Recounting Relations for Set Partitions with Restrictions

Mark Shattuck
Department of Mathematics
University of Tennessee
Knoxville, TN 37996-1300, USA
shattuck@math.utk.edu

Abstract

We provide combinatorial arguments of some relations between classical Stirling numbers of the second kind and two refinements of these numbers gotten by introducing restrictions to the distances among the elements in each block of a finite set partition.

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For a positive integer m, let M be the finite set given by $M := \{x_1, x_2, \ldots, x_m\}$. Then the number of partitions of M into n nonempty blocks is the classical Stirling number of the second kind $S(m,n)$. For two elements x_i and x_j from M, define their distance by $|i-j|$, the distance between their subscripts i and j. As in [1, 2], introduce restrictions to the distances among the elements in each block of M-partitions by $|i-j| > r$ and $|i-j| \neq r$ for any two elements x_i and x_j in the same block. Denote the corresponding refinements of $S(m,n)$ by $S_r(m,n)$ and $T_r(m,n)$, respectively; note that $S_r(m,n)$ and $T_r(m,n)$ both reduce to $S(m,n)$ when $r=0$. Chu and Wei [1] provide algebraic arguments for the following relations which express the $S_r(m,n)$ and $T_r(m,n)$ in terms of the ordinary Stirling number $S(m,n)$ and ask for combinatorial proofs:

$$S_r(m,n) = S(m-r, n-r), \tag{1.1}$$

$$T_r(m,n) = \sum_{k=1}^{r} \binom{r-1}{k-1} S(m-k, n-1), \tag{1.2}$$

and

$$T_r(m,n) = \sum_{k=r}^{m} (-1)^{m-k} \binom{m-r}{k-r} S(k,n), \tag{1.3}$$

where $m \geqslant n \geqslant r \geqslant 1$.

In this note, we provide the requested combinatorial arguments for (1)–(3). We'll make use of the following notation. If $m \geqslant n$ are positive integers, then let $[m]$ denote the finite set $\{1,2,...,m\}$, $[n,m]$ denote the set $\{n,n+1,...,m\}$, and $\Pi(m,n)$ denote the set of partitions of $[m]$ into n nonempty sets. Note that the set $\Pi(m,n)$ has cardinality $S(m,n)$. Let $\Pi_r(m,n)$ and $\Pi^r(m,n)$ consist of those members of $\Pi(m,n)$ enumerated by the $S_r(m,n)$ and $T_r(m,n)$, respectively. For instance, the partition $\{1,2,7\}$, $\{3,4\}$, $\{5,6\}$ belongs to $\Pi^3(7,3)$ (but not to $\Pi_3(7,3)$), while $\{1,5\}$, $\{2,6\}$, $\{3,7\}$, $\{4\}$ belongs to $\Pi_3(7,4)$. In what follows, set partitions will be represented with blocks arranged from left to right in increasing order according to the largest element in a block. For example, the partition $\{2,4,6\}$, $\{7,8\}$, $\{1,3\}$, $\{5\}$ belonging to $\Pi(8,4)$ would be written as $\{1,3\}$, $\{5\}$, $\{2,4,6\}$, $\{7,8\}$. We first prove (1).

A bijection for (1):

To avoid trivialities, assume $m > n > r$. We'll describe an $(m-r)$-step iterative sorting procedure for transforming members of $\Pi(m-r,n-r)$ into members of $\Pi_r(m,n)$. Suppose $\lambda \in \Pi(m-r,n-r)$. Append directly to the right of λ the r singleton blocks $\{m-r+1\},\{m-r+2\},...,\{m\}$.

Given $i \in [m-r]$, we'll either move i to one of the final r appended blocks or leave i in its present block within λ. We'll first consider $i = m-r$ and then successively consider $i = m-r-1, i = m-r-2, ..., i = 1$. Note that $i = m-r$ must not be moved since the resulting partition of $[m]$ is to belong to $\Pi_r(m,n)$.

We now describe how to make the subsequent decisions regarding placement. Given $i \in [m-r-1]$, let S_i denote the set $[i+1,i+r]$. Assume that all of the decisions regarding placement of each of the numbers $i+1, i+2, ..., m-r$ have already been made. Call a member j of S_i "unmoved" if $j \leqslant m-r$ and j was not moved to one of the r appended blocks when one decided on placement of j. Suppose that S_i contains exactly k unmoved members, each of these clearly going in separate blocks (since we are to create a member of $\Pi_r(m,n)$). Then exactly $r-k$ members of S_i are currently contained in the final r appended blocks (each necessarily in different blocks) and so $r-(r-k) = k$ of the final r blocks do not currently contain a member of S_i, i.e., i may be moved to any one of these k blocks.

Suppose first that i shares a block with an unmoved member of S_i, which implies $k \geqslant 1$. In this case, move i to one of the k blocks amongst the final r blocks not currently containing a member of S_i. Do so in such a way that if i shares a block with the ℓ^{th} smallest unmoved member of S_i, then i is moved to the ℓ^{th} leftmost block amongst the final r blocks not currently containing a member of S_i, $1 \leqslant \ell \leqslant k$. On the other hand, if i does not share a block with an unmoved member of S_i, then leave i in its present block.

