Packings and coverings for ten graphs with seven points, seven edges and an even-cycle *

Huijuan Zuo, Yinzhi Gao†
College of Mathematics and Information Science, Hebei Normal University
Shijiazhuang 050016, P.R. China

Abstract

Let λK_v be the complete multigraph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x,y\}$. Let G be a finite simple graph. A G-packingdesign (G-covering design) of λK_v, denoted by (v,G,λ)-PD ((v,G,λ)-CD), is a pair $(X,\mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in at most (at least) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design has more (fewer) blocks. In this paper, we have completely determined the packing number and covering number for the graphs with seven points, seven edges and an even circle.

Keywords: G-design; G-packing design; G-covering design; G-holey design.

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x,y\}$. A *t-partite graph* is one whose vertex set can be partitioned into t subsets $X_1, X_2, \cdots$, and X_t, such that two ends of each edge lie in distinct subsets, respectively. Such a partition $(X_1, X_2, \cdots, X_t)$ is called a *t-partition* of the graph. We denote the path of k vertices by P_k.

Let G be a finite simple graph. A *G-packing design*(*G-covering design*, *G-design*) of λK_v, denoted by (v,G,λ)-PD((v,G,λ)-CD, (v,G,λ)-GD), is

*Research supported by NSFC Grant 10771051 and Science Fundation of Hebei Normal University.

†E-mail: huijuanzuo@163.com

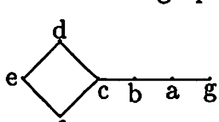
a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in at most (at least, exactly) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be *maximum* (*mimimum*) if no other such packing (covering) design has more (fewer) blocks. The number of blocks in a maximum packing design (mimimum covering design), denoted by $p(v, G, \lambda)$ $(c(v, G, \lambda))$, is called the *packing* (*covering*) *number*. It is well known that

$$p(v, G, \lambda) \leq \lfloor \frac{\lambda v(v-1)}{2e(G)} \rfloor \leq \lceil \frac{\lambda v(v-1)}{2e(G)} \rceil \leq c(v, G, \lambda), \qquad (*)$$

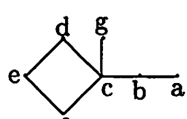
where $e(G)$ denotes the number of edges in G, and $\lfloor x \rfloor$ $(\lceil x \rceil)$ denotes the greatest (least) integer y such that $y \leq x$ $(y \geq x)$. A (v, G, λ)-*PD* $((v, G, \lambda)$-*CD*$)$ is called *optimal* and denoted by (v, G, λ)-*OPD* $((v, G, \lambda)$-*OCD*$)$ if the left (right) equality in $(*)$ holds. Obviously, there exists a (v, G, λ)-*GD* if and only if $p(v, G, \lambda){=}c(v, G, \lambda)$. A (v, G, λ)-*GD* can be regarded as a (v, G, λ)-*OPD* or a (v, G, λ)-*OCD*. The *leave-edge graph* $L_\lambda(\mathcal{D})$ of a packing design $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of $\mathcal{D}$ in λK_v. The number of edges in $L_\lambda(\mathcal{D})$ is denoted by $|L_\lambda(\mathcal{D})|$. Especially, when $\mathcal{D}$ is optimal, $|L_\lambda(\mathcal{D})|$ is called *leave-edge number* and is denoted by $l_\lambda(v)$. Similarly, the *repeat-edge graph* $R_\lambda(\mathcal{D})$ of a covering design $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of λK_v in $\mathcal{D}$. When $\mathcal{D}$ is optimal, $|R_\lambda(\mathcal{D})|$ is called the *repeat-edge number* and denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(\mathcal{D})$, $l_\lambda(v)$, $R_\lambda(\mathcal{D})$ and $r_\lambda(v)$ can be denoted by L_λ, l_λ, R_λ and r_λ, briefly. For some graphs, which have less vertices and less edges, the problem of their graph designs, packing designs and covering designs has already been researched (see [1]-[3] and [6]-[17]).

Let $(X_1, X_2, \cdots, X_t)$ be the t-partition of $\lambda K_{n_1, n_2, \cdots, n_t}$, and $|X_i|{=}n_i$. Denote $v{=}\sum_{i=1}^{t} n_i$ and $\mathcal{G}{=}\{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1, n_2, \cdots, n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to G and is called *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by G-$HD_\lambda(T)$, where $T{=}n_1^1 n_2^1 \cdots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. For HD_λ, the subscript can be omitted when $\lambda = 1$.

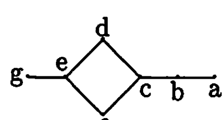
In this paper, we have completely determined the packing number and covering number for the graphs with seven points, seven edges and an even circle. The ten graphs are as follows:



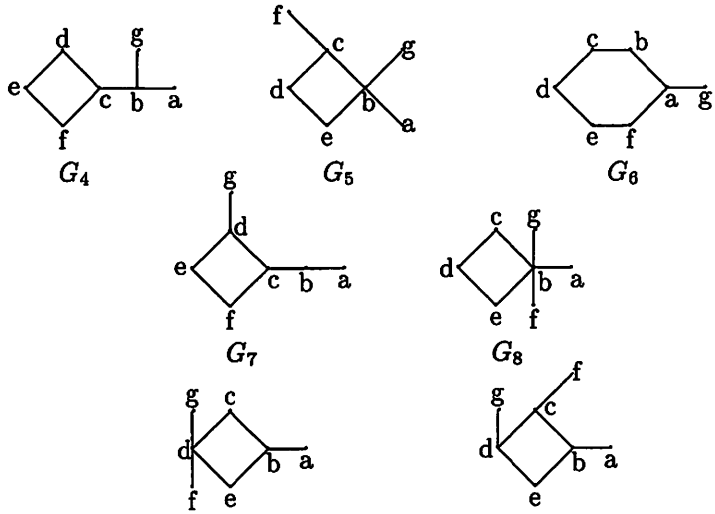
G_1



G_2



G_3



G_4 G_5 G_6 G_7 G_8 G_9 G_{10}

For convenience, we denoted the graphs as follows:

$G_1 : (a,b,c,d,e,f;g)$, $G_2 : (a,b,c,d,e,f;g)$, $G_3 : (a,b,c,d,e,f;g)$,
$G_4 : (a,b,c,d,e,f;g)$, $G_5 : (a,b,c,d,e;f;g)$, $G_6 : (g,a,b,c,d,e,f)$,
$G_7 : (a,b,c,d,e,f;g)$, $G_8 : (a,b,c,d,e;f;g)$, $G_9 : (a,b,c,d,e;f;g)$,
$G_{10} : (a,b,c,d,e;f;g)$.

Lemma 1.[5] *There exist G_i-$GD_\lambda(v)$ $\Longleftrightarrow$ $\lambda v(v-1) \equiv 0$ mod $7, v \geq 7$ and $(i,v,\lambda) \neq (8,7,1),(8,8,1),(9,7,1),(9,8,1),(10,7,1)$.*

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers $n, e, \lambda, s_1, \cdots, s_l$, nonnegative integer t, if there exists G-$OPD(n)$, G-$GD(et)$ and G-$HD(e^1 s_k^1), 1 \leq k \leq l$, then there exists G-$OPD(et+n)$, where $n = \sum\limits_{k=1}^{l} s_k x_k$ and each x_k is nonnegative integer. The conclusion still holds by replacing OPD with OCD.*

Proof. Let $n = \sum\limits_{k=1}^{l} s_k x_k$. Set $P = Z_e \times I_t$, $Q = \bigcup\limits_{1 \leq k \leq l} (S_k \times I_{x_k})$, where $|P| = et$, $|Q| = n$, $|S_k| = s_k$ and S_k are pairwise disjoint, $1 \leq k \leq l$. For any $i \in I_t$, $j \in I_{x_k}$, $1 \leq k \leq l$, suppose there exist

G-$OPD(n) = (Q, \mathcal{B})$, G-$GD(et) = (P, \mathcal{A})$,

G-$HD(e^1 s_k^1) = ((Z_e \times \{i\}) \cup (S_k \times \{j\}), \{Z_e \times \{i\}, S_k \times \{j\}\}, \mathcal{C}_{ijk})$,

then $\mathcal{B} \bigcup \mathcal{A} \bigcup (\bigcup\limits_{1 \leq k \leq l, i \in I_t, j \in I_{x_k}} \mathcal{C}_{ijk})$ form a G-$OPD(et+n)$ on the vertex set $P \bigcup Q$. ■

Lemma 2.2[5] *For G_i, $1 \le i \le 10$, there exist the following holey graphs design:* (1) G_i-$HD(4^1 7^1)$, (2) G_i-$HD(7^2)$.

