Edge-magic and Vertex-magic Total Labelings of Certain Cycles

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Abstract

We provide many new edge-magic and vertex-magic total labelings for the cycles C_{nk}, where $n \geq 3$ and $k \geq 3$ are both integers and n is odd. Our techniques are of interest since known labelings for C_k are used in the construction of those for C_{nk}. This provides significant new evidence for a conjecture on the possible magic constants for edge-magic and vertex-magic cycles.

1 Introduction

Let G be a graph with vertex set V and edge set E. A total labeling λ of G is a bijective map $\lambda : V \cup E \to \{1, 2, 3, \ldots, (|V| + |E|)\}$. The λ-weight, $wt_\lambda(v)$ of a vertex $v \in V$ is defined to be $wt_\lambda(v) = \lambda(v) + \Sigma\lambda(e)$, where the sum is over all edges e incident with v. The λ-weight $wt_\lambda(u, v)$ of an edge (u, v) with ends u and v is the sum $\lambda(u) + \lambda((u, v)) + \lambda(v)$. We say that λ is a *vertex-magic total labeling* (VMTL) if the vertex weight $wt_\lambda(v)$ does not depend on the choice of vertex v. In this case we write $h_\lambda = wt_\lambda(v)$ and we say that h_λ is the magic constant of λ. We say that λ is *edge-magic* if the edge weight $wt_\lambda(e)$ does not depend on the choice of edge e. In the case of cycles, one can easily obtain a VMTL from an edge-magic total labeling (and vice versa), by moving each vertex label over to one of its incident edges, and moving each edge label over to one of its end vertices, as shown in Figure 1. In [4], it was shown that every cycle has an edge-magic total labeling, and therefore also a VMTL. It was shown again in [3] with a different labeling. Similar work was done independently in [7] and [1]. In this paper we will only be concerned with cycles, and henceforth we will only discuss VMTLs, rather than edge-magic total labelings. Two "dual" VMTL's of C_5 are shown in Figure 2. This means that the labels in the second graph are obtained from the first by replacing each label $\lambda(x)$ with $|V| + |E| + 1 - \lambda(x)$. For regular graphs, including cycles, this will also be a magic labeling. Indeed, if we let h be the magic constant for λ and d be the degree of a regular graph, it is straightforward to show that

$(d+1)(|V|+|E|+1)-h$ would be the magic constant for the new labeling. To get an idea for the range of possible magic constants for C_n, assume λ is a VMTL and consider the sum $\Sigma_{v\in V}wt_\lambda(v)$. Note that the edge label $\lambda(u,v)$ is a summand in the defining equation for both $wt_\lambda(u)$ and $wt_\lambda(v)$. Hence $|V|h_\lambda = S_v + 2S_e$, where S_v is the sum of all vertex labels and S_e is the sum of all edge labels. Thus to get a large (respectively small) magic constant one uses large (respectively small) labels on the edges. For a cycle with n vertices,

$$S_v + 2S_e \leq (1+2+\cdots+n) + 2\big((n+1)+(n+2)+\cdots+2n\big)$$

and so,

$$|V|h_\lambda = nh_\lambda \leq \frac{n(n+1)}{2} + n(3n+1)$$

i.e.,

$$h_\lambda \leq \frac{7n+3}{2}.$$

A similar argument shows that

$$h_\lambda \geq \frac{5n+3}{2}.$$

The integral values of h_λ within this range are called the *feasible magic constants* for VMTL's of the graph C_n. The spectrum of the graph is the set of integers which can be realized as the magic constant of some VMTL. It is known [3] that C_5 has a spectrum of $\{14, 16, 17, 19\}$. The two feasible values 15 and 18 are missing. It is conjectured [3] that for all other cycles C_n, $n \neq 5$, the set of feasible values coincides with the spectrum. They verified the conjecture by computer up to $n = 10$. The goal of this paper is to provide a new tool for attacking the spectrum problem for cycles, as well as providing new direct evidence for this conjecture. Our methods are similar to some of those used in [6], where the spectrum problem for odd complete graphs was solved. For those graphs, the spectrum does indeed coincide with the set of feasible values [6]. In that paper, we found it convenient to find magic labelings using consecutive integers starting at 0, rather than 1. Then we obtained a VMTL by adding 1 to each label. This did not interfere with the magic property because of the regular property. We will do the same in this paper (c.f. Propostion 2.2).

