Embeddings of resolvable group divisible designs with block size 3 and for all λ*

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Abstract

This paper investigates the embedding problem for resolvable group divisible designs with block size 3. The necessary and sufficient conditions are determined for all $\lambda \geq 1$.

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1 Introduction

Let λ and v be positive integers, K and M be sets of positive integers. A *group divisible design* (GDD) of order v and index λ, denoted by GD($K, \lambda, M; v$), is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following conditions.
1. X is a v-set of *points*,
2. $\mathcal{G}$ is a set of subsets (called *groups*) of X which forms a partition of X, and $|G| \in M$ for each $G \in \mathcal{G}$,
3. $\mathcal{B}$ is a collection of subsets (called *blocks*) of X such that $|B| \in K$ for each $B \in \mathcal{B}$, and each pair of points from distinct groups occurs in exactly λ blocks,
4. $|G \cap B| \leq 1$ for each $G \in \mathcal{G}$ and each $B \in \mathcal{B}$.

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A GD$(K,\lambda,M;v)$ is also called a (K,λ)-GDD of type T where $T = \{|G| : G \in \mathcal{G}\}$ is a multiset. T is called the *group-type* or *type* of the GDD. We usually use an "exponential" notation to describe group-type: a type $1^i 2^j 3^k \cdots$ means i occurrences of 1, j occurrences of 2, etc.

When $\lambda = 1$, we simply write GD$(K,M;v)$ or K-GDD of type T. When $K = \{k\}$, or $M = \{m\}$, we simply write k for $\{k\}$, or m for $\{m\}$.

A GD$(k,\lambda,m;km)$ is called a *transversal design* and denoted by TD$(k,\lambda; m)$. When $\lambda = 1$, we simply write TD(k,m).

Let $(X,\mathcal{G},\mathcal{B})$ be a GD$(K,\lambda,M;v)$, $\mathcal{P} \subset \mathcal{B}$. If $\mathcal{P}$ forms a partition of X, then $\mathcal{P}$ is called a *parallel class*. If $\mathcal{B}$ can be partitioned into parallel classes, then the GDD is called *resolvable*, and denoted by RGD$(K,\lambda,M;v)$. A resolvable TD$(k,\lambda;m)$ is denoted by RTD$(k,\lambda;m)$. An RGD$(k,\lambda,1;v)$ is denoted by RB$(k,\lambda;v)$.

The existence problem for resolvable group divisible designs with block size three has been completely solved.

Theorem 1.1 [1, 11, 12, 13] *There exists an* RGD$(3,\lambda,g;v)$ *if and only if* $v \geq 3g$, $v \equiv 0 \pmod 3$, $v \equiv 0 \pmod g$, $\lambda(v-g) \equiv 0 \pmod 2$, *and* $(\lambda,g,v) \neq (1,2,12),(1,6,18),(2j+1,2,6),(4j+2,1,6), j \geq 0$.

Let $(X_1,\mathcal{G}_1,\mathcal{B}_1)$ be an RGD$(K,\lambda,M;v)$, and $(X_2,\mathcal{G}_2,\mathcal{B}_2)$ be an RGD$(K, \lambda,M;u)$. If $X_1 \subset X_2$, $\mathcal{G}_1 \subset \mathcal{G}_2$, $\mathcal{B}_1$ is a subcollection of $\mathcal{B}_2$, and each parallel class of $\mathcal{B}_1$ is a part of some parallel class of $\mathcal{B}_2$, then we say $(X_1,\mathcal{G}_1,\mathcal{B}_1)$ is embedded in $(X_2,\mathcal{G}_2,\mathcal{B}_2)$, or $(X_2,\mathcal{G}_2,\mathcal{B}_2)$ contains $(X_1,\mathcal{G}_1,\mathcal{B}_1)$ as a subdesign.

Several authors have studied the embedding problem for resolvable group divisible designs with block size 3. We summarize these known results in the following theorem.

Theorem 1.2 [3, 4, 5, 15, 14, 16, 17, 18, 19] (1) *An* RGD$(3,g;v)$ *can be embedded in an* RGD$(3,g;u)$ *if and only if* $u-g \equiv v-g \equiv 0 \pmod 2$, $u \equiv v \equiv 0 \pmod 3$, $u \equiv v \equiv 0 \pmod g$, $v \geq 3g$, $u \geq 3v$, *and* $(g,v) \neq (2,6),(2,12),(6,18)$.

(2) *An* RB$(3,\lambda;v)$ *can be embedded in an* RB$(3,\lambda;u)$ *if and only if* $\lambda(u-1) \equiv \lambda(v-1) \equiv 0 \pmod 2$, $u \equiv v \equiv 0 \pmod 3$, $u \geq 3v$, *and* $(\lambda,v) \neq (4j+2,6), j \geq 0$.

In this paper we will study the remaining cases for $\lambda > 1$ and completely solve the problem.

2 Recursive Constructions

An *incomplete group divisible design* (IGDD) is a quadruple $(X,H,\mathcal{G},\mathcal{B})$ which satisfies the following conditions.

1. X is the point set, $H \subset X$. H is called a *hole*,
2. $\mathcal{G}$ is a set of subsets (called *groups*) of X which forms a partition of X,
3. $\mathcal{B}$ is a collection of subsets (called *blocks*) of X such that each pair of points from distinct groups containing at least one member in $X \setminus H$ occurs in exactly λ blocks,
4. $|G \cap B| \leq 1$ for each $G \in \mathcal{G}$ and each $B \in \mathcal{B}$,
5. no block contains two members of H.

If $|B| \in K$ for each $B \in \mathcal{B}$, then an IGDD is called a (K, λ)-IGDD of type T where K is a given set of positive integers and $T = \{(|G|, |G \cap H|) : G \in \mathcal{G}\}$. T is called the *type* of the IGDD. As with GDDs, we use an "exponential" notation to describe the type. When $\lambda = 1$, we simply write K-IGDD. When $K = \{k\}$, we simply write k for $\{k\}$.

A (K, λ)-IGDD is said to be *resolvable* and is denoted by (K, λ)-IRGDD if its blocks can be partitioned into parallel classes and *holey parallel classes*, the latter partitioning $X \setminus H$.

In this paper, we will only use IRGDDs of type $(g, 0)^{m-n}(g, g)^n$ where $g > 0$ and $m > n > 0$. So, we will use $g^{(m,n)}$ to denote the types of such IRGDDs. It is obvious that a (k, λ)-IRGDD of type $g^{(m,n)}$ contains $\lambda g(m-n)/(k-1)$ parallel classes and $\lambda g(n-1)/(k-1)$ holey parallel classes. We note that a (k, λ)-IRGDD of type $g^{(m,1)}$ is just a (k, λ)-RGDD of type g^m.

It is easy to show that the necessary conditions for the existence of a $(3, \lambda)$-IRGDD of type $g^{(u/g, v/g)}$ are $u \geq 3v$, $u \equiv 0 \pmod 3$, $u \equiv v \equiv 0 \pmod g$, $\lambda(u-g) \equiv \lambda(v-g) \equiv 0 \pmod 2$, and any of the following.

1. $v = g$,
2. $v = 2g$, $\lambda g \equiv 0 \pmod 2$, and $g \equiv 0 \pmod 3$,
3. $v \geq 3g$, and $v \equiv 0 \pmod 3$.

