 13_1 25_1$	$1_2 4_2 28_2$	$0_2 6_2 25_2$
$5_2 12_2 27_2$	$2_2 11_2 23_2$	$24_1 39_1 21_2$	$17_1 33_1 7_2$
$46_1 10_1 34_2$	$56_1 22_1 47_2$	$16_1 42_1 14_2$	$35_1 43_2 56_2$
$4_1 31_2 45_2$	$50_1 19_2 37_2$	$23_1 42_2 8_2$	$0_1 7_1 a_0$
$12_1 18_2 a_1$	$9_2 17_2 a_2$	$55_1 55_2 \infty_1$	$54_1 48_2 \infty_2$
$31_1 40_2 \infty_3$	$47_1 33_2 \infty_4$	$51_1 46_2 \infty_5$	$40_1 53_2 \infty_6$
$36_1 50_2 \infty_7$	$34_1 15_2 \infty_8$	$52_1 41_2 \infty_9$	$26_1 44_2 \infty_{10}$
$19_1 39_2 \infty_{11}$	$9_1 49_2 \infty_{12}$	$43_1 35_2 \infty_{13}$	$44_1 20_2 \infty_{14}$
$53_1 30_2 \infty_{15}$	$18_1 22_2 \infty_{16}$	$27_1 38_2 \infty_{17}$	$21_1 36_2 \infty_{18}$
$45_1 10_2 \infty_{19}$	$41_1 51_2 \infty_{20}$	$38_1 54_2 \infty_{21}$	$49_1 24_2 \infty_{22}$
$48_1 26_2 \infty_{23}$	$37_1 16_2 \infty_{24}$		

3 Main Results

In this section, we will give some KPDs for small orders by direct constructions firstly, then we shall prove our main results by using Theorem 2.4.

Lemma 3.1 *There exists a* $KPD(\{3, 4^*, 5^*\}, v)$ *for any* $v \in \{57, 87, 99\}$.

Proof: We take the point set $Z_{(v-9)/2} \times \{1, 2\} \cup \{\infty_i | 1 \le i \le 3\} \cup \{a_i | i \in Z_3\} \cup \{b_i | i \in Z_3\}$. The blocks of the initial parallel class for each v are listed as follows, where the subscripts on a and b are developed modulo 3.

$v = 57$:

$0_1 0_2 1_1 2_2 4_1$	$1_2 4_2 5_2 10_1$	$5_1 9_2 21_1$	$3_2 13_2 18_2$	$3_1 8_1 18_1$
$2_1 12_2 19_1$	$6_1 11_2 22_2$	$6_2 16_1 22_1$	$7_1 14_2 20_2$	$7_2 15_2 a_0$
$9_1 20_1 a_1$	$10_2 23_1 a_2$	$14_1 17_2 b_0$	$16_2 23_2 b_1$	$15_1 17_1 b_2$
$8_2 11_1 \infty_1$	$13_1 19_2 \infty_2$	$12_1 21_2 \infty_3$		

$v = 87:$

$10_1 1_2 4_2 5_2$	$0_1 1_1 4_1 0_2 2_2$	$11_1 35_1 16_2$	$6_1 37_1 34_2$	$10_2 19_2 33_2$
$7_2 20_2 26_2$	$14_2 29_2 36_2$	$13_1 27_1 38_2$	$9_2 31_1 38_1$	$9_1 13_2 31_2$
$8_2 21_1 34_1$	$6_2 14_1 33_1$	$15_1 20_1 26_1$	$8_1 24_2 32_2$	$2_1 18_1 21_2$
$3_1 5_1 12_2$	$3_2 24_1 30_2$	$7_1 16_1 28_1$	$11_2 22_2 29_1$	$12_1 35_2 \infty_1$
$17_1 25_2 \infty_2$	$25_1 15_2 \infty_3$	$17_2 27_2 a_0$	$19_1 36_1 a_1$	$30_1 18_2 a_2$
$22_1 32_1 b_0$	$23_2 28_2 b_1$	$23_1 37_2 b_2$		