Lemma 2.3 *For $i \in \{4,5,7,8,9,10\}$, there exist G_i-$HD(5^1 7^1)$.*

Proof. Let $V(K_{5,7}) = X \bigcup Y$, where $X = \bar{Z}_5$, $Y = Z_5 \bigcup \{x, y\}$. For the graphs G_i, give the base block D_i as follows:

$D_4 = (x, \bar{0}, 4, \bar{1}, 3, \bar{3}; y)$ $D_5 = (x, \bar{0}, 4, \bar{1}, 2; y; \bar{4})$ $D_7 = (x, \bar{0}, 4, \bar{1}, 3, \bar{3}; y)$
$D_8 = (x, \bar{0}, 4, \bar{1}, 2; 0; y)$ $D_9 = (1, \bar{1}, 4, \bar{0}, 2; x; y)$ $D_{10} = (x, \bar{0}, 4, \bar{1}, 2; \bar{4}; y)$

Then $\{D_i \bmod 5\}$ are the blocks of G_i-$HD(5^1 7^1)$, $i \in \{4,5,7,8,9,10\}$. ■

Lemma 2.4 *For $1 \le i \le 10$ and $i \ne 6$, there exist G_i-$HD(6^1 7^1)$.*

Proof. Let $V(K_{6,7}) = X \bigcup Y$, where $X = \bar{Z}_6$, $Y = Z_6 \bigcup \{x\}$. For the graphs G_i, give the base block E_i as follows:

$E_1 = (\bar{0}, 5, \bar{1}, 4, \bar{3}, 3; x)$ $E_2 = (x, \bar{0}, 5, \bar{1}, 4, \bar{3}; \bar{5})$ $E_3 = (x, \bar{0}, 5, \bar{1}, 4, \bar{3}; \bar{4})$
$E_4 = (x, \bar{0}, 5, \bar{1}, 4, \bar{3}; 0)$ $E_5 = (x, \bar{0}, 5, \bar{1}, 3; 0; \bar{4})$ $E_7 = (x, \bar{0}, 5, \bar{1}, 4, \bar{3}; 1)$
$E_8 = (x, \bar{0}, 5, \bar{1}, 3; 0; 1)$ $E_9 = (x, \bar{0}, 5, \bar{1}, 3; 1; 2)$ $E_{10} = (x, \bar{0}, 5, \bar{1}, 3; \bar{5}; 2)$

Then $\{E_i \bmod 6\}$ are the blocks of G_i-$HD(6^1 7^1)$, $1 \le i \le 10$ and $i \ne 6$. ■

By Theorems 2.1 and Lemmas 2.2, 2.3 and 2.4, for G_1, G_2, G_3, there exist G_i-$GD(7t)$, G_i-$HD(7^2)$, G_i-$HD(6^1 7^1)$ and G_i-$HD(4^1 7^1)$, for $n \ge 9$, if there exist G_i-$OPD(n)$, $n = 4x + 6y + 7z$, there exist G_i-$OPD(7t+n)$. For the method, $n \in \{9, 10, 11, 12, 13, 16\}$, we have to give direct constructions. Similarly, we can give the desired small designs for other graphs. Now, let's list the following table:

Table 1

$v \equiv \pmod 7$	2		3		4		5		6	
G_1, G_2, G_3	9,16		10		11		12		13	
G_4, G_5, G_7	9		10		11		12		13	
G_6	9,16		10,17,24		11		12		13,20	
$v \equiv \pmod{14}$	2	9	3	10	4	11	5	12	6	13
G_8, G_9, G_{10}	16	9	17	10	18	11	19	12	20	13

In the following section, we shall construct the desired designs.

3 Packings and Coverings for $\lambda = 1$

Lemma 3.1 *There exist $(w, G_1, 1)$-OPD for $w = 9, 10, 11, 12, 13$ and 16.*

Proof. Let $(w, G_1, 1)$-$OPD = (X, \mathcal{B})$.

$\underline{w = 9}$: $X = Z_7 \cup \{A, B\}$
$B_1 = (5, 2, 0, 1, 6, 3; A)$, $B_2 = (A, 4, 5, 3, 1, B; 2)$,
$B_3 = (4, 2, 1, 5, 6, A; 0)$, $B_4 = (0, 6, 2, 3, 4, B; 5)$,
$B_5 = (4, 6, B, 3, A, 0; 1)$

$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=10}$: $X = Z_6 \cup \{A,B,C,D\}$

$B_1 = (D,0,2,4,B,5;3)$, $B_2 = (5,A,3,2,1,4;D)$,
$B_3 = (5,1,0,C,A,4;3)$, $B_4 = (4,C,1,A,D,B;5)$,
$B_5 = (D,2,C,3,0,5;4)$, $B_6 = (1,3,B,0,A,2;D)$.
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=11}$: $X = Z_9 \cup \{A,B\}$ (B_1,B_2,B_3 *are same as* $w=9$)

$B_4 = (4,6,B,3,8,0;1)$, $B_5 = (7,6,2,8,4,B;5)$,
$B_6 = (8,1,7,3,A,0;6)$, $B_7 = (B,A,7,2,3,4;8)$,
$L(\mathcal{B}) = \{(A,8),(8,5),(5,0),(0,6),(8,7),(7,B)\}$.

$\underline{w=12}$: $X = Z_8 \cup \{A,B,C,D\}$ (B_1,B_2,B_3 *are same as* $w=10$)

$B_4 = (4,C,1,6,7,B;5)$, $B_5 = (D,7,C,3,6,5;4)$,
$B_6 = (D,A,6,0,7,2;B)$, $B_7 = (A,1,D,6,C,2;7)$,
$B_8 = (1,3,B,0,A,2;7)$, $B_9 = (6,4,7,5,0,3;B)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=13}$: $X = Z_{11} \cup \{A,B\}$ ($B_1,\cdots,B_4$ *are same as* $w=9$)

$B_5 = (10,4,7,1,8,6;9)$, $B_6 = (7,3,8,2,9,4;A)$,
$B_7 = (2,7,9,8,B,3;10)$, $B_8 = (8,7,10,6,9,1;5)$,
$B_9 = (4,6,B,10,A,0;1)$, $B_{10} = (0,10,5,7,B,9;8)$,
$B_{11} = (0,9,A,8,10,3;7)$
$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=16}$: $X = Z_{14} \cup \{A,B\}$ ($B_1,\cdots,B_5$ *are same as* $w=9$)

$B_6 = (9,13,7,1,8,B;5)$, $B_7 = (9,4,7,3,8,5;B)$,
$B_8 = (13,0,7,8,11,9;6)$, $B_9 = (8,10,7,11,5,12;0)$,
$B_{10} = (11,6,7,A,13,2;12)$, $B_{11} = (13,4,8,6,10,9;11))$,
$B_{12} = (13,8,12,9,A,10;1)$, $B_{13} = (11,3,9,1,12,6;4)$,
$B_{14} = (0,11,10,3,13,5;12)$, $B_{15} = (9,0,10,B,12,4;2)$,
$B_{16} = (12,13,10,1,11,2;3)$, $B_{17} = (B,11,A,12,2,8;13)$
$L(\mathcal{B}) = \{(A,B)\}$.

Lemma 3.2 *There exist* $(w,G_2,1)$-*OPD for* $w = 9,10,11,12,13$ *and* 16.
Proof. Let $(w,G_2,1)$-$OPD = (X,\mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \cup \{A,B\}$

$B_1 = (5,2,0,B,3,A;1)$, $B_2 = (5,3,2,B,1,A;6)$,
$B_3 = (B,5,1,3,4,2;6)$, $B_4 = (3,0,4,5,6,B;A)$,
$B_5 = (1,4,6,0,5,A;3)$
$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=10}$: $X = Z_6 \cup \{A,B,C,D\}$

$B_1 = (B,D,0,A,C,1;4)$, $B_2 = (3,B,5,A,D,1;C)$,
$B_3 = (5,0,2,4,3,D;A)$, $B_4 = (5,2,1,B,0,3;A)$,
$B_5 = (5,D,4,B,2,C;1)$, $B_6 = (0,C,3,5,4,A;2)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=11}$: $X = Z_9 \cup \{A,B\}$ (B_1, B_2, B_3 *are same as* $w = 9$)
$B_4 = (3,7,4,5,6,B;A)$, $B_5 = (1,4,6,7,5,A;3)$,
$B_6 = (B,A,8,6,0,5;7)$, $B_7 = (7,B,8,4,0,3;1)$
$L(\mathcal{B}) = \{(7,1),(7,A),(7,2),(7,0),(8,2),(8,0)\}$.