The following two lemmas are obvious but important in solving the embedding problem.

Lemma 2.1 *Suppose there is a* (k, λ)-IRGDD *of type* $g^{(u/g, v/g)}$ *and an* RGD$(k, \lambda, g; v)$. *Then an* RGD$(k, \lambda, g; v)$ *can be embedded in an* RGD$(k, \lambda, g; u)$.

Lemma 2.2 *Suppose there exists a* (k, λ)-IRGDD *of type* $g^{(u/g, v/g)}$ *and an* RTD(k, m). *Then there exists a* (k, λ)-IRGDD *of type* $(mg)^{(u/g, v/g)}$.

A *frame* is a GD$(K, \lambda, M; v)$ $(X, \mathcal{G}, \mathcal{B})$, with the property that $\mathcal{B}$ can be partitioned into holey parallel classes, each of which forms a partition of $X \setminus G$, for some $G \in \mathcal{G}$. It is denoted by (K, λ)-frame of type T where T is the type of the underlying GDD.

The existence of $(3, \lambda)$-frame of type g^u has been determined.

Theorem 2.3 [1, 10, 18] *There exists a* $(3,\lambda)$*-frame of type* g^u *if and only if* $u \geq 4$, $g(u-1) \equiv 0 \pmod 3$, $\lambda g(u-1) \equiv 0 \pmod 2$, *and* $\lambda g u \equiv 0 \pmod 2$.

An *incomplete* (K,λ)*-frame* is a (K,λ)-IGDD $(X,H,\mathcal{G},\mathcal{B})$ where the block set $\mathcal{B}$ can be partitioned into holey parallel classes, each of which forms a partition of $X \setminus G$ for some $G \in \mathcal{G}$, or a partition of $X \setminus (G \cup H)$ for some $G \in \mathcal{G}$. When $K = \{k\}$ there are exactly $\frac{\lambda|G\setminus H|}{k-1}$ holey parallel classes that partition $X \setminus G$ and $\frac{\lambda|G\cap H|}{k-1}$ holey parallel classes that partition $X \setminus (G \cup H)$.

The following construction is called the fundamental incomplete frame construction (FIFC, see e.g. [6]).

Construction 2.4 (FIFC) *Let* $(X,H,\mathcal{G},\mathcal{B})$ *be a* K-IGDD *with index one and let* $w : X \to Z^+ \cup \{0\}$ *be a weight function on* X. *Suppose that for each block* $B \in \mathcal{B}$, *there exists a* (k,λ)*-frame of type* $\{w(x) : x \in B\}$. *Then there exists an incomplete* (k,λ)*-frame of type* $\{(\sum_{x\in G} w(x), \sum_{x\in G\cap H} w(x)) : G \in \mathcal{G}\}$.

Setting $H = \emptyset$ in Construction 2.4 gives Stinson's fundamental frame construction (SFFC, [18]).

Construction 2.5 (SFFC) *Let* $(X,\mathcal{G},\mathcal{B})$ *be a group divisible design with index one and let* $w : X \to Z^+ \cup \{0\}$ *be a weight function on* X. *Suppose for each block* $B \in \mathcal{B}$ *there exists a* (k,λ)*-frame of type* $\{w(x) : x \in B\}$. *Then there exists a* (k,λ)*-frame of type* $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.

The following "filling in holes" construction is a powerful tool in constructing IRGDDs (see [6]).

Construction 2.6 *Suppose there is a* (k,λ)*-frame of type* $T = \{t_i : i = 1,2,\ldots,n\}$. *Let* $t|t_i$ *and* $b > 0$. *Suppose there also exists a* (k,λ)-IRGDD *of type* $t^{(t_i/t+b,b)}$ *for* $i = 1,2,\ldots,n-1$, *then there exists a* (k,λ)-IRGDD *of type* $t^{(u/t+b,t_n/t+b)}$ *where* $u = \sum_{i=1}^{n} t_i$. *Furthermore, if there exists a* (k,λ)-IRGDD *of type* $t^{(t_n/t+b,b)}$, *then there exists a* (k,λ)-IRGDD *of type* $t^{(u/t+b,b)}$.

The following construction is a generalization of the construction used in [8, Lemma 5.4].

Construction 2.7 *Suppose there is an incomplete* (k,λ)*-frame of type* $t_1^x(t_2,t_2)^1(t_3,t_4)^1$. *Let* $g|t_i$ *and* $b > 0$. *Suppose there also exists a* (k,λ)*-IRGDD of type* $g^{(t_1/g+b,b)}$ *and a* (k,λ)*-IRGDD of type* $g^{(t_2/g+t_4/g+b,t_4/g+b)}$, *then there exists a* (k,λ)*-IRGDD of type* $g^{(u/g+b,t_3/g+b)}$ *where* $u = x \cdot t_1 + t_2 + t_3$.

Proof. Let $(X, H, \{G_1, G_2, \ldots, G_x, B, C\}, \mathcal{A})$ be an incomplete (k,λ)-frame of type $t_1^x(t_2,t_2)^1(t_3,t_4)^1$, where $|G_i| = t_1$ for $i = 1,2,\ldots,x$, $|B| = t_2$, $|C| = t_3$, $C = C' \cup C''$, $|C''| = t_4$, and $H = B \cup C''$. There are exactly $\frac{\lambda t_1}{k-1}$ holey parallel classes, denoted by $\mathcal{P}_{G_i,j}$, $j = 1,2,\ldots,\frac{\lambda t_1}{k-1}$, which partition $X \setminus G_i$ for $i = 1,2,\ldots,x$. There are $\frac{\lambda t_2}{k-1}$ holey parallel classes, denoted by $\mathcal{P}_{B,j}$, $j = 1,2,\ldots,\frac{\lambda t_2}{k-1}$, which partition $X \setminus (B \cup H)$. There are $\frac{\lambda(t_3-t_4)}{k-1}$ holey parallel classes, denoted by $\mathcal{P}_{C',j}$, $j = 1,2,\ldots,\frac{\lambda(t_3-t_4)}{k-1}$, which partition $X \setminus C$, and $\frac{\lambda t_4}{k-1}$ holey parallel classes, denoted by $\mathcal{P}_{C'',j}$, $j = 1,2,\ldots,\frac{\lambda t_4}{k-1}$, which partition $X \setminus (C \cup H)$.

Let Y be a set of size gb and $X \cap Y = \emptyset$.

Let $\mathcal{D}_i$ be a (k,λ)-IRGDD of type $g^{(t_1/g+b,b)}$ on $(G_i \cup Y, Y)$ for $i = 1,2,\ldots,x$, which has $\frac{\lambda t_1}{k-1}$ parallel classes, denoted by $\mathcal{Q}_{i,j}$, $j = 1,2,\ldots,\frac{\lambda t_1}{k-1}$, and $\frac{\lambda(gb-g)}{k-1}$ holey parallel classes, denoted by $\mathcal{Q}'_{i,j}$, $j = 1,2,\ldots,\frac{\lambda(gb-g)}{k-1}$.