$v = 99:$

$1_2 4_2 5_2 10_1$	$0_1 0_2 1_1 2_2 4_1$	$9_2 23_2 28_2$	$6_2 22_2 43_2$	$5_1 7_1 39_2$
$3_2 16_2 25_2$	$17_2 32_2 39_1$	$17_1 32_1 35_2$	$16_1 36_2 42_2$	$14_2 24_2 40_1$
$13_2 21_1 28_1$	$9_1 15_1 30_2$	$7_2 18_2 38_1$	$11_1 15_2 33_2$	$8_2 20_1 36_1$
$8_1 13_1 35_1$	$2_1 11_2 44_2$	$3_1 23_1 34_1$	$6_1 12_2 30_1$	$10_2 31_1 43_1$
$12_1 20_2 40_2$	$14_1 33_1 42_1$	$18_1 26_1 31_2$	$19_1 29_1 a_0$	$19_2 26_2 a_1$
$34_2 44_1 a_2$	$21_2 38_2 b_0$	$22_1 29_2 b_1$	$24_1 37_1 b_2$	$25_1 41_2 \infty_1$
$27_1 37_2 \infty_2$	$27_2 41_1 \infty_3$			

Lemma 3.2 *There exists a* $KPD(\{3, 4^*, 5^*\}, v)$ *for any* $v \in \{93, 105\}$.

Proof: We take the point set $Z_{(v-9)/2} \times \{1, 2\} \cup \{a_i, b_i, c_i | i \in Z_3\}$. The blocks of the initial parallel class for each v are listed as follows, where the subscripts on a, b and c are developed modulo 3.

$v = 93:$

$1_2 4_2 5_2 10_1$	$0_1 0_2 1_1 2_2 4_1$	$14_1 29_1 32_2$	$11_2 17_2 40_2$	$13_2 28_2 37_2$
$9_1 18_2 25_1$	$10_2 24_2 36_1$	$8_2 15_2 40_1$	$7_1 22_2 39_1$	$6_1 17_1 38_2$
$12_1 19_1 24_1$	$11_1 23_2 34_2$	$13_1 22_1 30_1$	$12_2 20_2 28_1$	$9_2 31_1 39_2$
$2_1 31_2 41_2$	$3_1 7_2 27_1$	$3_2 26_1 32_1$	$5_1 18_1 29_2$	$6_2 21_1 41_1$
$8_1 14_2 36_2$	$15_1 34_1 a_0$	$16_2 33_2 a_1$	$16_1 30_2 a_2$	$19_2 35_2 b_0$
$20_1 25_2 b_1$	$23_1 37_1 b_2$	$21_2 26_2 c_0$	$33_1 35_1 c_1$	$27_2 38_1 c_2$

$v = 105:$

$1_2 4_2 5_2 10_1$	$0_1 0_2 1_1 2_2 4_1$	$7_1 20_2 33_1$	$5_1 16_2 44_1$	$3_2 6_1 26_1$
$3_1 8_2 38_1$	$13_1 23_1 28_1$	$11_1 32_1 38_2$	$8_1 22_1 44_2$	$7_2 28_2 41_2$
$12_1 15_2 37_1$	$11_2 30_2 39_2$	$9_2 29_1 47_1$	$9_1 21_1 25_2$	$18_1 25_1 33_2$
$13_2 29_2 40_1$	$12_2 19_2 27_1$	$15_1 24_2 31_1$	$2_1 19_1 26_2$	$6_2 14_2 24_1$
$10_2 40_2 46_2$	$14_1 20_1 37_2$	$16_1 35_1 47_2$	$17_1 31_2 36_2$	$17_2 32_2 42_2$
$18_2 35_2 a_0$	$23_2 39_1 a_1$	$30_1 41_1 a_2$	$21_2 43_2 b_0$	$27_2 46_1 b_1$
$34_1 42_1 b_2$	$22_2 36_1 c_0$	$34_2 45_2 c_1$	$43_1 45_1 c_2$	

Lemma 3.3 *There exists a* $KPD(\{3, 4^*, 5^*\}, v)$ *for any* $v \in \{51, 75, 123\}$.

Proof: We take the point set $Z_{(v-9)/2} \times \{1, 2\} \cup \{\infty_i | 1 \le i \le 9\}$. The blocks of the initial parallel class for each v are listed as follows.