$\underline{w=12}$: $X = Z_8 \cup \{A,B,C,D\}$ (B_1, B_2, B_3 *are same as* $w = 10$)
$B_4 = (6,2,1,B,0,3;7)$, $B_5 = (5,D,4,B,7,C;1)$,
$B_6 = (0,C,3,5,4,7;2)$, $B_7 = (A,1,6,B,2,C;7)$,
$B_8 = (5,2,7,A,6,D;0)$, $B_9 = (7,5,6,3,A,4;0)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=13}$: $X = Z_{11} \cup \{A,B\}$ ($B_1, \cdots, B_4$ *are same as* $w = 9$)
$B_5 = (7,4,6,0,10,A;3)$, $B_6 = (9,2,7,A,8,6;10)$,
$B_7 = (9,1,8,7,B,10;2)$, $B_8 = (10,4,9,0,5,A;6)$,
$B_9 = (7,1,10,5,9,3;6)$, $B_{10} = (2,10,9,8,3,7;B)$,
$B_{11} = (1,4,8,5,7,0;B)$
$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=16}$: $X = Z_{14} \cup \{A,B\}$ ($B_1, \cdots, B_5$ *are same as* $w = 9$)
$B_6 = (7,1,9,2,8,3;0)$, $B_7 = (13,1,8,12,11,A;5)$,
$B_8 = (11,5,9,12,7,A;B)$, $B_9 = (12,6,13,7,B,8;A)$,
$B_{10} = (4,8,11,9,6,7;13)$, $B_{11} = (13,3,10,1,11,6;12)$,
$B_{12} = (6,8,7,4,10,5;3)$, $B_{13} = (9,4,11,0,12,3;2)$,
$B_{14} = (12,13,9,8,0,7;10)$, $B_{15} = (12,B,10,0,13,2;8)$,
$B_{16} = (7,2,12,4,13,5;1)$, $B_{17} = (12,A,10,13,B,11;7)$
$L(\mathcal{B}) = \{(A,B)\}$.

Lemma 3.3 *There exist* $(w, G_3, 1)$-*OPD for* $w = 9, 10, 11, 12, 13$ *and* 16.
Proof. Let $(w, G_3, 1)$-$OPD = (X, \mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \cup \{A,B\}$
$B_1 = (6,B,4,A,0,5;1)$, $B_2 = (A,5,2,B,0,6;4)$,
$B_3 = (2,3,1,A,6,4;5)$, $B_4 = (2,A,3,6,1,5;B)$,
$B_5 = (5,B,3,4,2,0;1)$
$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=10}$: $X = Z_6 \cup \{A,B,C,D\}$
$B_1 = (4,3,2,A,1,B;C)$, $B_2 = (5,0,A,C,2,D;4)$,
$B_3 = (4,C,3,D,5,1;A)$, $B_4 = (D,0,3,A,4,B;1)$,
$B_5 = (D,B,0,C,5,2;3)$, $B_6 = (B,5,4,D,1,0;2)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=11}$: $X = Z_9 \cup \{A,B\}$ ($B_1, \cdots, B_4$ *are same as* $w = 9$)
$B_5 = (8,B,7,4,2,0;1)$, $B_6 = (B,A,8,3,7,2;6)$,
$B_7 = (5,B,3,4,8,0;7)$
$L(\mathcal{B}) = \{(8,6),(8,5),(8,1),(5,7),(1,7),(7,A)\}$.

$\underline{w=12}$: $X = Z_8 \cup \{A,B,C,D\}$ (B_1, B_2, B_3 *are same as* $w = 10$)
$B_4 = (D,0,3,A,7,B;4)$, $B_5 = (D,B,0,C,6,2;4)$,

$B_6 = (7,5,6,D,1,0;2)$, $B_7 = (6,1,7,C,5,2;3)$,
$B_8 = (7,3,6,A,4,B;1)$, $B_9 = (B,5,4,D,7,0;6)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

<u>$w = 13$</u>: $X = Z_{11} \cup \{A,B\}$ ($B_1, \cdots, B_5$ *are same as* $w = 9$)
$B_6 = (9,B,8,4,7,0;1)$, $B_7 = (10,1,8,A,9,6;0)$,
$B_8 = (6,10,7,3,9,5;2)$, $B_9 = (8,2,7,9,10,B;0)$,
$B_{10} = (9,4,10,8,7,A;6)$, $B_{11} = (1,9,8,3,10,5;2)$
$L(\mathcal{B}) = \{(A,B)\}$.

<u>$w = 16$</u>: $X = Z_{14} \cup \{A,B\}$ ($B_1, \cdots, B_5$ *are same as* $w = 9$)
$B_6 = (11,0,7,2,13,1;12)$, $B_7 = (10,1,8,4,7,5;11)$,
$B_8 = (B,13,7,6,12,10;3)$, $B_9 = (12,11,13,9,7,A;3)$,
$B_{10} = (9,B,7,8,0,12;13)$, $B_{11} = (9,A,8,B,10,3;13)$,
$B_{12} = (13,6,8,11,A,12;10)$, $B_{13} = (3,13,8,10,6,9;11)$,
$B_{14} = (8,2,12,4,11,B;1)$, $B_{15} = (13,5,9,0,10,2;4)$,
$B_{16} = (12,1,9,3,11,10;2)$, $B_{17} = (13,4,9,11,5,12;10)$
$L(\mathcal{B}) = \{(A,B)\}$.

Lemma 3.4 *There exist* $(w, G_4, 1)$-*OPD for* $w = 9, 10, 11, 12$ *and* 13.
Proof. Let $(w, G_4, 1)$-$OPD = (X, \mathcal{B})$.

<u>$w = 9$</u>: $X = Z_7 \cup \{A,B\}$
$B_1 = (4,5,6,0,1,B;A)$, $B_2 = (2,A,1,5,4,3;0)$,
$B_3 = (6,3,0,B,2,4;A)$, $B_4 = (1,4,B,3,2,6;A)$,
$B_5 = (3,5,2,0,6,1;A)$
$L(\mathcal{B}) = \{(A,B)\}$.

<u>$w = 10$</u>: $X = Z_6 \cup \{A,B,C,D\}$
$B_1 = (5,C,1,A,0,B;3)$, $B_2 = (0,1,5,A,D,B;2)$,
$B_3 = (B,4,0,C,A,2;1)$, $B_4 = (D,5,2,B,3,4;0)$,
$B_5 = (0,D,3,A,4,5;1)$, $B_6 = (0,3,2,D,4,C;1)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

<u>$w = 11$</u>: $X = Z_9 \cup \{A,B\}$
$B_1 = (2,B,0,A,1,3;7)$, $B_2 = (1,0,2,A,B,6;4)$,
$B_3 = (3,B,4,5,8,7;1)$, $B_4 = (6,8,1,7,3,4;0)$,
$B_5 = (3,2,1,5,A,6;4)$, $B_6 = (8,3,5,7,0,6;A)$,
$B_7 = (B,5,2,7,A,8;0)$
$L(\mathcal{B}) = \{(A,4),(4,6),(6,3),(6,7),(4,8),(8,B)\}$.

<u>$w = 12$</u>: $X = Z_8 \cup \{A,B,C,D\}$
$B_1 = (3,B,0,A,C,4;6)$, $B_2 = (5,7,2,0,3,4;6)$,
$B_3 = (3,5,1,4,D,0;6)$, $B_4 = (6,3,2,B,D,1;7)$,
$B_5 = (6,0,C,2,A,3;7)$, $B_6 = (3,D,2,5,4,6;7)$,
$B_7 = (C,1,B,4,A,5;3)$, $B_8 = (B,7,1,A,D,6;4)$,
$B_9 = (D,5,C,6,A,7;0)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=13}$: $X = Z_{11} \cup \{A, B\}$
$\mathcal{B}: (A, 3, 0, 5, 1, 2; B)$ *mod* 11
$L(\mathcal{B}) = \{(A, B)\}$.

Lemma 3.5 *There exist* $(w, G_5, 1)$*-OPD for* $w = 9, 10, 11, 12$ *and* 13.

Proof. Let $(w, G_5, 1)$-$OPD = (X, \mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \cup \{A, B\}$
$B_1 = (6, 5, A, 4, B; 0; 1)$, $B_2 = (A, 6, B, 3, 0; 1; 2)$,
$B_3 = (A, 0, 4, 1, B; 2; 3)$, $B_4 = (1, 3, 6, 2, 5; A; 4)$,
$B_5 = (3, 2, 1, 5, 4; A; 0)$
$L(\mathcal{B}) = \{(A, B)\}$.

$\underline{w=10}$: $X = Z_6 \cup \{A, B, C, D\}$
$B_1 = (3, 0, A, C, 1; B; D)$, $B_2 = (A, 1, B, 4, 2; 3; D)$,
$B_3 = (4, C, 2, 3, 5; 0; B)$, $B_4 = (B, 3, D, 0, 4; C; 1)$,
$B_5 = (1, 4, 5, 2, D; A; B$, $B_6 = (1, 5, A, 2, 0; D; 3)$
$L(\mathcal{B}) = \{(A, B), (B, C), (C, D)\}$.

$\underline{w=11}$: $X = Z_9 \cup \{A, B\}$
$B_1 = (7, A, 3, 1, 0; 8; 2)$, $B_2 = (2, 1, 4, B, 7; 8; 3)$,
$B_3 = (7, 2, 5, 3, 8; 0; 4)$, $B_4 = (7, 3, 6, 8, 0; B; 5)$,
$B_5 = (2, 4, A, 5, 7; 8; 6)$, $B_6 = (B, 5, 1, 6, 0; 8; A)$,
$B_7 = (A, B, 0, 7, 8; 2; 4)$.
$L(\mathcal{B}) = \{(4, 6), (6, B), (B, 1), (7, 6), (6, 2), (2, A)\}$.

