Upper Bounds on the Total Domination Number

[1]Teresa W. Haynes and [2]Michael A. Henning*

[1]Department of Mathematics
East Tennessee State University
Johnson City, TN 37614-0002 USA
Email: haynes@etsu.edu

[2]School of Mathematical Sciences
University of KwaZulu-Natal
Pietermaritzburg, 3209 South Africa
Email: henning@ukzn.ac.za

Abstract

A total dominating set of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The total domination number of G is the minimum cardinality of a total dominating set in G. In this paper, we present several upper bounds on the total domination number in terms of the minimum degree, diameter, girth and order.

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1 Introduction

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [4] and is now well studied in graph theory (see, for example, [1, 5, 9]).

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The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8].

A *total dominating set* (TDS) of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS.

For notation and graph theory terminology we in general follow [7]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, the *open neighborhood* of S is $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is $N[S] = N(S) \cup S$. The *boundary* of S, denoted $\mathrm{B}(S)$, is $N(S) \setminus S$. The subgraph induced by S is denoted by $G[S]$. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). We denote the *girth* of G by $g(G)$. For disjoint subsets S and T of V, we define $G[S, T]$ as the set of edges of G joining S and T.

We call the tree obtained from a star $K_{1,n}$ by subdividing every edge exactly once a *subdivided star*, which we denote by $K^*_{1,n}$.

If G does not contain a graph F as an induced subgraph, then we say that G is F-free. In particular, we say a graph is *triangle-free* if it is K_3-free, *diamond-free* if it is $(K_4 - e)$-free, and *quadrilateral-free* if it is C_4-free.

In this paper, we present several upper bounds on the total domination number in terms of the minimum degree, diameter, girth and order.

2 Known Results

The decision problem to determine the total domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph. Cockayne, Dawes and Hedetniemi [4] obtained the following upper bound on the total domination number of a connected graph in terms of its order.

Theorem 1 ([4]) *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

A large family of graphs attaining the bound in Theorem 1 can be established using the following transformation of a graph. The 2-*corona* of a graph H is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint. The 2-corona of a connected graph has total domination number two-thirds its order. Brigham, Carrington, and Vitray [2] obtained the following characterization of connected graphs of order at least 3 with total domination number exactly two-thirds their order.

Theorem 2 ([2]) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or the 2-corona of some connected graph.*

If the minimum degree is at least 2, then the upper bound in Theorem 1 can be improved.

Theorem 3 ([9]) *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$.*

3 Preliminary Result

We shall need the following lemma about domination in bipartite graphs.

Lemma 4 *Let G be a bipartite graph with partite sets (X, Y) whose vertices in Y are of degree at least $\delta \geq 1$. Then there exists a set $A \subseteq X$ of size at most $\frac{1}{2}(|Y| + |X|/\delta)$ that dominates Y.*

Proof. Let $|X| = x$ and $|Y| = y$. The proof is by induction on $|V(G)| + |E(G)|$. The smallest graph described by the lemma is $K_{1,\delta}$, for which the statement holds. This establishes our base case.

If there exists a vertex v in Y of degree at least $\delta + 1$, then delete any edge e incident to v. The subset A of $G - e$ guaranteed by the inductive hypothesis dominates Y in G as desired. So we may assume the vertices in Y are all of degree exactly δ. If there exists an isolated vertex $v \in X$, then the set A in $G - v$ dominates Y in G as desired. So we may assume each vertex in X has degree at least 1. Since $y \geq 1$ and $x \geq \delta$, we have that $\frac{1}{2}(y + x/\delta) \geq 1$. Hence if there is a vertex in X that dominates Y, then the desired result follows readily. Thus we may assume that no vertex of X dominates Y.

If each vertex in X has degree 1, then G is a disjoint union of y copies of $K_{1,\delta}$, and so $x = \delta y$. For each vertex in Y, we now choose an adjacent vertex in X to form the set A. Then, $|A| = y = \frac{1}{2}(y + x/\delta)$, as desired. Hence we may assume that at least one vertex v in X has degree 2.

