On 3-Degeneracy of Some C_7-free Plane Graphs with Application to Choosability *

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Abstract

A graph G is said to be k-degenerate if for every induced subgraph H of G, $\delta(H) \leq k$. Clearly, planar graphs without 3-cycles are 3-degenerate. Recently, it was proved that planar graphs without 5-cycles or without 6-cycles are also 3-degenerate. And for every $k = 4$ or $k \geq 7$, there exist planar graphs of minimum degree 4 without k-cycles. In this paper, it is shown that each C_7-free plane graph in which any 3-cycle is adjacent to at most one triangle is 3-degenerate. So it is 4-choosable.

Key words: degenerate; choosable; cycle; plane graph; triangle.
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1 Introduction

In this paper, unless stated otherwise, *graph* means simple plane (finite) graph. Undefined symbols and concepts can be found in [1].

Let $G = (V, E, F)$ be a plane graph, where V, E and F denote the set of vertices, edges and faces of G, respectively. $N_G(v)$, or $N(v)$

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if there is no possibility of confusion, denotes the set of vertices adjacent to v in G, and ∂f denotes the set of vertices incident with the face f. The degree of a vertex v is denoted by $d(v)$. A vertex is called a *k-vertex* or a *k^+-vertex* if $d(v) = k$ or if $d(v) \geq k$, respectively. We denote by $\delta(G)$, the minimum degree of G. A face of a graph is said to be *incident* with all edges and vertices on its boundary. Two faces sharing an edge e are called *adjacent* at e. The *degree* of a face f of plane graph G, denoted by $d_G(f)$, is the number of edges incident with it, where each cut edge is counted twice. A *k-face* or a *k^+-face* is a face of degree k or of degree at least k, respectively. A *triangle* is synonymous with a *3-face*. A graph is called *C_i-free* graph if it contains no i-cycle.

A graph G is said to be *k-degenerate* if for every induced subgraph H of G, $\delta(H) \leq k$. Clearly, planar graphs without 3-cycles are 3-degenerate. Wang and Lih[8] proved that planar graphs without 5-cycles are 3-degenerate. Fijavž, Juvan, Mohar and Škrekovski[3] proved that planar graphs without 6-cycles are 3-degenerate. There exist planar graphs of minimum degree 4 without cycles of length 4. An example of such a graph is obtained by taking the line graph of a cubic planar graph of grith 5, e.g., the line graph of dodecahedron. Also, for every $k \geq 7$, there is a planar graph of minimum degree 4 without k-cycles. Such an example is the octahedron graph.

One of the main motivations to study degenerate graphs is the theory of graph colorings. A *list coloring* of G is an assignment of colors to V such that each vertex v receives from a prescribed list $L(v)$ of colors and adjacent vertices receive distinct colors. $L(G) = \{L(v)|v \in V\}$ is called a *color-list* of G. G is called *k-choosable* if G admits a list-coloring for all color-lists L with k colors in each list. Graph-choosability is a generalization of graph-colorability. It was first introduced by Vizing[7] and independently by Erdős, Rubin, and Taylar[2] nearly two decades ago. Thomasson [5,6] proved that every plane graph is 5-choosable and every plane graph with grith at least 5 is 3-choosable. Lam, Shiu, and Xu[4] proved that if G is free of k-cycle for some $k \in \{3, 4, 5, 6\}$, or if any two triangles in G have distance at least 2, then G is 4-choosable.

In this paper, we prove the following theorems.

Theorem 1. Every C_7-free plane graph in which any 3-cycle is adjacent to at most one triangle is 3-degenerate.

Theorem 2. Every C_7-free plane graph in which any 3-cycle is adjacent to at most one triangle is 4-choosable.

2 Proof of Theorems

Proof of Theorem 1. By contradiction. Let $G = (V, E, F)$ be a counterexample with $|V| + |E|$ minimal. Thus G is a connected C_7-free plane graph in which any 3-cycle is adjacent to at most one triangle, and with $\delta(G) \geq 4$.