Let $\mathcal{D}'$ be a (k,λ)-IRGDD of type $g^{(t_2/g+t_4/g+b,t_4/g+b)}$ on $(H \cup Y, C'' \cup Y)$, which has $\frac{\lambda t_2}{k-1}$ parallel classes, denoted by $\mathcal{R}_j$, $j = 1,2,\ldots,\frac{\lambda t_2}{k-1}$, and $\frac{\lambda(t_4+gb-g)}{k-1}$ holey parallel classes, denoted by $\mathcal{R}'_j$, $j = 1,2,\ldots,\frac{\lambda(t_4+gb-g)}{k-1}$.

Now we construct a (k,λ)-IRGDD of type $g^{(u/g+b,t_3/g+b)}$ on $(X \cup Y, C \cup Y)$ as follows. $\mathcal{P}_{G_i,j} \cup \mathcal{Q}_{i,j}$, $j = 1,2,\ldots,\frac{\lambda t_1}{k-1}$, $i = 1,2,\ldots,x$, form $\frac{\lambda x t_1}{k-1}$ parallel classes. The other $\frac{\lambda t_2}{k-1}$ ones come from $\mathcal{P}_{B,j} \cup \mathcal{R}_j$, $j = 1,2,\ldots,\frac{\lambda t_2}{k-1}$. $(\cup_{1\le i\le x}\mathcal{Q}'_{i,j}) \cup \mathcal{R}'_j$, $j = 1,2,\ldots,\frac{\lambda(gb-g)}{k-1}$, form $\frac{\lambda(gb-g)}{k-1}$ holey parallel classes. $\mathcal{P}_{C',j}$, $j = 1,2,\ldots,\frac{\lambda(t_3-t_4)}{k-1}$, form another $\frac{\lambda(t_3-t_4)}{k-1}$ ones. The remaining $\frac{\lambda t_4}{k-1}$ ones come from $\mathcal{P}_{C'',j} \cup \mathcal{R}'_{j+\lambda(gb-g)/(k-1)}$, $j = 1,2,\ldots,\frac{\lambda t_4}{k-1}$. □

Lemma 2.8 *If there is a* TD$(6,m)$, *and* $0 \le s \le m$, $m \le w \le 2m$, *then there exists a* $(3,2)$*-frame of type* $(3m)^4(6m-3s)^1(3w)^1$.

Proof. Take a TD$(6,m)$, give weight 3 to the points in the first four groups, weight 3 or 6 to the last two groups, then apply SFFC to get the desired frame. The required $(3,2)$-frames of types $3^4 6^2$ and $3^5 6^1$ are obtained by applying SFFC with weight 1 to 4-GDDs of types $3^4 6^2$ and $3^5 6^1$ (see [2]), respectively.□

Lemma 2.9 *Let* $m \ge 1$, $m \le w \le 2m$, *and* $(m,w) \notin \{(3,5),(4,7)\}$, *then there exists a* $(3,2)$*-frame of type* $(3m)^4(6m)^1(3w)^1$.

Proof. It is proved in [20] that if $m \ge 1$, $m \le w \le 2m$, and $(m,w) \notin \{(3,5),(4,7)\}$, then there exists a 4-GDD of type $(3m)^4(6m)^1(3w)^1$. Applying SFFC to this 4-GDD gives the desired frame. □

Theorem 2.10 [8] *There exists a* 4-GDD *of type* $6^u m^1$ *for every* $u \ge 4$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 3u-3$ *except for* $(u,m) = (4,0)$ *and* *except possibly for* $(u, m) \in \{(7,15), (11,21), (11,24), (11,27), (13,27), (13,33), (17,39), (17,42), (19,45), (19,48), (19,51), (23,60), (23,63)\}$.

Theorem 2.11 [9] *There exists a* 4-GDD *of type* g^4m^1 *with* $m > 0$ *if and only if* $g \equiv m \equiv 0 \pmod 3$ *and* $0 < m \leq 3g/2$.

3 Existence of $(3, 2)$-incomplete resolvable group divisible designs of type $3^{(u/3,v/3)}$

With the above preparations, now we prove the existence of $(3, 2)$-IRGDDs of type $3^{(u/3,v/3)}$. First we give the following lemma for the convenient of our description.

Lemma 3.1 *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a* K-GDD *of type* $T = \{g_i : i = 1, 2, \ldots, n\}$. *Let* $w > 0$, $3|wg_i$, *and* $b > 0$. *Suppose for each block* $B \in \mathcal{B}$ *there exists a* $(3, 2)$*-frame of type* $w^{|B|}$. *Suppose there also exists a* $(3, 2)$*-IRGDD of type type* $3^{(wg_i/3+b,b)}$ *for* $i = 1, 2, \ldots, n-1$, *then there exists a* $(3, 2)$*-IRGDD of type* $3^{(u/3+b,wg_n/3+b)}$ *where* $u = \sum_{i=1}^{n} wg_i$. *Furthermore, if there exists a* $(3, 2)$*-IRGDD of type* $3^{(wg_n/3+b,b)}$, *then there exists a* $(3, 2)$*-IRGDD of type* $3^{(u/3+b,b)}$.

Proof. Applying Construction 2.5 (SFFC) with weight w gives a $(3, 2)$-frame of type $\{wg_i : i = 1, 2, \ldots, n\}$. Then applying Construction 2.6 gives the result. □

Lemma 3.2 *For* $(u, v) \in \{(18,6), (21,6), (24,6), (27,6), (33,6), (39,6), (45,6), (51,6), (30,9), (54,15), (57,18), (72,21), (75,24), (81,24), (84,27), (96,27)\}$, *there exists a* $(3, 2)$-IRGDD *of type* $3^{(u/3,v/3)}$.

Proof. For $(u, v) \in \{(57,18), (75,24), (84,27), (96,27)\}$, see the proof of [17, Lemma 3.4], where the desired designs are constructed. For $(u, v) \in \{(54,15), (72,21), (81,24)\}$, take a $(\{2,3\}, 2)$-RGDD of type $3^{(u-v)/3}$ (see the proof of [17, Lemma 3.4]), then add v ideal points to the blocks of size 2. For the remaining cases, see the Appendix. □

Lemma 3.3 *For* $u \equiv v \equiv 3 \pmod 6$, *there exists a* $(3, 2)$-IRGDD *of type* $3^{(u/3,v/3)}$.

Proof. Take a 3-IRGDD of type $3^{(u/3,v/3)}$ and repeat its blocks twice to get the desired design. □

Lemma 3.4 *For any* $v \equiv 0 \pmod 3$ *with* $v \geq 9$ *and any* $k \geq 3$, *there exists a* $(3, 2)$-IRGDD *of type* $3^{(kv/3,v/3)}$.

Proof. Take a (3, 2)-RGDD of type v^k and form a (3, 2)-RGDD of type $3^{v/3}$ on all but one of its groups. This gives a (3, 2)-IRGDD of type $3^{(kv/3, v/3)}$. □

Lemma 3.5 [3] *For each odd* $x \geq 21$, $x \notin \{23, 27, 31\}$, *there exists a* GDD *on* x *points with block sizes at least four and group sizes from the set* $\{4, 5, 6, 7\}$.

Define $T_6 = \{n \geq 5\} \setminus \{6, 10, 14, 18, 22\}$. Then for every $n \in T_6$, there exists a TD$(6, n)$.

The following lemma is obvious.

Lemma 3.6 *If* $n \in T_6$, $n \geq 7$, *then there exists an* $n_1 > n$ *such that* $n_1 \in T_6$ *and* $4n_1 \leq 5n - 1$.