Let $G' = G - N[v]$ and let X' and Y' be the restriction of X and Y, respectively, to G'. Then, $|X'| = x - 1$ and $|Y'| \leq y - 2$. By the inductive hypothesis, there exists a subset A' of X' that dominates Y' in G' with $|A'| \leq \frac{1}{2}(|Y'| + |X'|/\delta) \leq \frac{1}{2}(y - 2 + (x-1)/\delta) < \frac{1}{2}(y + x/\delta) - 1$. Thus, the set $A = A' \cup \{v\}$ dominates Y in G with $|A| < \frac{1}{2}(y + x/\delta)$, as desired. □

4 Upper bounds in terms of minimum degree

Flach and Volkmann [6] proved that if G is a graph of order n with minimum degree $\delta \geq 2$, and if $A \subset V(G)$ is an arbitrary subset, then $\gamma(G) \leq \frac{1}{2}\left(n + |A| - (\frac{\delta-1}{\delta})|\mathrm{B}(A)|\right)$ where $\gamma(G)$ denotes the domination number of G. In this section, we present an analogous result for the total domination number.

Theorem 5 *If A is an arbitrary subset of vertices in a graph G of order n with minimum degree $\delta \geq 2$, then*

$$\gamma_t(G) < \frac{2}{3}\left(n + 2|A| - \left(\frac{\delta-2}{\delta-1} + \frac{1}{4(\delta-1)}\right)|B(A)|\right).$$

Proof. Let V_1 be the set of isolated vertices in $G - N[A]$ and let V_2 be the set of vertices that belong to a K_2-component of $G - N[A]$. Let $G_1 = G[N[A] \cup V_1 \cup V_2]$ and let $G_2 = G - V(G_1)$. Each component of G_2 has order at least 3. Thus by Theorem 1, $\gamma_t(G_2) \leq 2|V(G_2)|/3 = 2(n - |V_1| - |V_2| - |A| - |\mathrm{B}(A)|)/3$. If $V_1 = V_2 = \emptyset$, then

$$\begin{aligned}
\gamma_t(G) &\leq \gamma_t(G_1) + \gamma_t(G_2) \\
&\leq 2|A| + \tfrac{2}{3}\left(n - |A| - |\mathrm{B}(A)|\right) \\
&= \tfrac{2}{3}\left(n + 2|A| - |\mathrm{B}(A)|\right) \\
&= \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{\delta-1}\right)|\mathrm{B}(A)|\right) \\
&< \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)|\mathrm{B}(A)|\right),
\end{aligned}$$

which is the desired bound. Hence we may assume that $V_1 \cup V_2 \neq \emptyset$.

Let H be the bipartite subgraph of G with partite sets $(V_1 \cup V_2, N(V_1) \cup \mathrm{B}(V_2))$ and with edge set defined by $G[V_1 \cup V_2, N(V_1) \cup \mathrm{B}(V_2)]$. Then each vertex in $V_1 \cup V_2$ has degree at least $\delta - 1 \geq 1$ in H. Thus by Lemma 4, there exists a set $A' \subseteq N(V_1) \cup \mathrm{B}(V_2)$ of size at most $\frac{1}{2}(|V_1|+|V_2|+|N(V_1) \cup \mathrm{B}(V_2)|/(\delta-1))$ that dominates $V_1 \cup V_2$. Since $N(V_1) \cup \mathrm{B}(V_2) \subseteq \mathrm{B}(A)$, we have $|A'| \leq \frac{1}{2}(|V_1| + |V_2| + |\mathrm{B}(A)|/(\delta-1))$.

The set $A' \subseteq \mathrm{B}(A)$ can be extended to a TDS of G_1 by adding to it the set A and adding for each vertex, if any, in $A \setminus N(A')$ one of its neighbors. Since there are at most $|A|-1$ vertices in A not dominated by A', $\gamma_t(G_1) \leq 2|A| - 1 + |A'| \leq 2|A| - 1 + \frac{1}{2}(|V_1| + |V_2| + |\mathrm{B}(A)|/(\delta-1))$. Hence,

$$\begin{aligned}
\gamma_t(G) &\leq \gamma_t(G_1) + \gamma_t(G_2) \\
&\leq 2|A| - 1 + \tfrac{1}{2}\left(|V_1| + |V_2| + \tfrac{1}{\delta-1}|\mathrm{B}(A)|\right) + \\
&\qquad \tfrac{2}{3}\left(n - |V_1| - |V_2| - |A| - |\mathrm{B}(A)|\right) \\
&= \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)|\mathrm{B}(A)| - \tfrac{1}{4}(|V_1| + |V_2|) - \tfrac{3}{2}\right) \\
&< \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)|\mathrm{B}(A)|\right). \square
\end{aligned}$$

Taking the set A to consist of a singleton vertex of maximum degree in G, we have the following immediate consequence of Theorem 5.