$\underline{w=12}$: $X = Z_8 \cup \{A, B, C, D\}$
$B_1 = (7, A, 0, 2, C; 6; 1)$, $B_2 = (7, 2, A, 5, 6; 4; 3)$,
$B_3 = (6, 1, A, D, 2; C; 4)$, $B_4 = (1, 7, 4, 3, B; 6; 5)$,
$B_5 = (4, 0, 3, D, B; 6; 1)$, $B_6 = (7, 5, 1, B, 2; D; 4)$,
$B_7 = (5, C, 0, D, 6; 3; 7)$, $B_8 = (B, 4, D, 7, C; 6; 1)$,
$B_9 = (7, 3, 5, B, 6; 2; 0)$.
$L(\mathcal{B}) = \{(A, B), (B, C), (C, D)\}$.

$\underline{w=13}$: $X = Z_{11} \cup \{A, B\}$
$\mathcal{B}: (A, 0, 5, 1, 3; B; 4)$ *mod* 11
$L(\mathcal{B}) = \{(A, B)\}$.

Lemma 3.6 *There exist* $(w, G_6, 1)$*-OPD for* $w = 9, 10, 11, 12, 13, 16$, 17, 20 *and* 24.

Proof. Let $(w, G_6, 1)$-$OPD = (X, \mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \cup \{A, B\}$
$B_1 = (6, 0, 5, A, 1, 4, 2)$, $B_2 = (5, B, 4, A, 0, 1, 3)$,
$B_3 = (4, 5, 1, B, 2, 6, 3)$, $B_4 = (0, 4, 3, A, 2, 5, 6)$,
$B_5 = (A, 6, B, 0, 3, 2, 1)$
$L(\mathcal{B}) = \{(A, B)\}$.

$\underline{w=10}$: $X = Z_6 \cup \{A, B, C, D\}$

$B_1 = (5,2,4,A,0,C,3)$, $B_2 = (4,5,C,2,D,B,3)$,
$B_3 = (5,1,2,A,3,4,0)$, $B_4 = (B,0,5,A,C,4,D)$,
$B_5 = (C,1,B,2,0,3,D)$, $B_6 = (3,1,4,B,5,D,A)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

<u>$w = 11$</u>: $X = Z_9 \cup \{A,B\}$
$B_1 = (5,8,4,1,0,7,3)$, $B_2 = (4,5,7,8,2,6,3)$,
$B_3 = (5,A,8,1,3,4,0)$, $B_4 = (3,A,6,8,0,B,7)$,
$B_5 = (4,B,3,2,7,6,1)$, $B_6 = (8,B,A,1,2,5,6)$,
$B_7 = (B,5,1,7,4,2,0)$
$L(\mathcal{B}) = \{((3,0),(0,6),(6,4),(4,A),(A,2),(2,B)\}$.

<u>$w = 12$</u>: $X = Z_8 \cup \{A,B,C,D\}$
$B_1 = (3,0,C,7,1,A,6)$, $B_2 = (3,1,0,B,D,5,4)$,
$B_3 = (C,3,A,0,D,1,B)$, $B_4 = (5,2,1,6,3,D,7)$,
$B_5 = (6,7,A,4,C,5,0)$, $B_6 = (B,5,A,2,0,4,3)$,
$B_7 = (6,2,C,A,D,4,B)$, $B_8 = (5,6,B,7,3,2,D)$,
$B_9 = (2,4,6,C,1,5,7)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

<u>$w = 13$</u>: $X = Z_{11} \cup \{A,B\}$ ($B_1,\cdots,B_5$ *are same as* $w = 9$)
$B_6 = (10,7,0,8,1,9,2)$, $B_7 = (B,9,0,10,1,7,6)$,
$B_8 = (9,8,4,7,5,10,6)$, $B_9 = (B,8,2,10,4,9,5)$,
$B_{10} = (B,10,A,8,3,7,9)$, $B_{11} = (B,7,A,9,3,10,8)$
$L(\mathcal{B}) = \{(A,B)\}$.

<u>$w = 16$</u>: $X = Z_{14} \cup \{A,B\}$ ($B_1,\cdots,B_5$ *are same as* $w = 9$)
$B_6 = (8,0,7,3,12,4,9)$, $B_7 = (10,2,12,7,B,11,13)$,
$B_8 = (12,10,0,11,3,8,4)$, $B_9 = (A,7,2,11,6,13,5)$,
$B_{10} = (B,12,0,13,8,9,5)$, $B_{11} = (7,6,12,11,8,2,9)$,
$B_{12} = (7,1,10,5,11,A,13)$, $B_{13} = (3,10,7,13,12,9,11)$,
$B_{14} = (5,8,1,12,A,10,6)$, $B_{15} = (11,4,7,9,B,10,13)$,
$B_{16} = (13,9,A,8,7,11,1)$, $B_{17} = (12,8,10,9,3,13,B)$
$L(\mathcal{B}) = \{(A,B)\}$.

<u>$w = 17$</u>: $X = Z_{13} \cup \{A,B,C,D\}$ ($B_1,\cdots,B_6$ *are same as* $w = 10$)
$B_7 = (8,0,6,10,11,9,7)$, $B_8 = (12,1,9,C,11,A,6)$,
$B_9 = (6,3,11,D,8,A,12)$, $B_{10} = (A,9,10,4,11,1,8)$,
$B_{11} = (12,7,3,9,2,6,11)$, $B_{12} = (12,11,B,9,0,10,2)$,
$B_{13} = (C,7,1,10,5,8,2)$, $B_{14} = (7,4,8,6,12,D,9)$,
$B_{15} = (6,B,7,10,12,C,8)$, $B_{16} = (2,12,0,11,5,6,4)$,
$B_{17} = (11,8,7,6,C,10,3)$, $B_{18} = (9,5,7,A,10,B,12)$,
$B_{19} = (7,D,6,9,12,8,10)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

<u>$w = 20$</u>: $X = Z_{18} \cup \{A,B\}$ ($B_1,\cdots,B_{16}$ *are same as* $w = 16$)
$B_{17} = (12,8,15,9,3,17,B)$, $B_{18} = (3,14,0,16,12,15,1)$,

$B_{19} = (3,15,6,14,7,16,2)$, $B_{20} = (15,11,17,9,10,8,14)$,
$B_{21} = (B,16,15,17,A,14,4)$, $B_{22} = (14,17,6,16,A,15,5)$,
$B_{23} = (B,14,2,17,1,16,5)$, $B_{24} = (4,15,10,14,12,17,7)$,
$B_{25} = (11,16,9,14,13,17,10)$, $B_{26} = (17,16,3,13,B,15,14)$,
$B_{27} = (4,17,0,15,13,16,8)$
$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=24}$: $X = Z_{20} \cup \{A,B,C,D\}$ ($B_1,\cdots,B_{19}$ *are same as* $w = 17$)
$B_{20} = (15,0,13,6,17,7,14)$, $B_{21} = (17,1,13,9,15,14,19)$,
$B_{22} = (19,A,16,C,15,D,17)$, $B_{23} = (12,14,1,15,7,18,10)$,
$B_{24} = (C,19,1,14,A,15,12)$, $B_{25} = (18,3,14,B,16,8,13)$,
$B_{26} = (19,4,17,11,13,C,18)$, $B_{27} = (17,14,4,16,19,15,13)$,
$B_{28} = (14,5,15,7,19,D,18)$, $B_{29} = (17,18,14,16,5,19,3)$,
$B_{30} = (11,18,0,19,6,14,8)$, $B_{31} = (18,2,13,10,19,11,16)$,
$B_{32} = (6,16,0,17,8,19,9)$, $B_{33} = (8,15,1,18,9,14,11)$,
$B_{34} = (9.17.2.15.6.18.12)$, $B_{35} = (18,15,3,16,13,17,B)$,
$B_{36} = (10,17,3,19,B,18,16)$, $B_{37} = (7,13,4,15,10,16,12)$
$B_{38} = (B,13,5,17,19,18,A)$, $B_{39} = (13,D,14,C,17,15,16)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

Lemma 3.7 *There exist* $(w,G_7,1)$*-OPD for* $w = 9,10,11,12$ *and* 13.
Proof. Let $(w,G_7,1)$-$OPD = (X,\mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \cup \{A,B\}$
$B_1 = (6,A,0,1,5,2;B)$, $B_2 = (5,A,1,2,6,3;B)$,
$B_3 = (6,B,3,2,4,5;A)$, $B_4 = (5,6,4,3,0,B;A)$,
$B_5 = (B,5,0,4,1,6;A)$
$L(\mathcal{B}) = \{(A,B)\}$.

