ABSTRACT OVALS OF ORDER 9

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ABSTRACT. In this paper it is shown that there are exactly 5 non-isomorphic abstract ovals of order 9, all of them projective. The result has been obtained via an exhaustive search, based on the classification of the 1-factorizations of the complete graph with 10 vertices.

Keywords: Oval, Abstract Oval, 1-factorization.

1. INTRODUCTION

An oval Ω in a projective plane of order n is a set of $n+1$ points, no three of which are collinear. Ovals in projective planes have been intensively investigated since 1954. The starting point was the famous theorem of Segre, stating that in a finite desarguesian plane of odd order, every oval is an irreducible conic (see [9, Ch. 8] and the references therein).

Buekenhout [3] has shown how to define the structure of an oval without reference to the plane containing it. Buekenhout defines an abstract oval of order n as a pair (M, I) where M is a set of $n+1 \geq 3$ elements called *points*, and I is a set of n^2 permutations of M called *involutions*, such that

(i) every $\sigma \in I$ has order at most 2;
(ii) for any $(a_1, a_2), (b_1, b_2) \in M \times M$ with $a_i \neq b_j$ $(i, j = 1, 2)$, there exists a unique involution which permutes a_1 with a_2 and b_1 with b_2.

Note that by (ii) the identity involution belongs to I if and only if n is even. For n odd, any involution in I fixes either no or two points of M.

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Any oval Ω in a projective plane π gives rise to an abstract oval (Ω, I). The set I consists of the involutions $\{\sigma_P \mid P \in \pi \backslash \Omega\}$, where σ_P acts as follows: a point $Q \in \Omega$ is either permuted with the point of Ω collinear with P and Q if such a point exists, or fixed otherwise.

An isomorphism between two abstract ovals (M_1, I_1) and (M_2, I_2) is a bijection φ of M_1 onto M_2 such that $I_2 = \{\varphi f \varphi^{-1} \mid f \in I_1\}$. An abstract oval isomorphic to an oval in a projective plane is called projective. It's easy to prove that all abstract ovals of order $n \leq 5$ are projective. Also, there are no abstract ovals of order 6 ([3], [2]) and there is a unique abstract oval of order 7, which is projective [7]. Among the abstract ovals of order 8 there are exactly two projective abstract ovals [3] and the first two examples of non-projective abstract ovals ([8],[4]). By using some previous results by Mathon [11], Faina [6] proved that there are no other abstract ovals of order 8.

The aim of this paper is to classify all the abstract ovals of order 9. By [5] there are exactly 5 non-isomorphic projective ovals, which give rise to 5 non-isomorphic projective abstract ovals, listed in Table 1. We show that there are no other abstract ovals of order 9 by means of a computer assisted exhaustive search. The algorithm is based on the classification of the 1-factorizations of the complete graph with 10 vertices.

$\mathcal{O}_1$	conic in PG(2, 9)
$\mathcal{O}_2$	oval in the Hall plane of order 9
$\mathcal{O}_3$	oval in the dual Hall plane of order 9
$\mathcal{O}_4$	oval in the Hughes plane of order 9
$\mathcal{O}_5$	oval in the Hughes plane of order 9

Table 1

2. Abstract ovals of odd order and 1-factorizations of graphs

Firstly we recall some basic definitions from graph theory.

Let K_{2n} be the complete graph with $2n$ vertices. A *1-factor* of K_{2n} is a set of vertex disjoint edges which cover the vertices of K_{2n}. An edge disjoint set of 1-factors covering the edges of K_{2n} is said to be a *1-factorization* of K_{2n}. Note that a 1-factorization of K_{2n} consists of $(2n-1)$ 1-factors. An *automorphism* of a 1-factorization of K_{2n} is a permutation of the vertices of the graph that maps 1-factors onto 1-factors of the 1-factorization.

Now let (M, I) be an abstract oval of odd order n, and let K_{n+1} be the complete graph with $n+1$ vertices, labelled as $0, 1, \ldots, n$. Write $M = \{a_0, a_1, \ldots, a_n\}$, and define I_0 as the set of those involutions fixing a_0. To each $\sigma \in I_0$ can be associated a 1-factor f_σ of K_{n+1}, defined as follows: l is an edge of f_σ if either $l = \{i, j\}$ and $\sigma(a_i) = a_j$, or $l = \{0, i_0\}$ and a_{i_0} is fixed by σ and different from a_0. By property *(ii)* of abstract ovals, it follows that the edge set $F_{M,I}(a_0) = \{f_\sigma \mid \sigma \in I_0\}$ is a 1-factorization of K_{n+1}.