Our main result (Corollary 3.1) is the construction of many VMTL's for the graph C_{nk}, (n odd) having different magic constants, in terms of a VMTL for C_k. Since VMTL's for all cycles have already been constructed [4] and [3], our construction automatically uses these to produce new ones for C_{nk} provided that $n \geq 3$ is odd and $k \geq 3$ is any integer. Our constructions result in VMTL's with different magic constants then

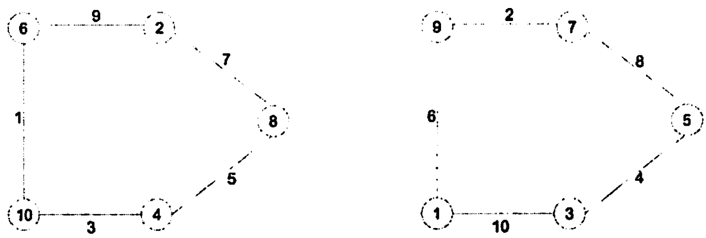

Figure 1: An edge-magic labeling (left) and a corresponding VMTL (right).

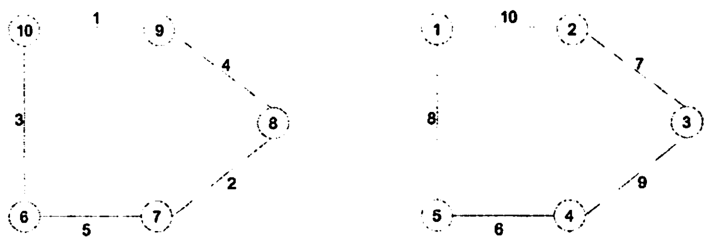

Figure 2: A VMTL with magic constant $h = 14$ (left) and a VMTL with magic constant $h = 19$ (right).

those found in both [4] and [3]. In section 2, we will provide the framework for our approach. It is interesting that from this point of view, the two different constructions in [4] and [3] for C_n, n odd, appear to be closely related. In section 4 we show how these methods can be used to almost immediately obtain VMTL's for most of the magic constants in the spectrum for C_9. An excellent exposition for both vertex-magic and edge-magic total labelings is [8]. The definition of VMTL, duality, feasible magic constants and other important ideas were introduced in [5]. Lists of graphs known to have edge-magic total labelings, and VMTL's can be found in [2].

2 Preliminaries

For the rest of this paper, m will denote a positive integer and $n = 2m + 1$ will be an odd integer. We will follow the methods of [6]. For the reader's convenience we will summarize some of the definitions and propositions here. The idea is that we use "magic surjections" to build VMTL's. These labelings can be easier to find. The proofs are easy exercises.

Definition 2.1 *[6] Let $\alpha : V \cup E \longrightarrow \{0, 1, 2, \ldots, (s-1)\}$ be a surjective map, with the magic property, i.e., $wt_\alpha(v)$ does not depend on the choice of vertex $v \in V$. Then we say that α is a magic s-surjection and $wt_\alpha(v)$ is called the magic constant of α, denoted by h_α.*

Definition 2.2 *[6] Let G be a graph with $|V \cup E| = st$. Let α be a magic s-surjection and let β be a magic t-surjection. We say that α and β are* compatible *if for each $q, r \in \mathbb{Z}$, $0 \le q \le s-1$, $0 \le r \le t-1$, we have $|\alpha^{-1}(q) \cap \beta^{-1}(r)| = 1$.*

Compatibility ensures that we can build VMTL's from the magic surjections, using the next two propositions.