$v = 51:$

$0_1 5_1 0_2 8_2$	$2_1 3_1 6_1 4_2 13_2$	$1_1 7_1 14_1$	$1_2 7_2 11_2$	$4_1 13_1 15_1$
$10_2 15_2 17_2$	$19_1 2_2 3_2$	$12_1 18_2 \infty_1$	$10_1 19_2 \infty_2$	$8_1 20_2 \infty_3$
$20_1 12_2 \infty_4$	$16_1 9_2 \infty_5$	$11_1 5_2 \infty_6$	$9_1 6_2 \infty_7$	$18_1 14_2 \infty_8$
$17_1 16_2 \infty_9$				

$v = 75:$

$0_1 0_2 11_2 12_2$	$3_1 6_1 16_1 7_2 13_2$	$1_1 12_1 13_1$	$4_1 18_1 20_1$	$2_1 9_1 17_1$
$1_2 3_2 17_2$	$2_2 5_2 15_2$	$9_2 16_2 24_2$	$7_1 11_1 6_2$	$21_1 26_1 19_2$
$25_1 31_1 21_2$	$29_1 5_1 23_2$	$23_1 28_2 32_2$	$10_1 26_2 31_2$	$14_1 20_2 29_2$
$28_1 30_2 \infty_1$	$22_1 25_2 \infty_2$	$19_1 27_2 \infty_3$	$30_1 10_2 \infty_4$	$8_1 22_2 \infty_5$
$24_1 8_2 \infty_6$	$32_1 18_2 \infty_7$	$27_1 14_2 \infty_8$	$15_1 4_2 \infty_9$	

$v = 123:$

$0_1 7_2 10_2 15_2$	$1_1 10_1 12_1 14_2 39_2$	$2_1 9_1 1_2$	$5_1 13_1 3_2$
$7_1 17_1 4_2$	$3_1 15_1 54_2$	$6_1 19_1 42_2$	$8_1 23_1 40_2$
$4_1 20_1 38_2$	$16_1 34_1 12_2$	$18_1 38_1 11_2$	$11_1 32_1 6_2$
$8_2 9_2 36_2$	$13_2 22_2 33_2$	$16_2 18_2 31_2$	$23_2 27_2 49_2$
$19_2 25_2 43_2$	$43_1 47_1 14_1$	$51_1 52_1 21_1$	$24_1 27_1 49_1$
$35_1 40_1 54_1$	$30_1 36_1 53_1$	$33_1 47_2 0_2$	$46_1 30_2 37_2$
$56_1 24_2 41_2$	$25_1 46_2 5_2$	$22_1 34_2 48_2$	$50_1 55_2 21_2$
$28_1 29_2 50_2$	$26_1 26_2 45_2$	$44_1 20_2 32_2$	$48_1 56_2 \infty_1$
$42_1 51_2 \infty_2$	$29_1 35_2 \infty_3$	$41_1 52_2 \infty_4$	$37_1 53_2 \infty_5$
$39_1 2_2 \infty_6$	$45_1 28_2 \infty_7$	$31_1 17_2 \infty_8$	$55_1 44_2 \infty_9$

Lemma 3.4 *There exists a* $KPD(\{3, 4^*, 5^*\}, v)$ *for any* $v \in \{63, 69, 81, 111, 117, 135\}$.

Proof: We take the point set $Z_{(v-9)/2} \times \{1, 2\} \cup \{\infty_i | 1 \le i \le 6\} \cup \{a_i | i \in Z_3\}$. The blocks of the initial parallel class for each v are listed as follows, where the subscripts on a are developed modulo 3.

$v = 63:$

$2_1 2_2 13_2 14_2$	$1_1 4_1 14_1 5_2 11_2$	$5_1 16_1 17_1$	$11_1 15_1 20_1$	$21_2 24_2 7_2$
$16_2 20_2 25_2$	$8_1 1_2 3_2$	$6_1 23_2 4_2$	$24_1 26_1 18_2$	$19_1 25_1 0_2$
$10_1 18_1 6_2$	$0_1 7_1 a_0$	$3_1 12_2 a_1$	$8_2 15_2 a_2$	$23_1 26_2 \infty_1$
$12_1 17_2 \infty_2$	$13_1 19_2 \infty_3$	$9_1 22_2 \infty_4$	$22_1 9_2 \infty_5$	$21_1 10_2 \infty_6$