Lemma 3.7 (a) *If there is a* TD(k, t), $k \geq 4$, *then for all* s, $4t \leq s \leq kt$, $s \neq 4t + 1$, *there exists a* (3, 2)-IRGDD *of type* $3^{(s+1, t+1)}$.

(b) *If there is a* TD(k, n), $1 \leq t \leq n$, $k \geq 5$, *then for all* s, $4n + t \leq s \leq (k-1)n + t$, $s \neq 4n + t + 1$, *there exists a* (3, 2)-IRGDD *of type* $3^{(s+1, t+1)}$.

Proof. For (a), take a TD(k, t), write $s - 4t$ as a sum $m(1) + m(2) + \cdots + m(k-4)$, where $m(i) = 0$ or $2 \leq m(i) \leq t$ for $i = 1, 2, \ldots, k-4$. Now truncate the ith group in the TD to $m(i)$ points for $i = 1, 2, \ldots, k-4$. This gives a GDD with block sizes at least four. Now apply Lemma 3.1 with $(w, b) = (3, 1)$.

The proof for (b) is similar. □

Lemma 3.8 *Let* $t \geq 1$, $n \in T_6$, $n \geq max\{7, t\}$, *then for all* $s \geq 4n + t$, $s \neq 4n + t + 1$, *there exists a* (3, 2)-IRGDD *of type* $3^{(s+1, t+1)}$.

Proof. By repeatedly using Lemma 3.7(b) (with $k = 6$) and Lemma 3.6, we obtain the result. □

Lemma 3.9 *For* $u \equiv 0 \pmod 3$, *and* $u \geq 18$, *there exists a* (3, 2)-IRGDD *of type* $3^{(u/3, 2)}$.

Proof. We write $u = 3s + 3$, $s \geq 5$. Lemma 3.2 covers $5 \leq s \leq 8$ and $s = 10, 12, 14, 16$. Lemma 3.8 (with n=7) covers s=29 and $s \geq 31$. For s odd, $s \geq 9$, apply Lemma 3.1 with a 4-GDD of type $6^x 3^1$, $x \geq 4$ (see Theorem 2.10), and $(w, b) = (1, 1)$. For s even, $s \geq 22$, $s \neq 24, 28, 32$, take a GDD in Lemma 3.5, and apply Lemma 3.1 with $(w, b) = (3, 2)$. For s=18, construct a (3,2)-IRGDD of type $3^{(6,2)}$ on the hole of a (3,2)-IRGDD of type $3^{(19,6)}$(see Lemma 3.2). For $s = 20, 24, 28$, take a (3, 2)-frame of type 15^4, 18^4, 21^4, apply Construction 2.6 with $b = 1$ to get a (3, 2)-IRGDD of type $3^{(21,6)}$, $3^{(25,7)}$, $3^{(29,8)}$, then construct a (3,2)-IRGDD of type $3^{(6,2)}$, $3^{(7,2)}$, $3^{(8,2)}$ on the hole, respectively. □

Lemma 3.10 *Let* $s = 4t + 1$, $t \geq 5$, *then there exists a* $(3, 2)$-IRGDD *of type* $3^{(s+1,t+1)}$.

Proof. Apply Lemma 3.1 with a 4-GDD of type $(3t-3)^4 12^1$ (see Theorem 2.11) and $(w, b) = (1, 2)$. □

Lemma 3.11 *For* $t = 10, 14, 18, 22$, $4t \leq s \leq 5t$, $s \neq 4t + 1$, *there exists a* $(3, 2)$-IRGDD *of type* $3^{(s+1,t+1)}$.

Proof. Take a TD(5,$t/2$), give weight 6 to the points in the first four groups, weight 0,3 or 6 to the last group, then apply SFFC to get a $(3, 2)$-frame of type $(3t)^4(3(s-4t))^1$. (the required (3,2)-frame of type $6^4 3^1$ is obtained by applying SFFC with weight 1 to a 4-GDD of type $6^4 3^1$.) Now apply Construction 2.6 with $b = 1$. □

Lemma 3.12 *Suppose there is a* TD$(6, t + 1)$, *where* $t \geq 6$. *Then there exists a* $(3, 2)$-IRGDD *of type* $3^{(5t+r+1,t+1)}$ *for* $r = 1, 2, 3, 5$.

Proof. It is proved in [3] that there exist a GD$(\{4, 5, t+r-3, t+1\}, \{3, 4, t\}; 5t + r)$ for $r = 1, 2, 3$, and a GD$(\{4, 5, 6, t\}, \{4, 5, t\}; 5t + 5)$. Apply Lemma 3.1 to this GDD with $(w, b) = (3, 1)$. □

Theorem 3.13 *Suppose* $t \geq 7$ *and* $s \geq 4t$. *Then there exists a* $(3, 2)$-IRGDD *of type* $3^{(s+1,t+1)}$.

Proof. When $t \geq 7$, $t \in T_6$, the result follows from Lemma 3.7(a) (with $k = 6$), Lemma 3.8 (with $n = t$), and Lemma 3.10. When $t \geq 7$, $t \notin T_6$, the result follows from Lemma 3.8 (with $n = t + 1$), Lemmas 3.10–3.12. □

Theorem 3.14 *For every* $u \equiv v \equiv 0 \pmod 3$ *with* $v \geq 24$ *and* $u \geq 4v-9$, *there exists a* $(3, 2)$-IRGDD *of type* $3^{(u/3,v/3)}$.

Proof. Write $v = 3t + 3$, $t \geq 7$, and $u = 3s + 3$. Then the condition $u \geq 4v - 9$ is equivalent to $s \geq 4t$. Apply Theorem 3.13 to get the result. □

Now we consider the cases $2 \leq t \leq 6$ (i.e., $v = 3t + 3$=9, 12, 15, 18, 21) and $s \geq 3t + 2$ (i.e., $u = 3s + 3 \geq 3v$). We have the following lemma.

Lemma 3.15 *For* $v \equiv 0 \pmod 3$, $v \leq 21$, *there exists a* $(3, 2)$-IRGDD *of type* $3^{(u/3,v/3)}$ *if and only if* $u \geq 3v$, *and* $u \equiv 0 \pmod 3$.

Proof. $v = 9$. Lemma 3.2 covers s=9. For s even, $s \geq 8$, see Lemma 3.3. For s odd, $s \geq 11$, apply Lemma 3.1 with a 4-GDD of type $6^x 9^1$, $x \geq 4$ (see Theorem 2.10), and $(w, b) = (1, 1)$.

$v = 12$. Lemma 3.8 (with n=7) covers s=31 and $s \geq 33$. Lemma 3.7 (b) (with k=6, n=5) covers s=23 and $25 \leq s \leq 28$. Lemma 3.4 covers $s = 11, 19$. For $s \equiv 0 \pmod 3$, $s \geq 12$, take a $(3,2)$-frame of type $9^{s/3}$ and apply Construction 2.6 with $b = 1$. For $s = 20, 29$, take a GD({4,5},{3,4};20) (see [3]), GD({4,5},{3,5};29) (see [17]), and apply Lemma 3.1 with $(w,b) = (3,1)$. For $s = 13, 14, 16, 17, 32$, take a 4-GDD of type $6^5 9^1$, $9^4 6^1$, $9^4 12^1$, $9^5 6^1$, $9^8 24^1$ (see [9]), and apply Lemma 3.1 with $(w,b) = (1,1)$. For s=22, take a $(3,2)$-frame of type $9^4 12^1 18^1$ (see Lemma 2.9), and apply Construction 2.6 with $b = 1$.