$\underline{w=10}$: $X = Z_6 \cup \{A,B,C,D\}$
$B_1 = (C,4,0,2,B,1;5)$, $B_2 = (D,4,1,3,C,2;0)$,
$B_3 = (B,D,2,A,4,3;1)$, $B_4 = (4,B,3,5,D,A;C)$,
$B_5 = (3,D,0,A,5,B;C)$, $B_6 = ((2,4,5,1,C,0;D)$
$L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\}$.

$\underline{w=11}$: $X = Z_9 \cup \{A,B\}$
$B_1 = (7,1,0,3,B,2;A)$, $B_2 = (A,B,1,4,8,3;7)$,
$B_3 = (7,3,2,5,A,4;8)$, $B_4 = (8,7,6,3,5,B;4)$,
$B_5 = (7,5,4,0,A,6;B)$, $B_6 = (7,2,1,5,0,8;6)$,
$B_7 = (7,0,6,2,A,1;8)$.
$L(\mathcal{B}) = \{(4,B),(B,8),(8,A),(A,7),(7,B),(6,8)\}$.

$\underline{w=12}$: $X = Z_8 \cup \{A,B,C,D\}$
$B_1 = (6,1,C,5,0,A;4)$, $B_2 = (2,5,6,7,C,0;A)$,
$B_3 = (6,3,0,1,A,D;4)$, $B_4 = (D,7,3,2,B,4;6)$,
$B_5 = (5,1,2,0,B,D;7)$, $B_6 = (4,2,7,5,3,B;D)$,
$B_7 = (0,4,A,3,C,2;D)$, $B_8 = (3,1,D,4,C,6;7)$,

$B_9 = (7, 1, B, 6, A, 5; 4)$
$L(\mathcal{B}) = \{(A, B), (B, C), (C, D)\}$.

<u>$w = 13$</u>: $X = Z_{11} \cup \{A, B\}$
$\mathcal{B} : (A, 3, 0, 5, 1, 2; B) \ mod\ 11$
$L(\mathcal{B}) = \{(A, B)\}$.

Theorem 3.8 *There exist $(v, G_i, 1)$-OPD and $(v, G_i, 1)$-OCD for $v \geq 7$ and $1 \leq i \leq 7$.*

Proof. By Theorem 2.1 and Lemma 3.1, $\cdots$, 3.7, the above conclusion holds. ■

Lemma 3.9 $p(7, G_8, 1)=1$, $c(7, G_8, 1)=4$; $p(8, G_8, 1)=3$, $c(8, G_8, 1)=5$.

Proof. We know that there is no $(v, G_8, 1)$-GD for $v = 7$, 8(see [5]).

(1) Obviously, G_8 is no subgraph of $K_7 \setminus G_8$. Thus, $p(7, G_8, 1) = 1$. However, there exists a $(7, G_8, 1)$-$CD = (G_8, \mathcal{B})$ as follows:

$\mathcal{B}=\{(6, 0, 3, 2, 5; 1; 4),\ (2, 1, 3, 6, 4; 5; 0),\ (3, 2, 4, 5, 6; 0; 1),\ (0, 6, 4, 3, 5; 1; 2)\}$
$R(\mathcal{B}) = \{(0, 1), (0, 6), (1, 2), (2, 3), (2, 6), (4, 6), (5, 6)\}$.

(2) There exists a $(8, G_8, 1)$-$PD = (G_8, \mathcal{A})$ as follows:

$\mathcal{A} = \{(2, 0, 3, 7, 4; 1; 6),\ (3, 1, 4, 6, 7; 5; 2),\ (6, 2, 7, 0, 5; 4; 3)\}$
$L(\mathcal{A}) = \{(1, 6), (6, 3), (3, 4), (4, 5), (5, 6), (5, 7), (3, 5)\}$.
$\mathcal{B} = \mathcal{A} \bigcup \{(1, 6, 3, 4, 5; 0; 2),\ (0, 7, 5, 3, 1; 2; 4)\}$
$R(\mathcal{B}) = \{(0, 6), (2, 6), (0, 7), (2, 7), (4, 7), (1, 7), (1, 3)\}$. ■

Lemma 3.10 *There exist $(w, G_8, 1)$-OPD for $w = 9, 10, 11, 12, 13, 16, 17, 18, 19$ and 20.*

Proof. Let $(w, G_8, 1)$-$OPD = (X, \mathcal{B})$.

<u>$w = 9$</u>: $X = Z_9$
$B_1 = (5, 0, 4, 2, 3; 1; 6)$, $B_2 = (1, 7, 5, 6, 3; 0; 8)$,
$B_3 = (7, 2, 6, 4, 5; 0; 8)$, $B_4 = (3, 8, 6, 7, 4; 0; 5)$,
$B_5 = (6, 1, 5, 3, 4; 2; 8)$
$L(\mathcal{B}) = \{(1, 3)\}$.

<u>$w = 10$</u>: $X = Z_9 \cup \{x_1\}$ ($B_1, \cdots, B_4$ *are same as* $w = 9$)
$B_5 = (1, x_1, 5, 3, 4; 0; 7)$, $B_6 = (4, 1, x_1, 6; 3; 5)$
$L(\mathcal{B}) = \{(3, x_1), (x_1, 8), (8, 1)\}$.

<u>$w = 11$</u>: $X = Z_{11}$
$B_1 = (3, 0, 5, 2, 4; 7; 10)$, $B_2 = (7, 1, 6, 3, 5; 2; 0)$,
$B_3 = (0, 2, 7, 4, 6; 3; 8)$, $B_4 = (9, 3, 8, 5, 7; 4; 1)$,
$B_5 = (10, 4, 9, 6, 8; 5; 1)$, $B_6 = (2, 10, 9, 0, 8; 1; 3)$,
$B_7 = (1, 9, 5, 10, 7; 2; 8)$
$L(\mathcal{B}) = \{(1, 8), (8, 7), (7, 6), (6, 0), (6, 5), (6, 10)\}$.

<u>$w = 12$</u>: $X = Z_{11} \cup \{x_1\}$ ($B_1, \cdots, B_6$ *are same as* $w = 11$)
$B_7 = (1, 9, 5, 6, 7; 2; 8)$, $B_8 = (8, x_1, 0, 6, 10; 2; 3)$,
$B_9 = (6, x_1, 5, 10, 7; 4; 9)$
$L(\mathcal{B}) = \{(7, 8), (8, 1), (1, x_1)\}$.

$\underline{w=13}$: $X = Z_{11} \cup \{x_1, x_2\}$ ($B_1, \cdots, B_7$ *are same as* $w = 11$)
$B_8 = (1, x_1, 0, 6, 10; 2; 3)$, $B_9 = (2, x_2, 1, 8, 7; 0; 3)$,
$B_{10} = (6, x_1, 5, 10, 7; x_2; 9)$, $B_{11} = (6, x_2, 8, x_1, 4; 5; 9)$
$L(\mathcal{B}) = \{(10, x_2)\}$.

$\underline{w=16}$: $X = Z_{11} \cup \{x_1, x_2, \cdots, x_5\}$ ($B_1, \cdots, B_{10}$ *are same as* $w = 13$)
$B_{11} = (6, x_2, 8, x_1, 4; 5; x_5)$, $B_{12} = (3, x_3, 0, x_5, 1; 2; 4)$,
$B_{13} = (1, x_4, 9, x_3, 10; 0; 4)$, $B_{14} = (5, x_5, 2, x_4, 3; 4; 6)$,
$B_{15} = (6, x_3, x_4, x_5, 8; 5; 7)$, $B_{16} = (6, x_4, x_2, x_3, x_1; 5; 7)$,
$B_{17} = (x_3, x_5, 10, x_2, 9; 7; x_1)$
$L(\mathcal{B}) = \{(8, x_4)\}$.

$\underline{w=17}$: $X = Z_{11} \cup \{x_1, x_2, \ , x_6\}$ ($B_1, \cdots, B_{15}$ *are same as* $w = 16$)
$B_{16} = (x_6, x_5, 10, x_2, 9; 7; x_3)$, $B_{17} = (1, x_6, 5, x_4, 6; 0; x_3)$,
$B_{18} = (4, x_6, 7, x_4, 8; 3; 9)$, $B_{19} = (x_4, x_6, x_2, x_3, x_1; 10; 2)$,
$L(\mathcal{B}) = \{(x_2, x_4), (x_4, x_1), (x_1, x_5)\}$.