Proposition 2.1 *[6] Let G be a graph with $|V \cup E| = st$. Let α be a magic s-surjection with magic constant h_α and let β be a magic t-surjection with magic constant h_β. Assume that α and β are compatible. Then*

1. *$t\alpha + \beta$ is a magic st-surjection with magic constant $th_\alpha + h_\beta$,*
2. *$s\beta + \alpha$ is a magic st-surjection with magic constant $sh_\beta + h_\alpha$.*

An example of compatible magic surjections for C_5 is shown in Figure 3.

Letting $s = 5$ and $t = 2$, we use each part of the above proposition to obtain a different magic 10-surjection. These can be used to obtain VMTL's using the next proposition.

Proposition 2.2 *[6] Let G be a regular graph of degree d with $|V \cup E| = l$. Then, λ is a magic l-surjection with magic constant h if and only if $\lambda + 1$ is a vertex-magic total labeling with magic constant $h + d + 1$.*

Notation 2.1 *Let $v_0, v_1, \ldots, v_{2m}$ be the vertices of C_n. In order that we may write each edge in the form (v_j, v_{j+1}) we also set $v_{2m+1} = v_0$ and $v_{-1} = v_{2m}$. For the rest of this paper, α will be a magic n-surjection of C_n defined by the rule:*

1. *$\alpha(v_j) = j$, for $0 \le j \le 2m$,*
2. *$\alpha(v_{2i-1}, v_{2i}) = m - i$, for $0 \le i \le m$,*
3. *$\alpha(v_{2i}, v_{2i+1}) = 2m - i$, for $0 \le i \le m$.*

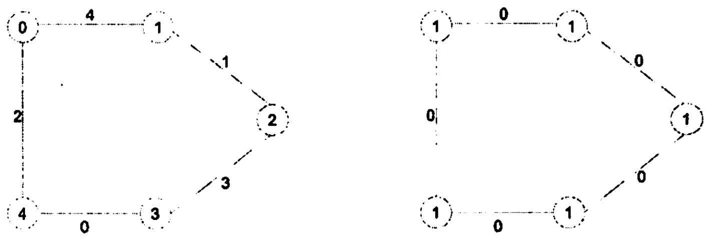

Figure 3: α for the graph C_5 (left) and a compatible 2-surjection (right).

We check that $\alpha(v_{-1}, v_0) = \alpha(v_{2m}, v_{2m+1}) = m$ (same edge!), and we see immediately that for $0 \le i \le m$,

$$\begin{aligned} wt_\alpha(v_{2i}) &= \alpha(v_{2i-1}, v_{2i}) + \alpha(v_{2i}) + \alpha(v_{2i}, v_{2i+1}) \\ &= (m-i) + 2i + (2m-i) = 3m. \end{aligned}$$

Furthermore, for $0 \le i \le m-1$ we have

$$\begin{aligned} wt_\alpha(v_{2i+1}) &= \alpha(v_{2i}, v_{2i+1}) + \alpha(v_{2i+1}) + \alpha(v_{2(i+1)-1}, v_{2(i+1)}) \\ &= (2m-i) + (2i+1) + \big(m-(i+1)\big) = 3m. \end{aligned}$$

Therefore α is indeed magic, as we claimed. We summarize a few properties of α as follows:

Proposition 2.3 *For the magic n-surjection α of C_n defined above:*

1. *the magic constant of α is $3m$,*
2. *each of the numbers $0, 1, 2, \ldots, 2m$ is the label of precisely one vertex of C_n,*
3. *each of the numbers $0, 1, 2, \ldots, 2m$ is the label of precisely one edge of C_n.*

The example of $n = 5$ is shown in Figure 3.

Proposition 2.4 *Let β be defined by $\beta(v_i) = 1$, for $0 \le i \le 2m$ and $\beta(v_i, v_{i+1}) = 0$, $0 \le i \le 2m$. Then β is a magic 2-surjection for C_n which is compatible with α.*

Proof idea. β gives the same label to all vertices (respectively edges). To ensure compatibility with α, we must check that all vertices (respectively edges) are given *different* labels by α. But this is the content of Proposition 2.3.