Let λ' denote the member of $\Pi_r(m,n)$ resulting after all the elements of $[m-r]$ have been sorted as described above. To illustrate the mapping $\lambda \mapsto \lambda'$, let $m=16$, $n=6$, $r=2$ and $\lambda = \{3,4\}, \{1,5,7,11\}, \{6,8,10,13\}, \{2,9,12,14\} \in \Pi(14,4)$. First append the singleton blocks $\{15\}$, $\{16\}$. The numbers 12, 8, and 5 would each be moved to the block containing 16 since they each share a block with the second smallest unmoved member of their respective S_i's. Since both 4 and 5 are unmoved and since 3 shares a block with 4, the number 3 would be moved to the block containing 15. The number 6 then would not be moved since 8 has already been moved. Thus, $\lambda' = \{4\}, \{1,7,11\}, \{6,10,13\}, \{2,9,14\}, \{3,15\}, \{5,8,12,16\} \in \Pi_2(16,6)$.

The mapping $\lambda \mapsto \lambda'$ may be reversed as follows. Let $\alpha \in \Pi_r(m,n)$. Starting with $i=1$, decide whether or not to move individual members i of $[m-r]$ from their present positions within α. If i does not share a block with a member of $[m-r+1,m]$ within α, then do not move i. On the other hand, suppose i shares a block with a member of $[m-r+1,m]$ and that i belongs to the block containing the ℓ^{th} smallest member of $[m-r+1,m]$ which does not share a block with a member of S_i. In this case, move i to the block containing the ℓ^{th} smallest member of S_i not sharing a block with a member of $[m-r+1,m]$. The resulting member of $\Pi(m,n)$ will end with the r singleton blocks $\{m-r+1\}, \{m-r+2\}, \ldots, \{m\}$, which we delete. Upon tracking the movement of individual elements, one can verify that the two procedures described are inverse operations.

A bijection for (2):

We'll show that the summation in (2) counts members of $\Pi^r(m,n)$ according to the number of elements of the set $[m-r+1,m-1]$ belonging to the same block as the element m. Note first that the summation in (2) clearly counts all ordered pairs of the form (T,λ), where $T \subseteq [m-r+1,m-1]$ and λ is a partition of the set $[m-1]-T$ with $n-1$ nonempty blocks, according to the cardinality, $k-1$, of the set T, where $1 \leqslant k \leqslant r$. Let $\alpha = (T,\lambda)$ be such an ordered pair with $|T| = k-1$. We'll convert α into a member of $\Pi^r(m,n)$ for which there are exactly $k-1$ elements of $[m-r+1,m-1]$ in the same block as the element m.

To do so, consider within each block of λ all maximal arithmetic progressions of the form $j, j+r, j+2r, \ldots, j+dr$. From each such sequence, extract the terms $j+(d-1)r, j+(d-3)r, \ldots$ and place all such terms in a single additional block. To this block, add the element m as well as the members of T. The resulting partition α' belongs to $\Pi^r(m,n)$.

To illustrate the mapping $\alpha \mapsto \alpha'$, let $m=18$, $n=5$, $r=5$ and $k=2$. Let $\alpha = (T,\lambda)$, where $T = \{16\} \subseteq \{14,15,16,17\}$ and $\lambda = \{1,3\}$, $\{8,13\}$, $\{2,5,6,10,11,14,15\}, \{4,7,9,12,17\}$. Then $\alpha' = \{1,3\}, \{13\}, \{2,5,11,14,15\}, \{7,9,17\}, \{4,6,8,10,12,16,18\} \in \Pi^5(18,5)$.

The mapping $\alpha \mapsto \alpha'$ may be reversed as follows. Let $\beta \in \Pi^r(m,n)$. First remove any members of the set $[m-r+1, m-1]$ belonging to the same block of β as the element m. Let i denote one of the remaining members of $[m-1]$ in this block, if there are any. Then $i \leqslant m-r$ and $i+r$ does not belong to this block. Add i to the block of β which contains $i+r$. Doing this for each i generates a partition of an $(m-k)$-set into $n-1$ blocks, where k is the number of elements of $[m-r+1, m]$ in the same block of β as the element m, $1 \leqslant k \leqslant r$. By tracking the movement of individual members of $[m]$, one can verify that this operation is the inverse of the prior one.

A sign-changing involution for (3):

First rewrite the binomial coefficient appearing in the summation in (3) as $\binom{m-r}{m-k}$. Note that $\binom{m-r}{m-k} S(k,n)$ counts pairs (T, λ), where T is a subset of $[m-r]$ of cardinality $m-k$ and λ is a partition of $[m]-T$ into n nonempty blocks. Each pair (T,λ) contributes ± 1 to the summation in (3) depending on the parity of $|T|$.

We match pairs of opposite sign as follows. Let $i \in [m-r]$ be the largest integer such that $i \in T$ or i and $i+r$ occur in the same block of λ.

(i) If $i \in T$ (and hence $i+r$ occurs in λ), then remove i from T and place it in the block of λ that contains $i+r$.

(ii) If i and $i+r$ occur in the same block of λ, then remove i from the block and place it in T.

When i exists, the process defines an involution which changes the size of T by ± 1. No i exists when T is the empty set and $\lambda \in \Pi^r(m,n)$. So all exceptions are positive and there are $T_r(m,n)$ of them.

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References

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