Euler's formula $|V| + |F| - |E| = 2$ can be rewritten as $(\frac{|E|}{4} - \frac{|V|}{2}) + (\frac{|E|}{4} - \frac{|F|}{2}) = -1$. It follows from $\sum_{v\in V} d(v) = \sum_{f\in F} d(f) = 2|E|$ that

$$\sum_{v\in V}(\frac{1}{8}d(v) - \frac{1}{2}) + \sum_{f\in F}(\frac{1}{8}d(f) - \frac{1}{2}) = -1.$$

For each $x \in V \cup F$, let $w(x) = \frac{1}{8}d(x) - \frac{1}{2}$ be a weight assigned to x. So the sum of the charges for all vertices and faces is -1. We are going to redistribute these charges, not changing their sum, so that the new charge $w^*(x)$ becomes non-negative for all $x \in V \cup F$. Thus a contradiction is produced below and henceforth the proof is complete.

$$0 \leq \sum_{x\in V\cup F} w^*(x) = \sum_{x\in V\cup F} w(x) = -1.$$

Weights will be transferred according to the following rules:

(R_1) From each 5-vertex v to an incident triangle f, transfer

$(R_{1.1})$ $\frac{1}{8}$, if v is incident with exactly one triangle.

$(R_{1.2})$ $\frac{1}{16}$, if v is incident with exactly two triangles.

$(R_{1.3})$ $\frac{1}{16}$, if v is incident with three triangles and f is adjacent to a triangle which is incident with v.

(R_2) From each 6^+-vertex v to an incident triangle f, transfer

$(R_{2.1})$ $\frac{1}{8}$, if v is incident with at least four 4^+-face.

$(R_{2.2})$ $\frac{1}{16}$, otherwise.

(R_3) From each 5-face f to an adjacent triangle f', transfer $\frac{1}{8}$.

(R_4) From each 6-face to an adjacent triangle, transfer $\frac{1}{8}$.

(R_5) Suppose that a 8^+-face f and a triangle f' are incident to $e = uv$, $w \in \partial f' \setminus \partial f$. From f to f', transfer

($R_{5.1}$) $\frac{1}{8}$, if f' are incident to exactly one triangle, one 4-face, and $d(u) = d(v) = 4$, or f' is adjacent to exactly two 4-faces, and $d(u) = d(v) = 4$.

($R_{5.2}$) $\frac{1}{16}$, otherwise.

We shall first make the following observations. Note that G is C_7-free plane graph in which any 3-cycle is adjacent to at most one triangle, and $\delta(G) \geq 4$.

(1) A 5-vertex v is incident with at most three triangles. If v is incident with three triangles f_1, f_2, f_3, there are two triangles, say f_1, f_2, such that f_3 is neither adjacent to f_1 nor f_2, and f_1, f_2 are adjacent to each other.

(2) If a 5-vertex v is incident with a 4-face, then v is incident with at most two triangles.

(3) A k-vertex, where $k \geq 6$, is incident with at most $\lfloor \frac{2}{3}k \rfloor$ triangles.

(4) A 5-face is adjacent to at most one triangle.

Proof. Otherwise, let f_1, f_2 be two triangles which are adjacent to a 5-face f. Since G does not contain 7-cycles, G must contain one of the following two structures(See Fig. 1).

Case 4.1 Case 4.2

Fig. 1

In Case (4.1) and Case (4.2), we can find a 3-cycle which is adjacent to two triangles, contradicting that any 3-cycle is adjacent to at most one triangle.

(5) A 6-face is adjacent to at most two triangles, and if a triangle f' is adjacent to 6-face f, then the vertices incident with f' are incident with 6-face f.

(6) If a triangle f is adjacent to exactly one triangle, one 4-face, then the remaining face which f is adjacent to is a 8^+-face, and it

must be the following structure(See Fig. 2).

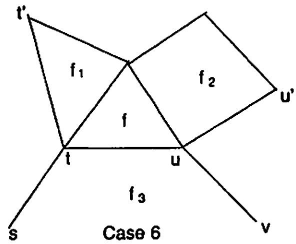

Fig. 2

(7) If a triangle f is adjacent to exactly two 4-faces f_1, f_2, then it must be one of the following two structures(See Fig. 3).