Corollary 6 *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then*

$$\gamma_t(G) < \frac{2}{3}\left(n + 2 - \left(\frac{\delta-2}{\delta-1} + \frac{1}{4(\delta-1)}\right)\Delta\right).$$

Archdeacon et al. [1] showed that if G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$. We remark that if $\delta = 3$ and Δ is large (namely, if $\Delta > 2(n+8)/5$), then the bound in Corollary 6 improves on the Archdeacon bound of one-half the order for the total domination number.

5 Upper bounds in terms of diameter

Volkmann [10] presented upper bounds on the domination number of a connected graph in terms of the diameter and minimum degree. In this section, we present analogous upper bounds on the total domination number. We shall prove:

Theorem 7 *If G is a connected graph of order n with diameter d and minimum degree $\delta \geq 2$, then*

$$\gamma_t(G) < \frac{2}{3}\left(n - \left(\frac{4\delta^2 - 15\delta + 8}{4(\delta - 1)}\right)\left(1 + \left\lfloor \frac{d}{3} \right\rfloor\right)\right).$$

Proof. Let $v_0, v_1, \ldots, v_d$ be a shortest path between two diametrical vertices v_0 and v_d of G. Let

$$A = \bigcup_{i=0}^{\lfloor d/3 \rfloor} \{v_{3i}\}.$$

Then, $|A| = 1 + \lfloor d/3 \rfloor$ and $N(A) \cap A = \emptyset$. Thus, $|\mathrm{B}(A)| \geq \delta |A|$. By Theorem 5,

$$\begin{aligned} \gamma_t(G) &\leq \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)|\mathrm{B}(A)|\right) \\ &\leq \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)\delta|A|\right) \\ &= \tfrac{2}{3}\left(n - \left(\tfrac{4\delta^2-15\delta+8}{4(\delta-1)}\right)|A|\right) \\ &= \tfrac{2}{3}\left(n - \left(\tfrac{4\delta^2-15\delta+8}{4(\delta-1)}\right)\left(1 + \lfloor \tfrac{d}{3} \rfloor\right)\right). \square \end{aligned}$$

If G is a triangle-free, then the result of Theorem 7 can be improved.

Theorem 8 *If G is a connected triangle-free graph of order n with diameter d and minimum degree $\delta \geq 2$, then*

$$\gamma_t(G) < \frac{2}{3}\left(n - \delta + \frac{15}{4}\right)\left(\left\lfloor \frac{d-1}{4} \right\rfloor + \left\lfloor \frac{d}{4} \right\rfloor + 2\right).$$

Proof. Let $v_0, v_1, \ldots, v_d$ be a shortest path between two diametrical vertices v_0 and v_d of G. Let

$$A = \left(\bigcup_{i=0}^{\lfloor d/4 \rfloor} \{v_{4i}\} \right) \cup \left(\bigcup_{i=0}^{\lfloor (d-1)/4 \rfloor} \{v_{4i+1}\} \right).$$

Then, $|A| = \lfloor d/4 \rfloor + \lfloor (d-1)/4 \rfloor + 2$. Since G is triangle-free, $|B(A)| \geq (\delta - 1)|A|$. By Theorem 5,

$$\begin{aligned} \gamma_t(G) &\leq \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)|B(A)|\right) \\ &\leq \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)(\delta-1)|A|\right) \\ &= \tfrac{2}{3}\left(n - \delta + \tfrac{15}{4}\right)|A| \\ &= \tfrac{2}{3}\left(n - \delta + \tfrac{15}{4}\right)\left(\lfloor \tfrac{d}{4} \rfloor + \lfloor \tfrac{d-1}{4} \rfloor + 2\right). \square \end{aligned}$$

If G is a quadrilateral-free and diamond-free graph, then the result of Theorem 7 can be improved.