$v = 15$. Lemma 3.2 covers s=17. Lemma 3.8 (with n=7) covers s=32 and $s \geq 34$. Lemma 3.7 (a) (with k=5) covers s=16 and $18 \leq s \leq 20$. Lemma 3.7 (b) (with k=6, n=5) covers s=24 and $26 \leq s \leq 29$. Lemma 3.3 covers s = 14, 22, 30. For s=15, take a $(3,2)$-frame of type $6^4 9^1 12^1$ (see Lemma 2.9), and apply Construction 2.6 with $b = 1$. For $s = 21, 23, 31, 33$, take a GD$(\{4,5\},\{4,5\};21)$, GD$(\{4,5\},\{3,4\};23)$, GD$(\{4,5\},\{3,4\},31)$, GD$(\{4,5\},\{4,5\},33)$ (see [17]), and apply Lemma 3.1 with $(w,b) = (3,1)$. For s = 25, take a GD$(\{5,6\},\{4,5\};25)$ (see [3]), and apply Lemma 3.1 with $(w,b) = (3,1)$.

$v = 18$. Lemma 3.2 covers s=18. Lemma 3.8 (with n=7) covers s=33 and $s \geq 35$. Lemma 3.4 covers s=17. Lemma 3.7 (a) (with k=6) covers s=20 and $22 \leq s \leq 30$. Lemma 3.10 covers s=21. For s=19, apply Lemma 3.1 with a 4-GDD of type $6^5 12^1 15^1$ (see [9]) and $(w,b) = (1,1)$. For s=31, apply Lemma 3.1 with a 4-GDD of type $6^6 9^1$ (see Theorem 2.10) and $(w,b) = (2,2)$. For $s = 32, 34$, delete 1 or 3 points from a group in a $\{5,7\}$-GDD of type 5^7, then apply Lemma 3.1 with $(w,b) = (3,1)$.

$v = 21$. Lemma 3.2 covers s=23. Lemma 3.8 (with n=7) covers s=34 and $s \geq 36$. Lemma 3.4 covers s=20, 27. Lemma 3.3 covers s = 22, 24, 26, 28, 30. Lemma 3.12 covers $31 \leq s \leq 33$ and s=35. For s = 21, apply Lemma 3.1 with a 4-GDD of type $3^5 6^1$ (see [2]) and $(w,b) = (3,1)$. For $s = 25$, apply Lemma 3.1 with a 4-GDD of type $15^4 12^1$ (see Theorem 2.11) and $(w,b) = (1,2)$. For s=29, apply Lemma 3.1 with a 4-GDD of type $18^4 15^1$ (see Theorem 2.11) and $(w,b) = (1,1)$. □

Theorem 3.16 *Suppose* $u \equiv v \equiv 0 \pmod 3$, $v \geq 30$, *and* $u \geq 3.5v$, *then there exists a* $(3,2)$-IRGDD *of type* $3^{(u/3,v/3)}$.

Proof. By Theorem 3.14 we only need to consider $u < 4v - 9$. We divide the proof into two cases.

Case 1: $v \equiv 3 \pmod 6$.

Take a $(3,2)$-frame of type $(3m)^4(6m)^1(3w)^1$, $m \leq w \leq 2m$, $m \geq 5$ (see Lemma 2.9), and apply Construction 2.6 with $b = 1$ to yield a $(3,2)$-IRGDD of type $3^{(u/3,v/3)}$, where $v = 6m + 3 \geq 33$, $u = 18m + 3w + 3$, and $3.5v - 7.5 \leq u \leq 4v - 9$.

Case 2: $v \equiv 0 \pmod 6$.

Take a $(3,2)$-frame of type $(3m)^4(6m-3)^1(3w)^1$, $m \le w \le 2m$, $m \ge 5$, $m \ne 6, 10, 14, 18, 22$ (see Lemma 2.8), and apply Construction 2.6 with $b = 1$ to yield a $(3,2)$-IRGDD of type $3^{(u/3,v/3)}$, where $v = 6m \ge 30$, $v \ne 36, 60, 84, 108, 132$, $u = 18m + 3w$, and $3.5v \le u \le 4v$.

Take a $(3,2)$-frame of type $(3m)^4(6m)^1(3w)^1$, $m \le w \le 2m$, $m \ge 5$ (see Lemma 2.9), and apply Construction 2.6 with $b = 2$ to yield a $(3,2)$-IRGDD of type $3^{(u/3,v/3)}$, where $v = 6m + 6 \ge 36$, $v = 18m + 3w + 6$, and $3.5v - 15 \le u \le 4v - 18$.

Take a $(3,2)$-frame of type $(3m)^4(6m-9)^1(3w)^1$, $m \le w \le 2m$, $m = 11$, $15, 19, 23$ (see Lemma 2.8), and apply Construction 2.6 with $b = 1$ to yield a $(3,2)$-IRGDD of type $3^{(u/3,v/3)}$, where $v = 6m - 6 = 60, 84, 108, 132$, and $u = 4v - 15, 4v - 12$.

For $v = 36$, $u = 4v - 12$, apply Lemma 3.1 with a 4-GDD of type $24^4 33^1$ (see Theorem 2.11) and $(w, b) = (1, 1)$.

For $v = 36$, $u = 4v - 15$, apply Lemma 3.1 with a 4-GDD of type $9^8 33^1 21^1$ (see [5]) and $(w, b) = (1, 1)$.

With the above discussion, the theorem is proved. □

Lemma 3.17 *For* $v \equiv 0 \pmod 3$, *and* $v \le 36$, *there exists a* $(3,2)$-IRGDD *of type* $3^{(u/3,v/3)}$ *if and only if* $u \equiv 0 \pmod 3$, *and* $u \ge 3v$.

Proof. For $v \le 21$, see Lemma 3.15.

$v = 24$, and $u \in \{72, 75, 78, 81, 84\}$. Lemma 3.2 covers $u = 75, 81$. Lemma 3.4 covers u=72. For u=78, take a $(3,2)$-frame of type 18^4 and apply Construction 2.6 $b = 2$. For u=84, apply Lemma 3.1 with a 4-GDD of type $15^4 21^1$ (see Theorem 2.11) and $(w, b) = (1, 1)$.

$v = 27$, and $u \in \{81, 84, 87, 90, 93, 96\}$. Lemma 3.3 covers $u = 81$, 87, 93. Lemma 3.2 covers $u = 84, 96$. For $u = 90$, take a $(3,2)$-frame of type 21^4 and apply Construction 2.6 with $b = 2$.

$v = 30$, and $u \in \{90, 93, 96, 99, 102\}$. For $u \ne 99$, take a $(3,2)$-frame of type $12^4 24^1 (3w)^1$, $4 \le w \le 8$, $w \ne 7$ (see Lemma 2.9), and apply Construction 2.6 with $b = 2$. For u=99, apply Lemma 3.1 with a 4-GDD of type $9^6 15^1 27^1$ (see [17]) and $(w, b) = (1, 1)$.