$\underline{w=18}$: $X = Z_{11} \cup \{x_1, x_2, \cdots, x_7\}$ ($B_1, \cdots, B_9$ *are same as* $w = 13$)
$B_{10} = (6, x_1, 5, 10, 7; x_4; 9)$, $B_{11} = (10, x_2, 8, x_1, 4; 9; x_5)$,
$B_{12} = (3, x_3, 0, x_5, 1; 2; 4)$, $B_{13} = (1, x_4, 9, x_3, 10; 0; 4)$,
$B_{14} = (x_7, x_5, 2, x_4, 3; 4; 5)$, $B_{15} = (4, x_7, 9, x_5, 10; 3; 5)$,
$B_{16} = (4, x_6, 7, x_4, 8; 3; 9)$, $B_{17} = (1, x_6, x_1, x_3, x_2; 0; x_4)$,
$B_{18} = (x_2, x_7, x_6, x_5, 6; 8; x_4)$, $B_{19} = (1, x_7, 7, x_5, x_1; 0; 2)$,
$B_{20} = (x_3, x_6, 5, x_4, 6; 10; 2)$, $B_{21} = (5, x_3, x_4, x_5, 8; x_7; 7)$,
$L(\mathcal{B}) = \{(x_1, x_2), (x_2, 6), (6, x_3), (x_3, x_5), (x_5, x_2), (x_2, x_4)\}$.

$\underline{w=19}$: $X = Z_{11} \cup \{x_1, x_2, \cdots, x_8\}$ ($B_1, \cdots, B_{19}$ *are same as* $w = 18$)
$B_{20} = (6, x_3, x_4, x_5, 8; 5; x_5)$, $B_{21} = (1, x_8, 6, x_2; 0; 2)$,
$B_{22} = (x_8, x_6, 5, x_4, 6; 10; 2)$, $B_{23} = (4, x_8, x_1, x_2, x_4; 3; 5)$,
$B_{24} = (9, x_8, x_7, x_3, 7; 8; 10)$
$L(\mathcal{B}) = \{(x_6, x_3), (x_3, x_8), (x_8, x_2)\}$.

$\underline{w=20}$: $X = Z_{11} \cup \{x_1, x_2, \ , x_9\}$ ($B_1, \cdots, B_{21}$ *are same as* $w = 19$)
$B_{22} = (x_3, x_6, 5, x_4, 6; 10; 2)$, $B_{23} = (4, x_8, x_1, x_2, x_4; 3; x_9)$,
$B_{24} = (1, x_9, 8, x_8, 9; 0; 2)$, $B_{25} = (4, x_9, 10, x_8, x_3; 3; 5)$,
$B_{26} = (x_1, x_9, x_7, x_3, 7; 6; x_4)$, $B_{27} = (7, x_8, x_2, x_9, x_6; x_7; 5)$,
$L(\mathcal{B}) = \{(x_9, x_5)\}$.

Theorem 3.11 *There exist* $(v, G_8, 1)$*-OPD and* $(v, G_8, 1)$*-OCD for* $v \geq 9$. *And,* $p(7, G_8, 1) = 1$, $c(7, G_8, 1) = 4$; $p(8, G_8, 1) = 3$, $c(8, G_8, 1) = 5$.
Proof. By Theorem 2.1 and Lemma 3.9, 3.10, we can give the above conclusion. ■

Lemma 3.12 $p(7, G_9, 1)=2$, $c(7, G_9, 1)=4$; $p(8, G_9, 1)=3$, $c(8, G_9, 1)=5$.
Proof. We know that there is no $(v, G_9, 1)$-*GD* for $v = 7, 8$ (*see*[5]).
(1) There exists a $(7, G_9, 1)$-$PD = (G_9, \mathcal{A})$ as follows:
$\mathcal{A} = \{(2, 0, 5, 1, 6; 3; 4), \ (1, 2, 4, 3, 5; 6; 0\}$.
$L(\mathcal{A}) = \{(0, 1), (0, 4), (2, 3), (2, 6), (4, 5), (4, 6), (5, 6)\}$.

$\mathcal{B} = \mathcal{A} \bigcup \{(0,1,3,6,4;5;2),\ \ (0,4,5,2,6;3;1)\}$
$R(\mathcal{B}) = \{(1,2),(2,5),(1,3),(1,4),(3,6),(4,6),(5,6)\}$.
(2) There exists a $(8, G_9, 1)$-$PD = (G_9, \mathcal{A})$ as follows:
$\mathcal{A} = \{(1,0,6,3,7;2;4),\ \ (2,1,6,4,7;5;0),$
$(0,2,6,5,7;1;3)\}$
$L(\mathcal{A}) = \{(0,3),(0,5),(1,3),(1,4),(2,4),(2,5),(6,7)\}$.
$\mathcal{B} = \mathcal{A} \bigcup \{(2,5,0,3,1;4;6),\ \ (0,7,6,4,5;1;2)\}$
$R(\mathcal{B}) = \{(1,5),(3,6),(4,6),(3,4),(4,5),(5,7),(0,7)\}$. ■

Lemma 3.13 *There exist* $(w, G_9, 1)$-*OPD for* $w = 9, 10, 11, 12, 13, 16, 17, 18, 19$ *and* 20.

Proof. Let $(w, G_9, 1)$-$OPD = (X, \mathcal{B})$.

<u>$w = 9$</u>: $X = Z_9$
$B_1 = (3,0,6,1,2;4;8)$, $B_2 = (0,1,5,2,3;7;8)$,
$B_3 = (8,7,3,5,4;0;6)$, $B_4 = (1,7,5,8,0;3;4)$,
$B_5 = (0,4,2,6,3;7;8)$
$L(\mathcal{B}) = \{(4,6)\}$.

<u>$w = 10$</u>: $X = Z_9 \cup \{x_1\}$ ($B_1, \cdots, B_4$ *are same as* $w = 9$)
$B_5 = (1,x_1,2,6,3;7;8)$, $B_6 = (3,4,6,x_1,0;5;7)$
$L(\mathcal{B}) = \{(2,4),(4,x_1),(x_1,8)\}$.

<u>$w = 11$</u>: $X = Z_9 \cup \{x_1, x_2\}$ ($B_1, \cdots, B_5$ *are same as* $w = 10$)
$B_6 = (3,4,6,x_2,0;2;1)$, $B_7 = (4,x_1,5,x_2,7;8;3)$
$L(\mathcal{B}) = \{(2,4),(4,x_2),(x_2,x_1),(x_1,0),(x_1,6),(x_1,8)\}$.

<u>$w = 12$</u>: $X = Z_9 \cup \{x_1, x_2, x_3\}$ ($B_1, \cdots, B_6$ *are same as* $w = 11$)
$B_7 = (7,x_1,6,x_3,4;2;1)$, $B_8 = (5,x_1,0,x_3,8;3;x_2)$
$B_9 = (x_1,x_3,5,x_2,7;3;8)$
$L(\mathcal{B}) = \{(2,4),(4,x_2),(x_2,x_1)\}$.

<u>$w = 13$</u>: $X = Z_9 \cup \{x_1, x_2, x_3, x_4\}$
($B_1, \cdots, B_5$ *and* B_7, B_8, B_9 *are same as* $w = 12$)
$B_6 = (0,x_2,1,x_4,2;3;4)$, $B_{10} = (3,4,0,x_4,6;5;7)$
$B_{11} = (2,4,x_1,x_4,x_2;8;x_3)$
$L(\mathcal{B}) = \{(6,x_2)\}$.

<u>$w = 16$</u>: $X = Z_9 \cup \{x_1, x_2,\ , x_7\}$ ($B_1, \cdots, B_{11}$ *are same as* $w = 13$)
$B_{12} = (x_2,x_7,0,x_6,1;2;3)$, $B_{13} = (7,x_7,4,x_6,5;8;x_1)$
$B_{14} = (x_2,x_5,8,x_7,x_1;x_3;x_4)$, $B_{15} = (x_4,x_6,x_7,x_5,x_3;0;1)$
$B_{16} = (6,x_7,2,x_5,3;x_4;x_6)$, $B_{17} = (x_2,x_6,6,x_5,7;4;5)$
$L(\mathcal{B}) = \{(6,x_2)\}$.

<u>$w = 17$</u>: $X = Z_9 \cup \{x_1, x_2, \cdots, x_8\}$ ($B_1, \cdots, B_{15}$ *are same as* $w = 16$)
$B_{16} = (3,x_5,x_4,x_8,x_6;x_1;x_3)$, $B_{17} = (7,x_5,4,x_8,5;8;x_1)$
$B_{18} = (6,x_7,2,x_8,3;x_2;x_5)$, $B_{19} = (x_2,x_6,6,x_8,7;0;1)$
$L(\mathcal{B}) = \{(2,x_5),(x_5,6),(6,x_2)\}$.

$\underline{w=18}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_9\}$
($B_1, \cdots, B_{11}$ *and* $B_{14}, \cdots, B_{16}$ *are same as* $w = 17$)
$B_{12} = (7, x_7, 0, x_6, 1; x_2; 3)$, $B_{13} = (x_2, x_7, 4, x_6, 5; 8; x_1)$
$B_{17} = (6, x_5, 4, x_8, 5; 8; x_1)$, $B_{18} = (3, x_8, x_2, x_9, x_5; 6; x_7)$,
$B_{19} = (6, x_7, 2, x_9, 3; 4; 5)$, $B_{20} = (7, x_8, 0, x_9, 1; x_3; x_4)$,
$B_{21} = (x_2, 6, x_8, x_9, x_6; 8; x_1)$
$L(\mathcal{B}) = \{(7, x_9), (7, x_5), (7, x_6), (x_5, 2), (x_6, 2), (2, x_8)\}$.