If one uses Proposition 2.1*(1)* together with Proposition 2.2, one obtains the construction in [4]. On the other hand, if one uses Proposition 2.1*(2)* together with Proposition 2.2, one obtains the construction in [3].

3 Inflation of Cycles

Let k be a positive integer $k \geq 3$, and assume that we are given a VMTL of C_k. By Proposition 2.2, we can subtract 1 from each label and obtain a magic $2k$-surjection which we will call β whose magic constant will be denoted by h. In this section we will "inflate" α to obtain a magic n-surjection α^* for C_{nk} with magic constant $3m$, and then "inflate" β to obtain a $2k$-surjection β^* for C_{nk} whose magic constant is h. We will prove that α^* and β^* are compatible. Once we have done this, Proposition 2.1 immediately yields the following.

Theorem 3.1 *Assume that C_k has a magic $2k$-surjection with magic constant h. Then*

1. *C_{nk} has a magic $2nk$-surjection with magic constant $nh + 3m$,*

2. *C_{nk} has a magic $2nk$-surjection with magic constant $6km + h$.*

A few applications of Proposition 2.2 give us

Corollary 3.1 *Assume that C_k has a VMTL with magic constant g. Then*

1. *C_{nk} has a VMTL with magic constant $ng - 3m$,*

2. *C_{nk} has a VMTL with magic constant $6km + g$.*

Now we will define α^* and β^*, and prove that they are compatible by looking at all things given the same label by β^* and showing that α^* gives them different labels. An example with $n = 5$ and $k = 3$ is shown in Figure 4 and Figure 5, for the reader's convenience. Let $v_0, v_1, \dots, v_{2m}$ be the vertices of C_n, with edge set (v_i, v_{i+1}) $0 \leq i \leq 2m$, and for convenience we set $v_{2m+1} = v_0$ as in section 2. Let $u_0, u_1, \dots, u_{k-1}$ be the vertices of C_k, with edge set (u_i, u_{i+1}) $0 \leq i \leq k-1$ and set $u_k = u_0$ and $\mu_{-1} = \mu_{k-1}$. Finally let $w_0, w_1, \dots, w_{nk-1}$ be the vertices of C_{nk} with edges (w_i, w_{i+1}), and for convenience we set $w_{nk+j} = w_j$, for $j = 0, 1, \dots, n-1$.

Let x be an integer, $0 \le x \le nk-1$. By the division algorithm, x can be written uniquely in the form $x = qn + r$, $0 \le q \le k-1$, $0 \le r \le n-1$. We define α^* as follows. $\alpha^*(w_x) = \alpha(v_r)$ and $\alpha^*(w_x, w_{x+1}) = \alpha(v_r, v_{r+1})$. Note that α^* inherits the magic property from α, and its magic constant is also $3m$. Now define $\beta^*(w_x) = \beta(u_q)$. Note that the vertices $w_{qn}, w_{qn+1}, \ldots, w_{qn+(n-1)}$, which are given the same label by β^* are given different labels by α^* (by Proposition 2.3(2)). Finally, set

$$\beta^*(w_x, w_{x+1}) = \begin{cases} \beta(u_q, u_{q+1}) & \text{if } r \text{ is even,} \\ \beta(u_{q-1}, u_q) & \text{if } r \text{ is odd.} \end{cases}$$