$v = 33$. By the proof of Theorem 3.16, we only need to consider $u \in \{99, 102, 105\}$. Lemma 3.3 covers u=99, 105. For u=102, take a 4-IGDD of type $9^6(15,15)^1(30,3)^1$ (see [17]), and apply FIFC to get an incomplete $(3,2)$-frame of type $9^6(15,15)^1(30,3)^1$, then apply Construction 2.7 with $b = 1$.

For $v = 36$, by Lemma 3.4 and the proof of Theorem 3.16, there's no remaining case. □

Theorem 3.18 *For* $u \equiv v \equiv 0 \pmod 3$, *and* $u \ge 3.4v$, *there exists a* $(3,2)$-IRGDD *of type* $3^{(u/3,v/3)}$.

Proof. Let m, x take the values listed in Table 1. By Theorem 3.16 and Lemma 3.17 there exists a (3,2)-IRGDD of type $3^{((3w+x)/3, x/3)}$ where $w = m, m+1, \ldots, 2m$. Take a (3,2)-frame of type $(3m)^4(6m)^1(3w)^1$ (see Lemma 2.9), and apply Construction 2.6 with $b = x/3$ to yield a (3,2)-IRGDD of type $3^{(u/3, v/3)}$, where $v = 6m+x$, $u = 18m+3w+x$, and $u_{min} \leq u \leq u_{max}$. The interval $[u_{min}, u_{max}]$ covers $[3.4v, 3.5v]$. □

v	m	x	t	u_{min}	u_{max}
$27t$	$4t$	$3t$	$t \geq 2$	$87t$	$99t$
$27t+3$	$4t+1$	$3t-3$	$t \geq 2$	$87t+18$	$99t+21$
$27t+6$	$4t+1$	$3t$	$t \geq 1$	$87t+21$	$99t+24$
$27t+9$	$4t+2$	$3t-3$	$t \geq 2$	$87t+39$	$99t+45$
$27t+12$	$4t+2$	$3t$	$t \geq 1$	$87t+42$	$99t+48$
$27t+15$	$4t+2$	$3t+3$	$t \geq 1$	$87t+45$	$99t+51$
$27t+18$	$4t+3$	$3t$	$t \geq 1$	$87t+63$	$99t+72$
$27t+21$	$4t+3$	$3t+3$	$t \geq 1$	$87t+66$	$99t+75$
$27t+24$	$4t+4$	$3t$	$t \geq 1$	$87t+84$	$99t+96$

Table 1

Lemma 3.19 *For* $v \equiv 3 \pmod 9$, $v \geq 12$, *and* $u = 3v+3$, *there exists a* (3,2)-IRGDD *of type* $3^{(u/3, v/3)}$.

Proof. Adjoin a group of size $9(s-1)/2$ at infinity to a 3-RGDD of type 9^s to get a 4-GDD of type $9^s(9(s-1)/2)^1$, where s is odd, and $s \geq 3$. Then apply Lemma 3.1 with $(w, b) = (1, 1)$. □

Lemma 3.20 *For* $v \equiv 0 \pmod 3$, $u = 3v, 3v+3, 3v+6$, *there exists a* (3,2)-IRGDD *of type* $3^{(u/3, v/3)}$.

Proof. For $u = 3v$, see Lemma 3.4. For $v \leq 36$, see Lemma 3.17.
Case 1 $u = 3v+3$.

Lemma 3.19 covers $v = 39, 48, 57, 66, 102$. For $v = 42, 54, 72$, take a 4-GDD of type $12^6 15^1 36^1$, $12^8 15^1 48^1$, $12^{11} 15^1 66^1$ (see [5]), and apply Lemma 3.1 with $(w, b) = (1, 2)$. For $v = 45, 51, 60, 105$, take a (3,2)-frame of type $18^4 36^1 21^1$, $21^4 42^1 21^1$, $24^4 48^1 27^1$, $42^4 84^1 45^1$ (see Lemma 2.9), and apply Construction 2.6 with b=3, 3, 4, 7, respectively. For $v = 63$, take a 4-GDD of type $18^6 21^1 54^1$ (see [17]), and apply Lemma 3.1 with $(w, b) = (1, 3)$. For $v = 69$, take a 4-IGDD of type $9^{14}(15,15)^1(66,3)^1$ (see [17]), and apply FIFC to get an incomplete (3,2)-frame of type $9^{14}(15,15)^1(66,3)^1$, then apply Construction 2.7 with $b = 1$. For $v = 99$, take a 4-IGDD of type $9^{16}(57,57)^1(96,24)^1$ (see [17]), and apply FIFC to get an incomplete (3,2)-frame of type $9^{16}(57,57)^1(96,24)^1$, then apply Construction 2.7 with $b = 1$. For $v = 108$, take a 4-IGDD of type $12^{16}(27,27)^1(102,6)^1$ (see [5]), and apply FIFC to get an incomplete (3,2)-frame of type $12^{16}(27,27)^1(102,6)^1$, then apply Construction 2.7 with $b=2$.

Take a 4-IGDD of type $9^{2n}(6t+9,6t+9)^1(9n+3t,3t)^1$, $n \equiv 0 \pmod 4$, $n \neq 4,88,124$, $0 \leq t \leq n-1$ (see [17]). Apply FIFC to get an incomplete (3,2)-frame of type $9^{2n}(6t+9,6t+9)^1(9n+3t,3t)^1$. Then apply Construction 2.7 with $b=1$. This covers $75 \leq v \leq 96$ and $v \geq 111$.

Case 2 $u = 3v+6$.

For $v \equiv 3 \pmod 6$, see Lemma 3.3.

For $v \equiv 0 \pmod 6$, $v \geq 42$, and $v \neq 54,66$, take a 4-GDD of type $6^s(3s-6)^1$ (see Theorem 2.10) and apply Lemma 3.1 with $(w,b)=(1,1)$. For $v=54$, take a 4-IGDD of type $12^8(18,18)^1(51,3)^1$ (see [17]), and apply FIFC to get an incomplete (3,2)-frame of type $12^8(18,18)^1(51,3)^1$, then apply Construction 2.7 with $b=1$. For $v=66$, take a (3,2)-frame of type $27^4 54^1 30^1$ (see Lemma 2.9) and apply Construction 2.6 with $b=4$. □

Lemma 3.21 *For every* $v \equiv 9 \pmod{15}$, $v \geq 39$, $u = 3v+9$, *there exists a* (3,2)-IRGDD *of type* $3^{(u/3,v/3)}$.

Proof. For v=39, take a 4-GDD of type $9^8 36^1 15^1$ (see [5]), and apply Lemma 3.1 with $(w,b)=(1,1)$. For v=54, take a (3,2)-frame of type 39^4 and apply Construction 2.6 with $b=5$. For v=69, take a 4-GDD of type $12^{11}15^1 66^1$ (see [5]), and apply Lemma 3.1 with $(w,b)=(1,1)$. For v=84, take a 4-IGDD of type $15^8(57,57)^1(81,21)^1$ (see [17]) and apply FIFC to get an incomplete (3,2)-frame of type $15^8(57,57)^1(81,21)^1$, then apply Construction 2.7 with $b=1$. For v=114, take a TD(8,11) and give all the points on the first six groups weight 3 and all the points on a seventh group weight 9, then give two points on the last group weight 6 and the remaining nine points weight 3, apply SFFC to get a (3,2)-frame of type $33^6 99^1 39^1$. Note that we need (3,2)-frames of types $3^7 9^1$ and $3^6 6^1 9^1$; the former is obtained by applying SFFC to a 4-GDD of type $3^7 9^1$ (see [2]), while the latter to a $\{4,7\}$-GDD of type $3^6 6^1 9^1$ (see [5]). Then apply Construction 2.6 with $b=5$. For v=159, take a 4-IGDD of type $12^{25}(27,27)^1(156,6)^1$ (see [5]), and apply FIFC to get an incomplete (3,2)-frame of type $12^{25}(27,27)^1(156,6)^1$, then apply Construction 2.7 with $b=1$. For v=189, take a (3,2)-frame of type 129^4 and apply Construction 2.6 with $b=20$. (the required (3,2)-IRGDD of type $3^{(63,20)}$ is obtained by letting $(m,w)=(8,11)$ in Lemma 2.9 and applying Construction 2.6 with $b=4$.)