Theorem 9 *If G is a connected $(C_4, K_4 - e)$-free graph of order n with diameter d and minimum degree $\delta \geq 2$, then*

$$\gamma_t(G) < \frac{2}{3}\left(n + \frac{11}{4} - \delta + \frac{3}{4(\delta-1)} - \left(\delta - \frac{15}{4}\right)\left\lfloor \frac{d}{2} \right\rfloor\right).$$

Proof. Let $v_0, v_1, \ldots, v_d$ be a shortest path between two diametrical vertices v_0 and v_d of G. Let

$$A = \bigcup_{i=0}^{\lfloor d/2 \rfloor} \{v_{2i}\}.$$

Then, $|A| = 1 + \lfloor d/2 \rfloor$. Since the graph G is C_4-free and diamond-free, there are $|A| - 1$ common neighbors of vertices in A, namely the vertices in the set

$$\bigcup_{i=0}^{\lfloor d/2 \rfloor - 1} \{v_{2i+1}\}.$$

Hence, $|B(A)| \geq \delta|A| - (|A| - 1) = (\delta - 1)|A| + 1$. By Theorem 5,

$$\begin{aligned}
\gamma_t(G) &\leq \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)|B(A)|\right) \\
&\leq \tfrac{2}{3}\left(n + 2|A| - \left(\tfrac{\delta-2}{\delta-1} + \tfrac{1}{4(\delta-1)}\right)((\delta-1)|A| + 1)\right) \\
&= \tfrac{2}{3}\left(n - 1 + \tfrac{3}{4(\delta-1)} - (\delta - 4 + \tfrac{1}{4})|A|\right) \\
&= \tfrac{2}{3}\left(n - 1 + \tfrac{3}{4(\delta-1)} - (\delta - \tfrac{15}{4})(1 + \lfloor \tfrac{d}{2} \rfloor)\right) \\
&= \tfrac{2}{3}\left(n + \tfrac{11}{4} - \delta + \tfrac{3}{4(\delta-1)} - (\delta - \tfrac{15}{4})\lfloor \tfrac{d}{2} \rfloor\right). \square
\end{aligned}$$

6 Upper bounds in terms of maximum degree and girth

In this section, we present an upper bound on the total domination number of a graph with girth at least 6 in terms of its maximum degree.

Theorem 10 *If G is a graph with $\delta(G) \geq 2$ and $g(G) \geq 6$, then $\gamma_t(G) \leq \frac{2}{3}(n + 2 - \Delta(G))$.*

Proof. Let v be a vertex of maximum degree $\Delta(G)$, and let $G_v = G[V - N[v]]$. Since $g(G) \geq 6$, $N(v)$ is independent and v is the only common neighbor of any pair of vertices in $N(v)$, that is, no component of G_v is an isolate. If any component of G_v has exactly two vertices, then since $\delta(G) \geq 2$ each of these vertices has a neighbor in $N(v)$ and a 5-cycle is formed, contradicting the fact that $g(G) \geq 6$. Hence every component of G_v has order at least three, and so by Theorem 1, $\gamma_t(G_v) \leq \frac{2}{3}|V(G_v)| = \frac{2}{3}(n - 1 - \Delta(G))$. Adding the vertex v along with a neighbor of v to a $\gamma_t(G_v)$-set produces a TDS of G, and so, $\gamma_t(G) \leq 2 + \frac{2}{3}(n - 1 - \Delta(G)) = \frac{2}{3}(n + 2 - \Delta(G))$. $\square$

If we restrict the girth of the graph G in the statement of Corollary 6 to be at least 6, then the upper bound of that result can be improved.

Theorem 11 *If G is a graph of order n with minimum degree $\delta \geq 2$ and girth $g(G) \geq 6$, then*

$$\gamma_t(G) \leq \frac{4}{7}\left(n - \frac{(\delta-2)^2}{2}\right),$$

unless $G \in \{C_6, C_{10}\}$.