$\underline{w=19}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_{10}\}$
($B_1, \cdots, B_{11}$ *and* $B_{14}, \cdots, B_{21}$ *are same as* $w = 18$)
$B_{12} = (x_2, x_7, 4, x_6, 5; 8; 1)$, $B_{13} = (3, x_6, x_2, x_{10}, 0; 2; 4)$
$B_{22} = (0, x_7, 1, x_{10}, 7; 5; 6)$, $B_{23} = (x_6, 2, x_5, x_{10}, x_8; 3; 8)$
$B_{24} = (x_5, 7, x_9, x_{10}, x_6; x_4; x_3)$
$L(\mathcal{B}) = \{(x_7, x_{10}), (x_{10}, x_1), (x_1, x_6)\}$.

$\underline{w=20}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_{11}\}$ ($B_1, \cdots, B_{22}$ *are same as* $w = 19$)
$B_{23} = (x_6, 2, x_5, x_{11}, x_8; 0; 1)$, $B_{24} = (x_8, x_{10}, x_1, x_{11}, x_7; 4; 5)$
$B_{25} = (x_5, x_{10}, 3, x_{11}, 8; 6; 7)$, $B_{26} = (x_5, 7, x_9, x_{11}, x_6; 2; x_{10})$
$B_{27} = (x_2, x_{11}, x_4, x_{10}, x_3; x_9; x_6)$
$L(\mathcal{B}) = \{(x_1, x_6)\}$.

Theorem 3.14 *There exist* $(v, G_9, 1)$-*OPD and* $(v, G_9, 1)$-*OCD for* $v \geq 9$. *And,* $p(7, G_9, 1) = 2$, $c(7, G_9, 1) = 4$; $p(8, G_9, 1) = 3$, $c(8, G_9, 1) = 5$.
Proof. By Theorem 2.1 and Lemma 3.12 and 3.13, we can give the above conclusion. ■

Lemma 3.15 $p(7, G_{10}, 1) = 2$, $c(7, G_{10}, 1) = 4$.
Proof. We know that there is no $(7, G_{10}, 1)$-*GD*(see [5]). There exists a $(7, G_{10}, 1)$-*PD* $= (G_{10}, \mathcal{A})$ as follows:
$\mathcal{A} = \{(5, 2, 1, 0, 6; 4; 3), \quad (1, 5, 4, 3, 6; 0; 2)\}$.
$L(\mathcal{A}) = \{(0, 2), (0, 5), (1, 3), (1, 6), (2, 4), (3, 5), (4, 6)\}$.
$\mathcal{B} = \mathcal{A} \bigcup \{(3, 5, 4, 2, 0; 6; 1), \quad (5, 6, 2, 3, 1; 0; 4)\}$
$R(\mathcal{B}) = \{(0, 2), (1, 2), (2, 3), (2, 6), (3, 4), (4, 5), (5, 6)\}$. ■

Lemma 3.16 *There exist* $(w, G_{10}, 1)$-*OPD for* $w = 9, 10, 11, 12, 13, 16, 17, 18, 19$ *and* 20.
Proof. Let $(w, G_{10}, 1)$-*OPD* $= (X, \mathcal{B})$.

$\underline{w=9}$: $X = Z_9$
$B_1 = (2, 0, 1, 7, 5; 8; 3)$, $B_2 = (4, 6, 1, 2, 8; 3; 5)$,
$B_3 = (0, 7, 2, 3, 6; 4; 5)$, $B_4 = (5, 8, 3, 4, 7; 0; 1)$,
$B_5 = (8, 4, 5, 6, 0; 1; 2)$
$L(\mathcal{B}) = \{(0, 8)\}$.

$\underline{w=10}$: $X = Z_9 \cup \{x_1\}$ (B_1, B_2, B_3 *are same as* $w = 9$)
$B_4 = (8, 3, 4, x_1, 0; 1; 5)$, $B_5 = (1, x_1, 2, 6, 3; 7; 8)$,
$B_6 = (4, 5, 6, x_1, 1; 2; 3)$
$L(\mathcal{B}) = \{(6, 0), (0, 8), (8, 7)\}$.

$\underline{w = 11}$: $X = Z_9 \cup \{x_1, x_2\}$
$B_1 = (0,7,2,3,6;4;5)$, $B_2 = (7,8,2,1,6;5;3)$,
$B_3 = (8,1,5,6,x_2;x_1;2)$, $B_4 = (2,0,5,7,1;x_2;3)$,
$B_5 = (2,x_2,0,4,x_1;3;5)$, $B_6 = (5,8,x_2,7,4;3;x_1)$,
$B_7 = (x_2,4,6,x_1,1;0;8)$
$L(\mathcal{B}) = \{(4,3),(3,x_1),(x_1,0),(0,8),(8,3),(x_1,2)\}$.

$\underline{w = 12}$: $X = Z_9 \cup \{x_1, x_2, x_3\}$ (B_1, B_2, B_3 *are same as* $w = 11$)
$B_4 = (4,x_1,6,x_3,1;0;5)$, $B_5 = (0,x_2,7,x_3,3;4;x_1)$,
$B_6 = (4,8,x_1,2,x_2;7;0)$, $B_7 = (x_1,x_2,4,0,x_3;5;3)$,
$B_8 = (x_1,3,4,x_3,8;6;2)$, $B_9 = (x_2,5,7,1,0;3;4)$
$L(\mathcal{B}) = \{(5,8),(8,0),(0,x_1)\}$.

$\underline{w = 13}$: $X = Z_9 \cup \{x_1, x_2, x_3, x_4\}$ ($B_1, \cdots, B_6$ *are same as* $w = 12$)
$B_7 = (x_1,x_2,4,0,x_3;1;3)$, $B_8 = (x_3,8,3,x_4,0;x_1;2,$
$B_9 = (4,5,7,x_4,8;3;x_2)$, $B_{10} = (3,4,x_3,x_4,6;2;1)$
$B_{11} = (1,0,5,x_4,x_1;x_2;4)$
$L(\mathcal{B}) = \{(1,7)\}$.

$\underline{w = 16}$: $X = Z_9 \cup \{x_1, \cdots, x_7\}$ ($B_2, \cdots, B_{10}$ *are same as* $w = 13$)
$B_1 = (x_7,7,2,3,6;4;5)$, $B_{11} = (3,x_7,4,x_5,2;x_4;0)$,
$B_{12} = (5,0,x_7,x_5,1;x_2;7)$, $B_{13} = (8,x_6,1,x_7,6;7;x_4)$
$B_{14} = (2,x_6,5,x_5,3;x_4;6)$, $B_{17} = (x_7,x_3,x_6,x_2,x_5;x_1;5)$
$B_{16} = (x_5,x_4,x_6,x_7,x_1;7;8), B_{15} = (7,0,x_6,x_5,x_1;4;8)$
$L(\mathcal{B}) = \{(5,x_7)\}$.

$\underline{w = 17}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_8\}$ ($B_1, \cdots, B_{13}$ *are same as* $w = 16$)
$B_{14} = (0,x_1,x_5,x_8,x_7;8;3)$, $B_{15} = (x_2,x_5,x_4,x_8,x_3;x_1;7)$,
$B_{16} = (x_3,x_7,x_6,x_8,8;7;6)$, $B_{17} = (2,x_6,5,x_5,3;x_7;6)$
$B_{18} = (x_3,x_6,x_2,x_8,x_1;5;1)$, $B_{19} = (x_5,x_6,0,x_8,4;7;2)$
$L(\mathcal{B}) = \{(x_8,5),(5,x_4),(x_4,x_6)\}$.

$\underline{w = 18}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_9\}$ ($B_1, \cdots, B_{16}$ *are same as* $w = 17$)
$B_{17} = (x_4,5,x_9,3,x_6;6;x_5)$, $B_{18} = (5,x_2,x_8,x_9,x_6;1;x_3)$,
$B_{19} = (x_8,5,x_5,x_9,x_7;6;1)$, $B_{20} = (x_3,x_6,x_1,x_9,x_4;x_8;7)$,
$B_{21} = (x_5,x_6,0,x_9,2;7;8)$
$L(\mathcal{B}) = \{(2,x_8),(0,x_8),(x_8,4),(4,x_9),(x_6,4),(x_2,x_9)\}$.

$\underline{w = 19}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_{10}\}$ ($B_1, \cdots, B_{18}$ *are same as* $w = 18$)
$B_{12} = (x_2,x_7,4,x_6,5;8;1)$, $B_{13} = (3,x_6,x_2,x_{10},0;2;4)$
$B_{22} = (0,x_7,1,x_{10},7;5;6)$, $B_{23} = (x_6,2,x_5,x_{10},x_8;3;8)$
$B_{24} = (x_5,7,x_9,x_{10},x_6;x_4;x_3)$
$L(\mathcal{B}) = \{(x_7,x_{10}),(x_{10},x_1),(x_1,x_6)\}$.