For $v \equiv 0 \pmod 6$, $v \geq 144$, take a 4-IGDD of type $12^{s-3}(57,57)^1(6(s-3)+18,18)^1$, $s \equiv 3 \pmod 5$ and $s \geq 23$(see [5]), and apply FIFC to get an incomplete (3,2)-frame of type $12^{s-3}(57,57)^1(6(s-3)+18,18)^1$, then apply Construction 2.7 with $b=2$.

For $v \equiv 3 \pmod 6$, $v \geq 99$, and $v \neq 159,189$, take a 4-GDD of type $12^s 15^1(6s)^1$, $s \in \{16,21\} \cup \{n \equiv 1 \pmod 5 : n \geq 36\}$ (see [5]), and apply

Lemma 3.1 with $(w, b) = (1, 1)$. □

Lemma 3.22 *For $v \equiv 0 \pmod 3$ and $v \leq 192$, there exists a* $(3,2)$-IRGDD *of type* $3^{(u/3, v/3)}$ *if and only if $u \equiv 0 \pmod 3$ and $u \geq 3v$.*

Proof. By Lemma 3.17 and Theorem 3.18, we only need to consider $39 \leq v \leq 192$ and $u < 3.4v$.

Let m and x take the values listed in Table 2. The condition $3 \leq v \leq 192$ implies $t \leq 12$ in Table 2. By Lemma 3.17 there exists a $(3,2)$-IRGDD of type $3^{((3w+x)/3, x/3)}$ where $w = m, m+1, \ldots, 2m$. Take a $(3,2)$-frame of type $(3m)^4(6m)^1(3w)^1$, $m \leq w \leq 2m$ (see Lemma 2.9), and apply Construction 2.6 with $b = x/3$ to yield a $(3,2)$-IRGDD of type $3^{(u/3, v/3)}$, where $v = 6m + x$, $u = 18m + 3w + x$, $u_{min} \leq u \leq u_{max}$, and $u_{max} \geq 3.4v$.

The missing cases in Table 2 are covered by Lemmas 3.20–3.21. □

v	m	x	t	u_{min}	u_{max}	missing cases
$15t$	$2t$	$3t$	$3 \leq t \leq 12$	$45t$	$51t$	no
$15t+3$	$2t+1$	$3t-3$	$2 \leq t \leq 12$	$45t+18$	$51t+21$	$3v, 3v+3, 3v+6$
$15t+6$	$2t+1$	$3t$	$2 \leq t \leq 12$	$45t+21$	$51t+24$	$3v$
$15t+9$	$2t+2$	$3t-3$	$2 \leq t \leq 12$	$45t+39$	$51t+45$	$3v, 3v+3, 3v+6, 3v+9$
$15t+12$	$2t+2$	$3t$	$2 \leq t \leq 12$	$45t+42$	$51t+48$	$3v, 3v+3$

Table 2

Theorem 3.23 *There exists a* $(3,2)$-IRGDD *of type* $3^{(u/3, v/3)}$ *if and only if $u \equiv v \equiv 0 \pmod 3$ and $u \geq 3v$.*

Proof. By induction on t. Write $v = 15t + s$, $s = 0, 3, 6, 9, 12$. We have proved the cases $t \leq 12$ in Lemma 3.22. Now suppose that for every $t \leq t_1$ there exists a $(3,2)$-IRGDD of type $3^{(u/3, v/3)}$ where $v = 15t + s$, $s = 0, 3, 6, 9, 12$, $u \equiv 0 \pmod 3$, and $u \geq 3v$. For the values of m, x listed in Table 3, we have $x \leq 3t \leq 15t_1$. By our induction hypothesis, there exists a $(3,2)$-IRGDD of type $3^{((3w+x)/3, x/3)}$ where $w = m, m+1, \ldots, 2m$. Now take a $(3,2)$-frame of type $(3m)^4(6m)^1(3w)^1$, $m \leq w \leq 2m$ (see Lemma 2.9), and apply Construction 2.6 with $b = x/3$ to yield a $(3,2)$-IRGDD of type $3^{(u/3, v/3)}$, where $v = 6m + x$, $u = 18m + 3w + x$, $u_{min} \leq u \leq u_{max}$, and $u_{max} \geq 3.4v$. The missing cases in Table 3 are covered by Lemma 3.20 and Lemma 3.21. Thus the result for $t_1 < t \leq 5t_1$ is proved. This completes the proof. □

v	m	x	t	u_{min}	u_{max}	missing cases
$15t$	$2t$	$3t$	$t_1 < t \le 5t_1$	$45t$	$51t$	no
$15t+3$	$2t+1$	$3t-3$	$t_1 < t \le 5t_1$	$45t+18$	$51t+21$	$3v, 3v+3,$ $3v+6$
$15t+6$	$2t+1$	$3t$	$t_1 < t \le 5t_1$	$45t+21$	$51t+24$	$3v$
$15t+9$	$2t+2$	$3t-3$	$t_1 < t \le 5t_1$	$45t+39$	$51t+45$	$3v, 3v+3,$ $3v+6, 3v+9$
$15t+12$	$2t+2$	$3t$	$t_1 < t \le 5t_1$	$45t+42$	$51t+48$	$3v, 3v+3$

Table 3

4 Main Results

Theorem 4.1 *There exists a* $(3,\lambda)$-IRGDD *of type* $g^{(u/g,v/g)}$ *if and only if* $u \ge 3v$, $u \equiv 0 \pmod 3$, $u \equiv v \equiv 0 \pmod g$, $\lambda(u-g) \equiv \lambda(v-g) \equiv 0 \pmod 2$, *and any of the following conditions is satisfied.*

1. $v = g$, *and* $(\lambda, u, v) \ne (1,12,2), (1,18,6), (2j+1,6,2), (4j+2,6,1)$, $j \ge 0$,
2. $v = 2g$, $\lambda g \equiv 0 \pmod 2$, *and* $g \equiv 0 \pmod 3$,
3. $v \ge 3g$, *and* $v \equiv 0 \pmod 3$.

Proof. A $(3,\lambda)$-IRGDD of type $g^{(u/g,1)}$ is just a $(3,\lambda)$-RGDD of type $g^{u/g}$. Thus we only need to consider the cases $v \ge 2g$.