Let (M', I') be another abstract oval of order n, with $M' = \{a'_0, a'_1, \ldots, a'_n\}$. Any isomorphism φ of (M, I) onto (M', I') such that $\varphi(a_0) = a'_0$ induces an isomorphism φ^* of the 1-factorizations $F_{M,I}(a_0)$ and $F_{M',I'}(a'_0)$, defined by $\varphi^*(i) = j \Leftrightarrow \varphi(a_i) = a'_j$. This proves the following lemma, which will play a crucial role in the sequel.

Lemma 2.1. *Let $\mathcal{F}$ be a complete set of representatives for the isomorphism classes of 1-factorizations of K_{n+1}. Then for any isomorphism class $\mathcal{A}$ of abstract ovals of order n there exist $F \in \mathcal{F}$ and $(M, I) \in \mathcal{A}$, such that F is isomorphic to $F(M, I)(a_i)$ for some $i \in \{1, \ldots, n\}$.*

3. An algorithm for the classification of abstract ovals of order 9

From now on (M, I) denotes a generic abstract oval of order 9. Without loss of generality assume $M = \{0, 1, \ldots, 9\}$. Let $\mathcal{F}_{10}$ be a complete set of representatives for the isomorphism classes of 1-factorizations of K_{10}. It is known (see e.g. [1]) that the size of $\mathcal{F}_{10}$ is 396.

By Lemma 2.1, any abstract oval of order 9 is isomorphic to an abstract oval such that the involutions of I_i correspond to the 1-factors of F, for some $F \in \mathcal{F}_{10}$ and for some $i = 0, \ldots, 9$.

Then the basic idea of the algorithm is the following: fixed a 1-factorization $F \in \mathcal{F}_{10}$ and an integer $i \in \{0, 1, \ldots, 9\}$, find all abstract ovals of order 9 such that I_i corresponds to F. In the sequel, the following definition will be useful.

Definition 3.1. Let σ_1 and σ_2 be two permutations on $\{0, 1, \ldots, 9\}$. Then σ_1 and σ_2 are said to be *compatible* if $\sigma_1(j) = \sigma_2(j)$ for at most one $j \in \{0, 1, \ldots, 9\}$.

Note that by property (ii) of an abstract oval, any two involutions have to be compatible.

We are now in a position to give a more detailed description of the algorithm. As a matter of terminology, an abstract oval of order 9 in which I_i corresponds to a 1-factorization $F \in \mathcal{F}_{10}$, will be said an (F, i)-abstract oval. The algorithm consists of three steps:

(A) Fix $F \in \mathcal{F}_{10}$ and $i \in \{0, 1, \ldots, 9\}$.

(B) For any $j \in \{0, 1, \ldots, 9\}$, $j \neq i$, find all the sets of involutions that can possibly coincide with I_j for some (F, i)-abstract oval. More precisely, find all the sets L of 9 permutations on $\{0, 1, \ldots, 9\}$ such that:
   (a) each $\sigma \in L$ has order 2 and fixes j;
   (b) each $\sigma \in L$ is compatible with any $\tau \in I_i$;
   (c) the involutions in L are pairwise compatible.

   Define $\mathcal{I}_j$ as the set containing all such L.

(C) Consider all the sets I of type $I = (\bigcup_{j \neq i} L^{(j)}) \bigcup I_i$ with $L^{(j)} \in \mathcal{I}_j$. For each of such I, check whether (M, I) is an abstract oval, i.e. whether the involutions in I are pairwise compatible.

Step (B) is worth some more comments. The set $\mathcal{I}_j$ is constructed as follows. First, find all permutations satisfying both (a) and (b). Then, define a graph G_j whose vertices are such permutations, two of which being adjacent if and only if they correspond to compatible permutations. Finally, search all the complete subgraphs of G_j with 9 vertices. In fact, such subgraphs correspond to the elements of $\mathcal{I}_j$.

The algorithm was implemented using the computer algebra package MAGMA. As a result, we obtained 67 abstract ovals, each of them isomorphic to some projective oval, namely

3 abstract ovals isomorphic to $\mathcal{O}_1$;

28 abstract ovals isomorphic to $\mathcal{O}_2$;

30 abstract ovals isomorphic to $\mathcal{O}_3$;

3 abstract ovals isomorphic to $\mathcal{O}_4$;

3 abstract ovals isomorphic to $\mathcal{O}_5$.

Hence the following result is proved:

Theorem 3.2. *There are exactly* 5 *non-isomorphic abstract ovals of order 9, all of them projective.*

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