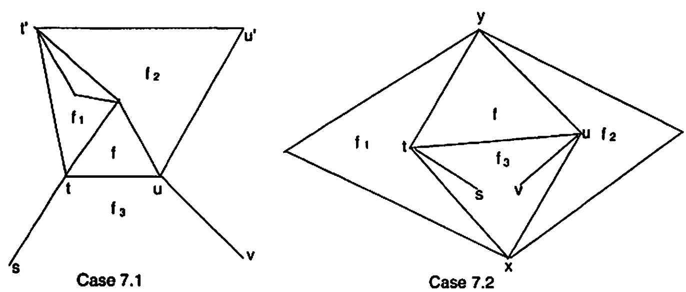

Fig. 3

In Case (7.1), $d(f_3) \geq 8$. In Case (7.2), $d(f_3) \geq 5$. And if $d(f_3) = 5$, then f_3 is adjacent to exactly one triangle.

We shall now establish the following claim. Suppose that st, tu and uv are three consecutive edges on the boundary of a face f with $d(f_3) \neq 6$ (See Fig. 2 and Fig. 3).

Claim. Suppose that the face f_3, where $d(f_3) \neq 6$, is adjacent to a triangle f at tu. If $\frac{1}{8}$ is transferred from face f_3 across tu to f, then f_3 is adjacent to a 4^+-face at st and at uv, respectively. So 0 is transferred across st and uv.

Proof. If $d(f_3) = 5$, it must be Case (7.2). From the Observation (4), we are done. Otherwise, $d(f_3) \geq 8$.

If Case (6) happens, then f_3 is adjacent to a 4^+-face at st and at uv, respectively. Otherwise, if f_3 is adjacent to any triangle at st, then the triangle must be stt', because of $d(t) = 4$. That is impossible because G is C_7-free. Therefore the weight to be transferred across st is 0. Similarly, the face adjacent to f_3 at uv is a 4^+-face.

If Case (7.1) happens, then f_3 is adjacent to a 4^+-face at st and at uv, respectively. Otherwise, if f_3 is adjacent to any triangle at st, then the triangle must be stt', because of $d(t) = 4$. That is impossible because any 3-cycle is adjacent to at most one triangle. If f_3 is adjacent to any triangle at uv, then the triangle must be $uu'v$, because of $d(u) = 4$. That is impossible, because G is C_7-free.

If Case (7.2) happens, then f_3 is adjacent to a 4^+-face at st and at uv, respectively. Otherwise, if f_3 is adjacent to any triangle at st, then the triangle must be stx, because of $d(t) = 4$. That is impossible because any 3-cycle is adjacent to at most one triangle in G. Therefore the weight to be transferred across st is 0. Similarly, the face adjacent to f_3 at uv is a 4^+-face and the weight to be transferred across uv is 0.

We shall now show that $w^*(x) \geq 0$ for all $x \in V \cup F$. Suppose that v is a k-vertex. Clearly $w^*(v) = w(v) = 0$ if $k = 4$. Now we consider $k = 5$. If v is incident with exactly one triangle, then $w^*(v) = w(v) - \frac{1}{8} = 0$; If v is incident with two triangles, then $w^*(v) = w(v) - 2 \times \frac{1}{16} = 0$; If v is incident with three triangles, then $w^*(v) = w(v) - 2 \times \frac{1}{16} = 0$. Assume that $k \geq 6$. If v is incident with at least four 4^+-face, then $w^*(v) \geq w(v) - (d(v) - 4) \times \frac{1}{8} = 0$. Otherwise, because of Observation (3), $w^*(v) \geq w(v) - \frac{2}{3} \times \frac{k}{16} \geq 0$.