$v = 69:$

$2_1 2_2 13_2 14_2$	$1_1 4_1 14_1 5_2 11_2$	$5_1 13_1 17_1$	$6_1 11_1 12_1$	$16_2 21_2 24_2$
$26_2 0_2 9_2$	$20_1 22_1 18_2$	$18_1 27_1 17_2$	$8_1 19_1 27_2$	$9_1 23_1 25_2$
$21_1 4_2 6_2$	$26_1 10_2 20_2$	$16_1 19_2 3_2$	$0_1 7_1 a_0$	$3_1 12_2 a_1$
$8_2 15_2 a_2$	$24_1 29_2 \infty_1$	$25_1 1_2 \infty_2$	$10_1 28_2 \infty_3$	$15_1 7_2 \infty_4$
$29_1 22_2 \infty_5$	$28_1 23_2 \infty_6$			

$v = 81:$

$0_1 0_2 1_1 2_2 4_1$	$1_2 4_2 5_2 10_1$	$12_1 33_1 35_1$	$11_2 21_1 28_1$	$7_1 14_2 29_2$
$6_2 16_2 32_1$	$5_1 13_1 19_1$	$3_2 15_2 26_1$	$2_1 27_1 31_2$	$11_1 16_1 28_2$
$10_2 22_1 33_2$	$9_1 17_2 23_2$	$7_2 18_2 34_2$	$3_1 19_2 26_2$	$6_1 18_1 24_2$
$8_1 17_1 34_1$	$8_2 13_2 30_2$	$9_2 30_1 a_0$	$12_2 20_2 a_1$	$15_1 31_1 a_2$
$14_1 35_2 \infty_1$	$20_1 25_2 \infty_2$	$21_2 29_1 \infty_3$	$22_2 25_1 \infty_4$	$24_1 27_2 \infty_5$
$23_1 32_2 \infty_6$				

$v = 111:$

$4_1 11_2 14_2 19_2$	$2_1 11_1 13_1 15_2 40_2$	$29_1 36_1 48_1$	$45_1 50_1 9_1$
$38_1 42_1 12_1$	$21_1 24_1 44_1$	$22_2 32_2 44_2$	$38_2 45_2 10_2$
$12_2 21_2 36_2$	$0_2 6_2 20_2$	$14_1 22_1 31_2$	$30_1 46_1 27_2$
$26_1 32_1 25_2$	$47_1 18_1 42_2$	$35_1 49_1 33_2$	$6_1 19_1 49_2$
$41_1 8_1 29_2$	$39_1 5_1 8_2$	$43_1 16_1 39_2$	$17_1 35_2 17_2$
$34_1 5_2 24_2$	$25_1 30_2 41_2$	$28_1 34_2 47_2$	$31_1 43_2 13_2$
$23_1 37_2 3_2$	$27_1 1_2 2_2$	$10_1 46_2 50_2$	$0_1 1_1 a_0$
$3_1 4_2 a_1$	$7_2 9_2 a_2$	$15_1 23_2 \infty_1$	$7_1 18_2 \infty_2$
$20_1 48_2 \infty_3$	$33_1 16_2 \infty_4$	$40_1 26_2 \infty_5$	$37_1 28_2 \infty_6$

$v = 117:$

$4_1 11_2 14_2 19_2$	$2_1 11_1 13_1 15_2 40_2$	$25_1 28_1 51_1$	$41_1 45_1 7_1$
$16_1 21_1 40_1$	$6_1 18_1 31_1$	$30_2 39_2 0_2$	$25_2 26_2 48_2$
$31_2 41_2 52_2$	$27_2 33_2 53_2$	$23_1 29_1 22_2$	$30_1 37_1 16_2$
$19_1 27_1 17_2$	$49_1 5_1 46_2$	$36_1 50_1 32_2$	$33_1 48_1 28_2$
$9_1 26_1 29_2$	$44_1 8_1 38_2$	$22_1 43_1 3_2$	$52_1 20_1 44_2$
$42_1 50_2 13_2$	$35_1 2_2 20_2$	$47_1 47_2 12_2$	$32_1 6_2 10_2$
$39_1 1_2 8_2$	$17_1 23_2 35_2$	$15_1 24_2 37_2$	$14_1 45_2 5_2$
$46_1 18_2 34_2$	$0_1 1_1 a_0$	$3_1 4_2 a_1$	$7_2 9_2 a_2$
$38_1 43_2 \infty_1$	$10_1 21_2 \infty_2$	$24_1 36_2 \infty_3$	$34_1 51_2 \infty_4$
$12_1 49_2 \infty_5$	$53_1 42_2 \infty_6$		