For λ odd, or λ, g even, take a 3-IRGDD of type $g^{(u/g,v/g)}$ (see [17]), and repeat its blocks λ times to yield a $(3,\lambda)$-IRGDD of type $g^{(u/g,v/g)}$. For λ even and $g \equiv 1,5 \pmod 6$, $g > 1$, take a $(3,2)$-IRGDD of type $1^{(u,v)}$ (i.e., an IRB$(3,2;u,v)$, see [16]), and repeat its blocks $\lambda/2$ times to yield a $(3,\lambda)$-IRGDD of type $1^{(u,v)}$, then apply Lemma 2.2 with an RTD$(3,g)$. For λ even and $g \equiv 3 \pmod 6$, $g > 3$, take a $(3,2)$-IRGDD of type $3^{(u/3,v/3)}$, and repeat its blocks $\lambda/2$ times to yield a $(3,\lambda)$-IRGDD of type $3^{(u/3,v/3)}$, then apply Lemma 2.2 with an RTD$(3,g/3)$. This completes the proof.□

Combining Lemma 2.1 and Theorem 4.1 gives the following theorem.

Theorem 4.2 *An* RGD$(3,\lambda,g;v)$ *can be embedded in an* RGD$(3,\lambda,g;u)$ *if and only if* $\lambda(u-g) \equiv \lambda(v-g) \equiv 0 \pmod 2$, $u \equiv v \equiv 0 \pmod 3$, $u \equiv v \equiv 0 \pmod g$, $v \ge 3g$, $u \ge 3v$, *and* $(\lambda, g, v) \ne (1,2,12), (1,6,18), (2j+1,2,6), (4j+2,1,6)$, $j \ge 0$.

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Appendix

Several (3,2)-IRGDDs of type $3^{(u/3,v/3)}$ $(X, H, \mathcal{G}, \mathcal{B})$ are constructed as follows. Let $X = Z_{u-v} \cup \{\infty_i : i \in Z_v\}$, $H = \{\infty_i : i \in Z_v\}$, $\mathcal{G} = \{\{0, (u-v)/3, 2(u-v)/3\} + i : i \in Z_{u-v}\}$. The $v-3$ holey parallel classes are obtained by developing the blocks in $\mathcal{P}_1$ mod $u-v$. (Each block $\{a, b, c\}$ in $\mathcal{P}_1$ satisfies $\{a, b, c\} \equiv \{0, 1, 2\}$ (mod 3), so from $\{a, b, c\}$ we can form 3 holey parallel classes.) The $u-v$ parallel classes are obtained by developing the base parallel class in $\mathcal{P}_2$ mod $u-v$.

$(u, v) = (18, 6)$. $\mathcal{P}_1$: $\{0, 1, 2\}$.
$\mathcal{P}_2$: $\{0, 2, \infty_0\}$, $\{3, 6, \infty_1\}$, $\{7, 10, \infty_2\}$,
$\{4, 9, \infty_3\}$, $\{8, 1, \infty_4\}$, $\{5, 11, \infty_5\}$.

$(u, v) = (21, 6)$. $\mathcal{P}_1$: $\{0, 1, 2\}$.
$\mathcal{P}_2$: $\{0, 2, 6\}$, $\{1, 4, \infty_0\}$, $\{7, 10, \infty_1\}$, $\{9, 13, \infty_2\}$,
$\{8, 14, \infty_3\}$, $\{5, 12, \infty_4\}$, $\{11, 3, \infty_5\}$.

$(u, v) = (24, 6)$. $\mathcal{P}_1$: $\{0, 1, 2\}$.
$\mathcal{P}_2$: $\{0, 2, 7\}$, $\{1, 4, 8\}$, $\{9, 12, \infty_0\}$, $\{13, 17, \infty_1\}$,
$\{10, 15, \infty_2\}$, $\{3, 11, \infty_3\}$, $\{16, 6, \infty_4\}$, $\{5, 14, \infty_5\}$.

$(u, v) = (27, 6)$. $\mathcal{P}_1$: $\{0, 1, 2\}$.
$\mathcal{P}_2$: $\{0, 2, 5\}$, $\{1, 4, 10\}$, $\{3, 7, 13\}$, $\{11, 15, \infty_0\}$, $\{14, 19, \infty_1\}$,
$\{8, 16, \infty_2\}$, $\{12, 20, \infty_3\}$, $\{9, 18, \infty_4\}$, $\{17, 6, \infty_5\}$.

$(u, v) = (33, 6)$. $\mathcal{P}_1$: $\{0, 1, 2\}$.
$\mathcal{P}_2$: $\{0, 2, 5\}$, $\{1, 4, 8\}$, $\{3, 7, 15\}$, $\{13, 18, 26\}$, $\{11, 17, 24\}$, $\{16, 22, \infty_0\}$,
$\{9, 19, \infty_1\}$, $\{10, 20, \infty_2\}$, $\{12, 23, \infty_3\}$, $\{14, 25, \infty_4\}$, $\{21, 6, \infty_5\}$.

$(u, v) = (39, 6)$. $\mathcal{P}_1$: $\{0, 1, 2\}$.
$\mathcal{P}_2$: $\{0, 2, 16\}$, $\{1, 4, 8\}$, $\{3, 6, 18\}$, $\{7, 11, 20\}$, $\{12, 17, 24\}$, $\{13, 19, 27\}$, $\{23, 29, 5\}$,

$\{25,30,\infty_0\}$, $\{14,22,\infty_1\}$, $\{21,31,\infty_2\}$, $\{32,9,\infty_3\}$, $\{15,28,\infty_4\}$, $\{10,26,\infty_5\}$.

$(u,v)=(45,6)$. $\mathcal{P}_1$: $\{0,1,2\}$.
$\mathcal{P}_2$: $\{0,2,19\}$, $\{1,4,8\}$, $\{3,6,21\}$, $\{7,11,23\}$, $\{5,10,16\}$, $\{20,25,34\}$,
$\{18,24,33\}$, $\{15,22,32\}$, $\{29,37,9\}$, $\{27,35,\infty_0\}$,
$\{28,38,\infty_1\}$, $\{14,26,\infty_2\}$, $\{17,31,\infty_3\}$, $\{36,13,\infty_4\}$, $\{12,30,\infty_5\}$.

$(u,v)=(51,6)$. $\mathcal{P}_1$: $\{0,1,2\}$.
$\mathcal{P}_2$: $\{0,2,22\}$, $\{3,6,24\}$, $\{1,4,8\}$, $\{9,13,26\}$, $\{10,15,29\}$, $\{12,17,23\}$,
$\{27,33,43\}$, $\{28,35,44\}$, $\{34,42,11\}$, $\{31,39,7\}$, $\{21,30,40\}$, $\{5,16,\infty_0\}$,
$\{20,32,\infty_1\}$, $\{25,37,\infty_2\}$, $\{19,36,\infty_3\}$, $\{41,14,\infty_4\}$, $\{18,38,\infty_5\}$.

$(u,v)=(30,9)$. $\mathcal{P}_1$: $\{0,1,2\}$; $\{0,4,8\}$.
$\mathcal{P}_2$: $\{0,3,8\}$, $\{2,4,\infty_0\}$, $\{6,9,\infty_1\}$, $\{11,16,\infty_2\}$, $\{12,18,\infty_3\}$, $\{14,20,\infty_4\}$,
$\{10,19,\infty_5\}$, $\{13,1,\infty_6\}$, $\{5,15,\infty_7\}$, $\{7,17,\infty_8\}$.