Let the face f be a triangle. If f is adjacent to a 6-face, then $w^*(f) \geq w(f) + \frac{1}{8} = 0$. If there is a vertex $v \in \partial f$ incidents with at least four 4^+-faces, then $w^*(f) \geq w(f) + \frac{1}{8} = 0$. Otherwise, let $f = uvw$ be adjacent to f_1, f_2, f_3, respectively, where $d(f_1) \leq d(f_2) \leq d(f_3)$, and $\partial f \bigcap \partial(f_1) = \{u, w\}$, $\partial f \bigcap \partial(f_2) = \{w, v\}$, $\partial f \bigcap \partial(f_3) = \{u, v\}$. If $d(f_2) \geq 5$, by (R_3) and (R_4), $w^*(f) \geq w(f) + 2 \times \frac{1}{16} = 0$. If $d(f_1) = 3, d(f_2) = 4$, then Case (6) happens. By Observation (6), $d(f_3) \geq 8$. If $d(u) = d(v) = 4$, then $w^*(f) = w(f) + \frac{1}{8} = 0$ by $(R_{4.1})$. If $d(v) = 5$, then v is incident with at most two triangles because of Observation (2). $w^*(f) \geq w(f) + 2 \times \frac{1}{16} = 0$. If $d(v) \geq 6$, then $w^*(f) \geq w(f) + 2 \times \frac{1}{16} = 0$ by (R_2). If $d(u) = 5$ and u is incident with two triangles, then $w^*(f) \geq w(f) + 2 \times \frac{1}{16} = 0$ by $(R_{1.2})$. If $d(u) = 5$ and u is incident with three triangles, then $w^*(f) \geq w(f) + 2 \times \frac{1}{16} = 0$ by $(R_{1.3})$. If $d(u) \geq 6$, then $w^*(f) \geq$

$w(f)+2\times\frac{1}{16}=0$ by (R_2). Now we consider that $d(f_1)=d(f_2)=4$, then Case (7.1) or Case (7.2) happens. Assume that Case (7.1) happens. If $d(u)=d(v)=4$, then $w^*(f)=w(f)+\frac{1}{8}=0$ by $(R_{4.1})$. If $d(u)=5$, then u is incident with at most two triangles because of Observation (2). $w^*(f)\geq w(f)+2\times\frac{1}{16}=0$. If $d(u)\geq 6$, then $w^*(f)\geq w(f)+2\times\frac{1}{16}=0$ by (R_2). If $d(v)=5$, then v is incident with at most two triangles because of Observation (2). $w^*(f)\geq w(f)+2\times\frac{1}{16}=0$. If $d(v)\geq 6$, then $w^*(f)\geq w(f)+2\times\frac{1}{16}=0$. Assume that Case (7.2) happens. Proofs are similar, but, simpler, and are therefore omitted.

If f is a 4-face, then $w^*(f)=w(f)=0$.

If f is a 5-face, by Observation (4), then $w^*(f)\geq w(f)-\frac{1}{8}=0$.

If f is a 6-face, by Observation (5), then $w^*(f)\geq w(f)-\frac{2}{8}=0$.

Let f be a k-face, $k\geq 8$. Assume that $e_1,e_2,...,e_k$ are consecutive edge on the boundary of f, and z_i is the weight transferred from f across e_i, for $1\leq i\leq k$. If $z_i=1/8$, then $z_{i-1}=z_{i+1}=0$, by Claim, where z_{k+1} is identified with z_1, z_0 is identified with z_k. So $z_i+z_{i+1}\leq\frac{1}{8}$ for all $i\in\{1,2,...,k\}$. Then $w^*(f)=w(f)-\sum_{i=1}^{k}z_i=w(f)-\frac{1}{2}\sum_{i=1}^{k}(z_i+z_{i+1})\geq w(f)-\frac{1}{2}\sum_{i=1}^{k}1/8=w(f)-\frac{k}{16}\geq 0$. That is complete the proof of Theorem 1.

Proof of Theorem 2. By induction on the order of $G=(V,E)$. It is trivial if $|V|=1$. Assume that the theorem holds for $|V|<n$, where $n\geq 2$. Suppose that $|V|=n$, by Theorem 1, $\delta(G)\leq 3$. Let $v\in V$ such that $d(v)=\delta(G)$. By the induction assumption, $G-v$ is 4-choosable. There must exist a color $\alpha\in L(v)$ which is not appear in $N(v)$, color v with α. Then G is 4-choosable.

In [4], it is proved that every planar graph without 4-cycles is 4-choosable. As we see, there is a planar graph G without 4-cycles, and G is not 3-degenerate. For further researching, we can consider the following problem:

Problem. Every plane graph without 7-cycles is 4-choosable.

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