$v = 135:$

$0_1 7_2 10_2 15_2$	$3_1 12_1 14_1 16_2 41_2$	$20_1 62_2 5_2$	$13_2 25_2 42_2$
$16_1 40_2 49_2$	$13_1 32_2 50_2$	$34_1 41_1 61_1$	$21_2 37_2 48_2$
$32_1 48_1 26_2$	$39_1 44_2 1_2$	$56_1 6_1 23_1$	$4_1 25_1 30_1$
$47_2 54_2 12_2$	$8_1 14_2 36_2$	$28_1 31_1 59_1$	$44_1 58_1 10_1$
$33_1 43_1 29_2$	$52_1 1_1 33_2$	$47_1 59_2 20_2$	$29_1 54_1 17_2$
$19_1 37_1 58_2$	$2_1 19_2 45_2$	$49_1 0_2 4_2$	$22_1 46_1 6_2$
$18_1 38_2 53_2$	$62_1 21_1 55_2$	$36_1 42_1 34_2$	$45_1 5_1 35_2$
$30_2 31_2 61_2$	$43_2 56_2 3_2$	$53_1 57_1 52_2$	$24_1 27_2 46_2$
$11_1 51_2 2_2$	$27_1 35_1 24_2$	$7_1 26_1 57_2$	$60_1 60_2 \infty_1$
$15_1 23_2 \infty_2$	$9_1 18_2 \infty_3$	$55_1 8_2 \infty_4$	$17_1 28_2 \infty_5$
$40_1 22_2 \infty_6$	$50_1 51_1 a_0$	$38_1 39_2 a_1$	$9_2 11_2 a_2$

Obviously, any $KPD(\{3,4^*,5^*\},v)$ constructed in Lemmas 3.1-3.4 i also an $IKPD(\{3,4^*,5^*\},v,9)$. Now we are in a position to prove ou main results.

Theorem 3.5 *There exists a $KPD(\{3,4^*,5^*\},v)$ containing $(v-9)/2$ parallel classes for every $v \equiv 3 \pmod 6$ with $v \geq 51$.*

Proof: For $v \geq 243$, let $v' = v - 9$. By Lemma 2.2, there exists a Kirkman frame of type $42^a 36^b 30^c$, where $v' = 42a + 36b + 30c$, $a \geq 4$, $b, c \geq 0$ or $a = 0$, $b \geq 4$, $c \geq 0$. To use an $IKPD(\{3,4^*,5^*\},39,9)$, an $IKPD(\{3,4^*,5^*\},51,9)$, and an $IKPD(\{3,4^*,5^*\},45,9)$ from Lemmas 2.6-2.7 and Lemma 3.3, to fill all holes of size $6u \in \{30,36,42\}$, we obtain an $IKPD(\{3,4^*,5^*\},v,9)$ which contains $(v-9)/2$ parallel classes, so it is actually a $KPD(\{3,4^*,5^*\},v)$. Now it remains to consider the cases while $v \leq 237$. For $v = 231, 237$, with frames of types $48^4 36^1$ and $48^4 30^1$ from Lemma 2.3, to apply Lemma 2.4 with $h = 9$ to fill in "holes" by using $IKPD(\{3,4^*,5^*\},57,9)$s, $IKPD(\{3,4^*,5^*\},45,9)$s and $IKPD(\{3,4^*,5^*\},39,9)$s, thus we obtain an $IKPD(\{3,4^*,5^*\},231,9)$ and an $IKPD(\{3,4^*,5^*\},237,9)$ which are also KPDs. A similar construction using frames of types $42^4(6x)^1$, $36^4(6x)^1$ and $30^4(6x)^1$, $5 \leq x \leq 7$, solves the cases when $159 \leq v \leq 171$, $183 \leq v \leq 195$ and $207 \leq v \leq 219$. Also the case for any $v \in \{129,153,177,201,225\}$ comes from Kirkman frames of types 42^4, 36^4, 30^4, 48^4 and 54^4. The required $IKPD$s come from Lemmas 3.1-3.4 and 2.6-2.7. Adding 9 new points to a Kirkman frame of type $30^4 18^1$ from Lemma 2.3 and filling in $IKPD(\{3,4^*,5^*\},39,9)$s and a $KP(27)$ kills the case $v = 147$. Adding 27 new points to an $IKPD(\{3,4^*,5^*\},141,27)$ from Lemma 2.8 and filling in a $KP(\{3,4^*,5^*\},27)$ covers the case $v = 141$. The case for $v \in \{n \mid 51 \leq n \leq 123, n \equiv 3 (mod 6)\} \cup \{135\}$ comes from Lemmas 3.1-3.4. This completes the proof.