$\underline{w=20}$: $X = Z_9 \cup \{x_1, x_2, \cdots, x_{11}\}$ ($B_1, \cdots, B_{22}$ *are same as* $w = 19$)
$B_{23} = (x_6, 2, x_5, x_{11}, x_8; 0; 1)$, $B_{24} = (x_8, x_{10}, x_1, x_{11}, x_7; 4; 5)$
$B_{25} = (x_5, x_{10}, 3, x_{11}, 8; 6; 7)$, $B_{26} = (x_5, 7, x_9, x_{11}, x_6; 2; x_{10})$
$B_{27} = (x_2, x_{11}, x_4, x_{10}, x_3; x_9; x_6)$
$L(\mathcal{B}) = \{(x_1, x_6)\}$.

Theorem 3.17 *There exist* $(v, G_{10}, 1)$*-OPD and* $(v, G_{10}, 1)$*-OCD for* $v \geq 8$. *And,* $p(7, G_{10}, 1) = 2$, $c(7, G_{10}, 1) = 4$.
Proof. By Theorem 2.1 and Lemma 3.15, 3.16, we can give the above conclusion. ■

4 Packings and Coverings for $\lambda > 1$

Lemma 4.1 *Given positive integers* v, λ, *and* μ. *Let* X *be a* v*-set.*

(1) Suppose there exist both a (v, G, λ)-OPD $= (X, \mathcal{D})$ (with leave-edge graph $L_\lambda(\mathcal{D})$) and a (v, G, μ)-OPD $= (X, \mathcal{E})$ (with leave-edge graph $L_\mu(\mathcal{E})$). If $|L_\lambda(\mathcal{D})| + |L_\mu(\mathcal{E})| = l_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-OPD and its leave-edge graph is just $L_\lambda(\mathcal{D}) \bigcup L_\mu(\mathcal{E})$;

(2) Suppose there exist both a (v, G, λ)-OCD $= (X, \mathcal{D})$ (with repeat-edge graph $R_\lambda(\mathcal{D})$) and a (v, G, μ)-OCD $= (X, \mathcal{E})$ (with repeat-edge graph $R_\mu(\mathcal{E})$). If $|R_\lambda(\mathcal{D})| + |R_\mu(\mathcal{E})| = r_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-OCD and its repeat-edge graph is just $R_\lambda(\mathcal{D}) \bigcup R_\mu(\mathcal{E})$;

(3) Suppose there exist both a (v, G, λ)-OPD $= (X, \mathcal{D})$ (with leave-edge graph $L_\lambda(\mathcal{D})$) and a (v, G, μ)-OCD $= (X, \mathcal{E})$ (with repeat-edge graph $R_\mu(\mathcal{E})$). If $L_\lambda(\mathcal{D}) \supset R_\mu(\mathcal{E})$ and $|L_\lambda(\mathcal{D})| - |R_\mu(\mathcal{E})| = l_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-OPD and its leave-edge graph is just $L_\lambda(\mathcal{D}) - R_\mu(\mathcal{E})$;

(4) Suppose there exist both a (v, G, λ)-OCD $= (X, \mathcal{D})$ (with repeat-edge graph $R_\lambda(\mathcal{D})$) and a (v, G, μ)-OPD $= (X, \mathcal{E})$ (with leave-edge graph $L_\mu(\mathcal{E})$). If $R_\lambda(\mathcal{D}) \supset L_\mu(\mathcal{E})$ and $|R_\lambda(\mathcal{D})| - |L_\mu(\mathcal{E})| = r_{\lambda+\mu}$, then there exists a $(v, G, \lambda+\mu)$-OCD and its repeat-edge graph is just $R_\lambda(\mathcal{D}) - L_\mu(\mathcal{E})$.

Lemma 4.2 *There exist* (v, G_i, λ)*-OPD and* (v, G_i, λ)*-OCD for* $v \equiv 2$, 6 $(mod\ 7)$ *and* $\lambda > 1$.
Proof. By Lemma 4.1, we have the following table:

λ	1	2	3	4	5	6	
l_λ	1	2	3	4	5	6	$(L_\lambda = L_1 + L_{\lambda-1})$
r_λ	6	5	4	3	2	1	$(R_\lambda = R_{\lambda-1} - L_1)$

,

where $L_1 = P_2$ and $R_1 = G_i - P_2$ by section 3. ■

Lemma 4.3 *There exist* (v, G_i, λ)*-OPD and* (v, G_i, λ)*-OCD for* $v \equiv 3$, 5 $(mod\ 7)$ *and* $\lambda > 1$.

Proof. By Lemma 4.1 and section 3, we have the following table:

λ	1	2	3	4	5	6
l_λ	3	6	2	5	1	4
L_λ	P_4	L_1+L_1	L_1-R_2	L_1+L_3	L_3-R_2	L_1+L_5
r_λ	4	1	5	2	6	3
R_λ	P_5	R_1-L_1	R_1+R_2	R_2+R_2	R_2+R_3	R_2+R_4

.

■

Lemma 4.4 *There exist (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $v \equiv 4$ (mod 7) and $\lambda > 1$.*
Proof. By Lemma 4.1, we have the following table:

λ	1	2	3	4	5	6	
l_λ	6	5	4	3	2	1	$(L_\lambda = L_{\lambda-1} - R_1)$
r_λ	1	2	3	4	5	6	$(R_\lambda = R_1 + R_{\lambda-1})$

,

where $L_1 = G_i - P_2$ and $R_1 = P_2$ by section 3. ■

Theorem 4.5 *There exist (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $\lambda > 1$ and any v.*
Proof. By Theorems 3.11, 3.14 and 3.17 and Lemmas 4.2, 4.3 and 4.4, we arrive at the conclusion. ■

Acknowledgements. The authors would like to thank the referees for their careful reading of the paper and helpful comments.

References

[1] Qingde Kang, Yanfang Zhang and Huijuan Zuo, *Packings and coverings of λK_v by k-circuits with one chord*, Discrete Mathematics, 279(2004), 287-315.

[2] Qingde Kang, Huijuan Zuo and Yanfang Zhang, *Decompositions of λK_v into k-circuits with one chord*, Australasian Journal of Combinatorics, 30(2004), 229-246.

[3] Zhiqin Wang and Qingde Kang, *Optimal packings and coverings of λK_v with graph $K_5 - P_5$*, Bulletin of the Institute of Combinatorics and its Applications, 41(2004), 22-41.

[4] Qingde Kang and Zhihe Liang, Optimal packings and coverings of λDK_v with k-circuits, JCMCC.

[5] Yinzhi Gao, Huijuan Zuo and Qingde Kang, *Decomositions for ten graphs with seven points, seven edges and an even-cycle*(in Chinese), Acta Mathematicae Applicatae Sinica, 27(2004), 646-658.

[6] Y. Roditty, Packing and Covering of the complete graphs with a graph G of four vertices or less, J. C. T. series A 34 (1983), 231-243.

[7] Y. Roditty, Packing and Covering of the complete graph, II : The trees of order six, Ars Combinatoria, 19 (1985), 81-94.

[8] Y. Roditty, Packing and Covering of the complete graphs with a graph G: The forest of order five, Int. J. Math. & Sci. Math. 9 (1986), 277-282.

[9] Y. Roditty, Packing and Covering of the complete graph, IV: The trees of order seven, Ars Combinatoria, 35 (1993), 33-64.

[10] Y. Roditty, Packing and Covering of the complete graph, V: The forests of order six and their multiple copies, Ars Combinatoria, 44 (1996), 55-64.

[11] Liang Zhihe and Kang Qingde, *Packings of the complete directed graphs with k-circuits*, Appl. Math. J. Chinese Univ., A13(1998), No.4, 486; B13(1998), No.4, 463-472.

[12] Kang Qingde and Wang Zhiqin, *Maximum $K_{2,3}$-packing and minimum $K_{2,3}$-covering designs of λK_v*, Journal of Mathematical Research and Exposition, 25(2005), 1-16.

[13] Shan Xiuling and Kang Qingde, *Construction of a class of graph designs*, Appl. Math. J. Chinese Univ.(A), 2005, 20(1), 97-102.

[14] Shan Xiuling and Kang Qingde, *Decompositions into even cycles with one chord*, Appl. Math. J. Chinese Univ.(A), 2005, 20(4), 489-498.

[15] Kang Qingde, Yuan Landang and Liu Shuxia, *Graph designs for all graphs with six vertices and eight edges*, Acta Mathematicae Applicatae Sinica, 21(2005), 469-484.

[16] Kang Qingde, Du Yanke and Tian Zihong, *Decomposition of λK_v into some graph with six vertices and seven edges*, Journal of Statistical Planning and Inference, 136(2006), 1394-1409.

[17] Shan Xiuling and Kang Qingde, *Decompositions of complete graph into $(2k-1)$-circles with one chord*, Journal of Mathematical Research and Exposition, 26(2006), 56-62.