Proof. If $\delta = 2$, then the result follows from Theorem 3. Hence we may assume that $\delta \geq 3$. Since G is triangle-free, the open neighborhood of every vertex is an independent set of vertices, and so $\beta(G) \geq \Delta$. Let S be an independent set of $\delta - 2 \geq 1$ vertices. Since G is triangle-free and C_4-free, every vertex in $N(S)$ is adjacent to at least one vertex of $V(G) \setminus N[S]$. Let $F = G - N[S]$.

For each vertex $v \in V(F)$, let $N_v = N(v) \cap N(S)$. Since G is C_4-free, $|N_v| \leq \delta - 2$ for every $v \in V(F)$, and so $\delta(F) \geq 2$. Furthermore, if $|N_v| = \delta - 2$ for some $v \in V(F)$, then $G[S \cup N_v \cup \{v\}] = K^*_{1,\delta-2}$. In particular, the vertex v is at distance 2 from every vertex of S.

Suppose that F contains two adjacent vertices u and v both having degree 2 (in F). Then, $\delta \leq d_G(v) = d_F(v) + |N_v| \leq 2 + (\delta - 2) = \delta$. Consequently, $|N_v| = \delta - 2$. Similarly, $|N_u| = \delta - 2$. Thus each of u and v is at distance 2 from every vertex of S, and so there is a common cycle of length at most 5 containing both u and v, contradicting the fact that $g(G) \geq 6$. Hence, no two adjacent vertices of F both have degree 2. In particular, F has no cycle component. Thus, since $\delta(F) \geq 2$, we have by Theorem 3 that $\gamma_t(F) \leq 4|V(F)|/7 = 4(n - |N[S]|)/7$.

Let S^* be the set of vertices in $N(S)$ that are adjacent to two or more vertices of S. Since S is an independent set and G is C_4-free, every pair of vertices in S has at most one common neighbor. Thus, $|S^*| \leq \binom{\delta-2}{2}$. It follows that

$$\begin{aligned} |N[S]| &= |S| + |N(S)| \\ &= |S| + \left(\sum_{x \in S} d_G(x)\right) - |S^*| \\ &\geq (\delta - 2) + \delta(\delta - 2) - \binom{\delta-2}{2} \\ &= \tfrac{1}{2}(\delta - 2)(\delta + 5). \end{aligned}$$

For each $x \in S$, let x' be a neighbor of x in G. Let $S' = \cup\{x'\}$ where the union is taken over all vertices $x \in S$. Then, $|S'| \le |S|$. Every $\gamma_t(F)$-set can be extended to a TDS of G by adding to it the vertices in the set $S \cup S'$. Hence,

$$
\begin{aligned}
\gamma_t(G) &\le 2|S| + \gamma_t(F) \\
&\le 2(\delta - 2) + \frac{4}{7}(n - |N[S]|) \\
&\le 2(\delta - 2) + \frac{4}{7}\left(n - \frac{1}{2}(\delta - 2)(\delta + 5)\right) \\
&\le \frac{4}{7}\left(n - \frac{(\delta - 2)^2}{2}\right). \quad \square
\end{aligned}
$$

We remark that if the maximum degree Δ is sufficiently large relative to the order n and minimum degree δ, namely if $\Delta \ge 2 + \frac{1}{7}(n + 3(\delta - 2)^2)$, then the upper bound for the total domination number given in Theorem 10 improves on that given in Theorem 11. However if the maximum degree is small relative to the order and minimum degree, then the upper bound for the total domination number given in Theorem 11 improves on that given in Theorem 10. In particular, for a regular graph, the upper bound for the total domination number given in Theorem 11 improves on that given in Theorem 10.

7 Upper bounds in terms of order and girth

Brigham and Dutton [3] showed that if G is a graph of order n, minimum degree $\delta \ge 2$ and girth $g \ge 5$, then $\gamma(G) \le \lceil (3n - g)/6 \rceil$. This bound was recently improved slightly by Volkmann [10] who showed that if the graph G is neither a cycle nor one of two exceptional graphs, then this upper bound can be reduced by 1.

Our aim in this section is to establish an upper bound on the total domination of a graph with girth at least 7 in terms of its order and girth. First we recall the total domination number of a cycle C_n on n vertices.

Observation 12 [9] *For* $n \ge 3$, $\gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

By Observation 12, $\gamma_t(C_n) \le (n+2)/2$ with equality if and only if $n \equiv 2 \pmod 4$.