4 Conclusions

Resolvable packings have been studied extensively and found to have a number of applications. Especially, many researchers have given some applications in threshold schemes (see, e.g. [3,7,15]). In this paper, we have determined the existence of a $KPD(\{3,4^*,5^*\},v)$ for any $v \equiv 3 \pmod 6$ with $v \geq 51$. These results can be used to construct some new $(2,w)$-threshold schemes. For the existence of $KPD(\{3,4^*,5^*\},v)$'s, it is easy to see that there is a $KPD(\{3,4^*,5^*\}, v)$ for $v \in \{9,15\}$, but we are not able to give a $KPD(\{3,4^*,5^*\},v)$ for $v \in \{21,27,33,39,45\}$. Moreover, how to construct a $KPD(\{3,4^*,5^*\},v)$ for $v \equiv 0 \pmod 6$ in general is also an open problem. In this case, we give the construction of a $KPD(\{3,4^*,5^*\},54)$ with 23 maximum parallel classes to close this paper.

Point set: $Z_{23} \times \{1,2\} \cup \{\infty_i \mid 1 \leq i \leq 8\}$

Base blocks:

$0_1 9_1 10_1 22_2$	$1_1 5_1 8_2 16_2 19_2$	$2_1 8_1 12_2$	$4_1 9_2 13_2$	$16_1 18_2 1_2$
$3_1 11_1 14_1$	$15_1 20_1 22_1$	$5_2 14_2 15_2$	$20_2 2_2 4_2$	$21_1 21_2 \infty_1$
$6_1 7_2 \infty_2$	$17_1 0_2 \infty_3$	$18_1 11_2 \infty_4$	$12_1 6_2 \infty_5$	$7_1 3_2 \infty_6$
$13_1 10_2 \infty_7$	$19_1 17_2 \infty_8$			

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References

[1] R. Baker, R.M. Wilson, Nearly Kirkman triple systems, Utilitas Math. 11 (1977) 289-296.

[2] A. Brouwer, Two new Kirkman triple systems, Utilitas Math. 13 (1978) 311-314.

[3] H. Cao, B. Du, Kirkman packing designs $KPD(\{w, s^*\}, v)$ and related threshols schemes, Discrete Math. 281 (2004) 83-95.

[4] H. Cao, Y. Tang, On Kirkman packing designs $KPD(\{3, 4\}, v)$, Discrete Math. 279 (2004) 121-133.

[5] H. Cao, L. Zhu, Kirkman packing designs $KPD(\{3, 5^*\}, v)$, Designs, Codes and Cryptography 26 (2002) 127-138.

[6] A. Cerny, P. Horakm, W.D. Wallis, Kirkman's school projects, Discrete Math. 167/168 (1997) 189-196.

[7] D. Chen and D.R. Stinson, Recent results on combinatorial constructionos for threshold schems, Austral. J. Combin. Vol. 1 (1990) 29-48.

[8] C.J. Colbourn, J.H. Dinitz (eds), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996.

[9] C. J. Colbourn, A.C.H. Ling, Kirkman school project designs, Discrete Math. 203 (1999) 49-56.

[10] A. Kotzig, A. Rosa, Nearly Kirkman triple systems, Proceedings of the Fifth SE Conference on combinatorics, Graph Theory and Computing, Boca Ratan, FL, 1974, pp. 607-614.

[11] N.C.K. Phillips, W.D. Wallis, R.S. Rees, Kirkman packing and covering designs, JCMCC 28 (1998), 299-325.

[12] D.R. Ray-Chaudhuri, R.M. Wilson, Solution of Kirkman's schoolgirl problem, Amer. Math. Soc. Symp. Pure. Math. 19 (1971) 187-204.

[13] R. Rees, D. Stinson, On resolvable group-divisible designs with block size 3, Ars Combin. 23 (1987)107-120.

[14] D.R. Stinson, Frames for Kirkman triple systems, Discrete Math. 65 (1987) 289-300.

[15] D.R. Stinson, S.A. Vanstone, A combinatorial approach to threshold schems, SIAM Discrete Math. 1 (1988) 230-236.