In particular $\beta^*(w_{qn+(n-1)}, w_{(q+1)n}) = \beta(u_q, u_{q+1})$. We see that for each vertex w_x, the summands for the β^*-weight are, in some order $\beta(u_{q-1}, u_q)$, $\beta(u_q)$ and $\beta(u_q, u_{q+1})$. Hence β^* is magic with magic constant h. We already observed that α^* gives n different labels to the n vertices with the same β^*-label. To complete the proof that α^* and β^* are compatible, we need to identify the n edges which are given the same label by β^* and show that α^* gives them different labels. From the "r is even" part of the definition of β^*, we see that the $m+1$ edges

$$(w_{qn}, w_{qn+1}), (w_{qn+2}, w_{qn+3}), \ldots, (w_{qn+2m}, w_{(q+1)n})$$

all have the same β^*-label, namely $\beta(u_q, u_{q+1})$. However, β^* also labels the m edges

$$(w_{(q+1)n+1}, w_{(q+1)n+2}), (w_{(q+1)n+3}, w_{(q+1)n+4}), \\ \ldots, (w_{(q+1)n+n-2}, w_{(q+1)n+n-1})$$

with the very same label $\beta(u_q, u_{q+1})$ due to the second part of the definition of β^* (where r is odd). Therefore, we must show that α^* gives n different labels to these n edges. By the definition of α^*, these labels are:

$$\alpha(v_0, v_1), \alpha(v_2, v_3), \ldots, \alpha(v_{n-1}, v_n), \alpha(v_1, v_2), \alpha(v_3, v_4), \ldots, \alpha(v_{n-2}, v_{n-1}).$$

These labels are indeed all different by Proposition 2.3(3). This proves compatibility of α^* and β^*, and the result follows.

4 Concluding Remarks

So far we have used $n = 5$ as a favorite example in our figures. However, Proposition 2.1 is particularly effective for the case $n = 3$. While it is an easy exercise to find all 6-surjections (and hence VMTL's) for C_3, we point out that all four of them are consequences of Proposition 2.1, using α

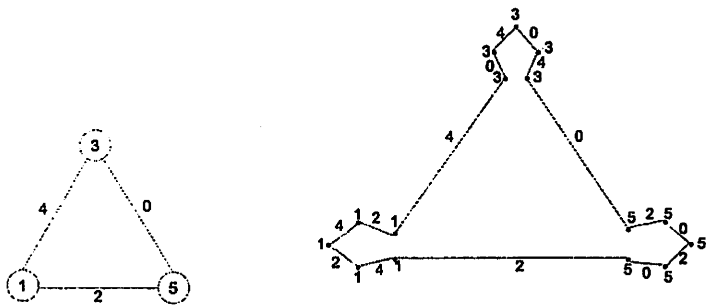

Figure 4: A 6-surjection, β for C_3 (left) and the corresponding 6-surjection β^* for C_{15} (right).

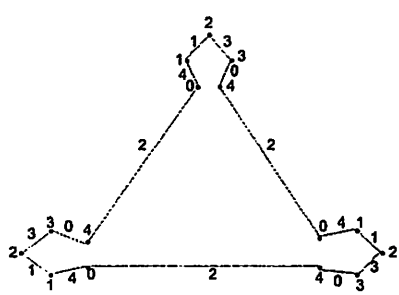

Figure 5: α^* for the case $n = 5$, $k = 3$.

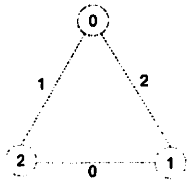

Figure 6: The 3-surjection α for C_3.

Figure 7: Two 2-surjections for C_3.

(Figure 6) and the two 2-surjections shown in Figure 7. For each resulting choice of 6-surjection, we can immediately obtain magic 18-surjections for C_9 by the method of inflation (section 3). This results in a VMTL with each of the following magic constants: 24, 27, 28, 29, 30 and 33. There are only 4 other feasible magic constants that require another method. Similarly it is not too difficult to find VMTL's for each feasible magic constant of C_4. Inflation then immediately provides VMTL's for C_{12} with each of the following magic constants: 33, 36, 37, 38, 39, and 42. There are only 6 other feasible magic constants and two of these, one with magic constant 32 and its dual with magic constant 43, are done in [3]. It also works very nicely for $n = 3$, $k = 7$, as we can obtain VMTL's for with magic constants 54, 57, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 72, and 75, leaving only 8 others to a possible other method. We would welcome another method that will fill in the gaps.

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