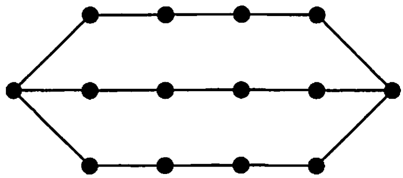

Figure 1: The graph G_1

We are now in a position to present our main result of this section.

Theorem 13 *If $G \neq C_n$ is a connected graph with order n, girth $g \ge 7$, and $\delta(G) \ge 2$, then $\gamma_t(G) \le (4n-g)/6$ unless $G = G_1$, where G_1 is the graph shown in Figure 1, in which case $\gamma_t(G) = 8 = (4n+2-g)/6$.*

Proof. Let C be a g-cycle in G, and let $H = G - V(C)$. By Observation 12, $\gamma_t(C) \le (g+2)/2$ with equality if and only if $g \equiv 2 \pmod 4$. Since $g > 4$, every vertex of H is adjacent to at most one vertex on the cycle C. Thus, since $\delta(G) \ge 2$, the graph H has minimum degree $\delta(H) \ge \delta(G) - 1 \ge 1$.

Suppose H has a K_2-component consisting of two vertices w and z. Then each of w and z has exactly one neighbor on C, say w' and z', respectively. Since $g \ge 7$, the two w'–z' paths on C both have length at least 4. Replacing a w'–z' path on C by the path w', w, z, z' produces a cycle of length less than g, a contradiction. Hence every component of H has order at least 3.

By Theorem 1, $\gamma_t(H) \le 2|V(H)|/3 = 2(n-g)/3$. Hence, $\gamma_t(G) \le \gamma_t(C) + \gamma_t(H) \le (g+2)/2 + 2(n-g)/3 = (4n-g)/6 + 1$. With a bit more work, we show that this bound of $(4n-g)/6+1$ can be reduced by 1, unless $G = G_1$.

Since G is connected, there is an edge uv in G with $u \in V(C)$ and $v \in V(H)$. Among all such vertices v of H that are joined to a vertex of C, we choose v so that

(1) v has smallest possible degree in H.
(2) Subject to (1), v belongs to a component of H of order congruent to 0 modulo 3 if possible.
(3) Subject to (2), v belongs to a $\gamma_t(H)$-set, if possible.

Let S_u be a $\gamma_t(C)$-set containing the vertex u, and let D_u be a $\gamma_t(C-u)$-set. Then, $|S_u| \le (g+2)/2$ with equality if and only if $g \equiv 2 \pmod 4$, while $|D_u| \le (g+1)/2$ with equality if and only if $g \equiv 3 \pmod 4$. Among all $\gamma_t(H)$-sets, let S_v be chosen so that, if possible, $v \in S_v$. With this choice of the set S_v, if $v \notin S_v$, then there is no $\gamma_t(H)$-set that contains the vertex v. It follows that in this case, every component of $H-v$ has order at least 3, and so $\gamma_t(H-v) \le 2|V(H-v)|/3 = 2(n-g-1)/3$. If $v \notin S_v$, let D_v be a $\gamma_t(H-v)$-set. Then, $|D_v| \le 2(n-g-1)/3$. We consider three possibilities.

Case 1. $\gamma_t(H) \le 2(n-g-1)/3$. If $\gamma_t(C) \le (g+1)/2$, then $\gamma_t(G) \le \gamma_t(H)+\gamma_t(C) \le 2(n-g-1)/3+(g+1)/2 = (4n-g-1)/6$. Hence we may assume that $\gamma_t(C) = (g+2)/2$, and so $g \equiv 2 \pmod 4$. Thus, $|S_u| = (g+2)/2$ and $|D_u| = g/2$. If $v \in S_v$, then the set $S_v \cup D_u$ is a TDS of G, and so $\gamma_t(G) \le 2(n-g-1)/3+g/2 = (4n-g-4)/6$. Hence we may assume that $v \notin S_v$. Thus the set $D_v \cup S_u$ is a TDS of G. If $|D_v| < 2(n-g-1)/3$, then $\gamma_t(G) \le (2(n-g)-3)/3+(g+2)/2 = (4n-g)/6$, as desired. Hence we may assume that $|D_v| = 2(n-g-1)/3$, that is, $\gamma_t(H-v) = 2|V(H-v)|/3$. Therefore, by Theorem 2, every component of $H-v$ is the 2-corona of some connected graph (of girth at least g). Let H_v be the component of H containing v. Note that $|V(H_v)| \ge 4$.

Suppose $H_v = P_4$. Let w be the end-vertex of this P_4 different from v. Then w has exactly one neighbor on C, say x. Since $g \equiv 2 \pmod 4$ and $g \ge 7$, we must have that $g = 10$ and that x is the vertex at maximum distance 5 from u on the cycle C. Hence, $G[V(C) \cup V(H_v)] = G_1$. If $H \neq H_v$, then by our earlier assumption, every component of $H-H_v$ is the 2-corona of some connected graph. In particular, every every component of $H-H_v$ has order congruent to 0 modulo 3 and contains a vertex of degree 1 (that is joined to a vertex of C). But this would contradict our choice of the vertex v. Hence $H = H_v$, whence $G = G_1$.

Suppose that $H_v \neq P_4$. As observed earlier, every component of $H-v$ is the 2-corona of some connected graph. Hence it follows from our choice of v, that $H = H_v$. Thus $|V(H)| \ge 7$ and since v is in no $\gamma_t(H)$-set, we deduce that $H-v$ is the 2-corona of some (not necessary connected) graph F and that the only possible vertices of $H-v$ adjacent to v belong to the graph F. But then every vertex of degree 1 in $H-v$ that does not belong to F is a vertex of degree 1 in H that is easily seen to belong to some $\gamma_t(H)$-set, contradicting our choice of v.

Case 2. $\gamma_t(H) = (2(n-g)-1)/3$. Thus, $n-g \equiv 2 \pmod 3$. Suppose that $v \notin S_v$. Then, the set $D_v \cup S_u$ is a TDS of G. However, $|D_v| \le \lfloor 2(n-g-1)/3 \rfloor = (2(n-g-1)-2)/3$, and so $\gamma_t(G) \le (2(n-g)-4)/3+(g+2)/2 =$

$(4n-g-2)/6$. Hence we may assume that $v \in S_v$. Then the set $S_v \cup D_u$ is a TDS of G. If $|D_u| \leq g/2$, then $\gamma_t(G) \leq (2(n-g)-1)/3 + g/2 = (4n-g-2)/6$. Hence we may assume that $|D_u| = (g+1)/2$, implying that $g \equiv 3 \pmod 4$. But then the set $S_v \cup \{u\}$ can be extended to a TDS of G by adding to it $(g-3)/2$ vertices from the path $C - N[u] \cong P_{g-3}$, and so $\gamma_t(G) \leq (2(n-g)-1)/3 + 1 + (g-3)/2 = (4n-g-5)/6$.

Case 3. $\gamma_t(H) = 2(n-g)/3$. Then, by Theorem 2, H is the 2-corona of some graph (of girth at least g). Thus every vertex of H is in some $\gamma_t(H)$-set. In particular, $v \in S_v$. If $g \equiv 2 \pmod 4$, then $|D_u| = g/2$, and so $\gamma_t(G) \leq |S_v \cup D_u| = 2(n-g)/3 + g/2 = (4n-g)/6$. If $g \equiv 3 \pmod 4$, then $\gamma_t(G) \leq |S_v \cup \{u\}| + \gamma_t(C - N_C[u]) = 2(n-g)/3 + 1 + \gamma_t(C_{g-3}) = 2(n-g)/3+1+(g-3)/2 = (4n-g-3)/6$. If $g \equiv 1 \pmod 4$, then $|D_u| = (g-1)/2$, and so $\gamma_t(G) \leq |S_v \cup D_u| = 2(n-g)/3 + (g-1)/2 = (4n-g-3)/6$. If $g \equiv 0 \pmod 4$, then $\gamma_t(G) \leq |S_u \cup S_v| = g/2 + 2(n-g)/3 = (4n-g)/6$. Hence in the case where $\gamma_t(H) = 2(n-g)/3$, we have $\gamma_t(G) \leq (4n-g)/6$. □

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