Cyclic m-Cycle Systems of $K_{n,n}$ for $m \leq 30$*

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Abstract. Let $K_{n,n}$ denote the complete bipartite graph with n vertices in each part. In this paper, it is proved that there is no cyclic m-cycle system of $K_{n,n}$ for $m \equiv 2 \ (mod\ 4)$ and $n \equiv 2 \ (mod\ 4)$. As a consequence, necessary and sufficient conditions are determined for the existence of cyclic m-cycle systems of $K_{n,n}$ for all integers $m \leq 30$.

Keywords: (cyclic) m-cycle system, difference system.

MSC: 05C38

1 Introduction

An *m-cycle system* of a graph G is a set $\mathscr{B}$ of m-cycles in G whose edges partition the edge set of G. Let $K_{n,k}$ denote the complete bipartite graph with partite sets of sizes n and k. The existence problem for m-cycle systems of $K_{n,k}$ was completely settled in [7].

Theorem 1.1 *There exists an m-cycle system of $K_{n,k}$ if and only if m, n and k are even, $n, k \geq m/2$ and m divides nk.*

An *m-cycle system* $\mathscr{B}$ of a graph G with vertex set $\mathbb{Z}_v$ is *cyclic* if for each $B = (b_1, b_2, \cdots, b_m) \in \mathscr{B}$ we have $B+1 = (b_1+1, b_2+1, \cdots, b_m+1) \in \mathscr{B}$. In the sequel, any graph of order v will be considered as a graph with vertex set $\mathbb{Z}_v$. It is immediate to see that $2m$ must divide $v(v-1)$ if there exists an m-cycle system of complete graph K_v.

The existence problem for cyclic m-cycle systems of complete graphs has attracted much interest. For m even and $v \equiv 1 \ (mod\ 2m)$, cyclic m-cycle systems of K_v were constructed for $m \equiv 0 \ (mod\ 4)$ in [5] and for

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$m \equiv 2$ (*mod* 4) in [6]. For m odd and $v \equiv 1$ (*mod* $2m$), cyclic m-cycle systems of K_v were found in [3,1,4]. For $v \equiv m$ (*mod* $2m$), it is proved in [2] that there exists a cyclic m-cycle system of K_v for all $m \notin M$ but $m \neq 3$, where $M = \{p^e \mid p \text{ is prime}, e > 1\} \bigcup \{15\}$, and in [9] for all $m \in M$. It is proved in [3] that there exists a cyclic m-cycle system of complete k-partite graph $K_{k\times m}$ for any pair of odd integers (k, m) but $(k, m) \neq (3, 3)$. For the existence of cyclic m-cycle systems of $K_{n,n}$, necessary and sufficient conditions were given in [8] for the case $m \equiv 0$ (*mod* 4) and $m/4$ square-free and the case $m \equiv 2$ (*mod* 4) with $m \geq 6$ and m square-free.

In this paper, it is proved that there is no cyclic m-cycle system of $K_{n,n}$ for $m \equiv 2$ (*mod* 4) and $n \equiv 2$ (*mod* 4). As a consequence, necessary and sufficient conditions are determined for the existence of cyclic m-cycle systems of $K_{n,n}$ for all integers $m \leq 30$.

2 Preliminaries

The main method used in this paper is the *difference* method, which enables us to construct cyclic cycle systems in a quite effective way. First, we provide some basic definitions and related properties.

Definition 2.1 *The type of a cycle B is the cardinality of the set $\{j \in \mathbb{Z}_v \mid B = B + j\}$.*

If $B = (b_1, b_2, \cdots, b_m)$ is a m-cycle of type d, let ∂B denote the multiset $\{\pm(b_i - b_{i-1}) \mid i = 1, 2, \cdots, m/d\}$ where $b_m = b_0$.

Definition 2.2 *Let $\mathcal{F} = \{B_1, B_2, \cdots, B_l\}$ be a set of m-cycles and d_i be the type of B_i for $i = 1, 2, \cdots, l$. If each element in $\mathbb{Z}_v \backslash \{0\}$ appears exactly once in the multiset $\partial\mathcal{F} = \bigcup_i \partial B_i$, then $\mathcal{F}$ is called a (K_v, C_m)-difference system((K_v, C_m)-DS for short).*

Throughout this paper we let the two partite sets of $K_{n,n}$ be $\{0, 2, 4, \cdots, 2n-2\}$ and $\{1, 3, 5, \cdots, 2n-1\}$. Then we can define $(K_{n,n}, C_m)$-DS analogously.

Definition 2.3 *Let $\mathcal{F} = \{B_1, B_2, \cdots, B_l\}$ be a set of m-cycles and d_i be the type of B_i for $i = 1, 2, \cdots, l$. If each element in $\pm\{1, 3, 5, \cdots, n-1\}$ appears exactly once in the multiset $\partial\mathcal{F} = \bigcup_i \partial B_i$, then $\mathcal{F}$ is called a $(K_{n,n}, C_m)$-DS.*

It is obvious that (a, b) is an edge of $K_{n,n}$ if and only if $a - b = j$ for some $j \in \pm\{1, 3, 5, \cdots, n-1\}$. So we have the following assertion:

Proposition 2.4 *If $\mathcal{F} = \{B_1, B_2, \cdots, B_l\}$ is a $(K_{n,n}, C_m)$-DS, then the cycles $\{B_i + j \mid i = 1, 2, \cdots, l, j = 1, 2, \cdots, 2n/d_i\}$ form a cyclic m-cycle system of $K_{n,n}$ where d_i is the type of B_i.*

Let B be a m-cycle of type d and

$$(B) = \{B + j \mid j = 1, 2, \cdots, 2n/d\}.$$

Then (B) is called the *orbit* generated by B and B is called a *base cycle* of (B). It is clear that for giving a cyclic m-cycle system of $K_{n,n}$ it is enough to give a set $\mathcal{F}$ of representatives for the orbits of its cycles. Let B be a m-cycle in $\mathcal{F}$ and let (a, b) be an edge of $K_{n,n}$. Then the number of cycles in the orbit of B admitting (a, b) as an edge, is exactly equal to the number of times that $a - b$ appears in ∂B. It follows that $\mathcal{F}$ is a $(K_{n,n}, C_m)$-DS. So this allows us to state the following assertion:

Proposition 2.5 *There exists a cyclic m-cycle system of $K_{n,n}$ if and only if there exists a $(K_{n,n}, C_m)$-DS.*

Now we state three theorems proved in [8] for later use.

Theorem 2.6 *Let m, n be positive integers, $m \equiv 2 \ (mod\ 4)$ with $m \geq 6$. There exists a cyclic m-cycle system of $K_{n,n}$ for $n \equiv 0 \ (mod\ 2m)$.*

Theorem 2.7 *Let m, n be positive integers, $m \equiv 0 \ (mod\ 4)$ and $m/4$ is square-free. There exists a cyclic m-cycle system of $K_{n,n}$ if and only if $n \equiv 0, m/2, m$ or $3m/2 \ (mod\ 2m)$.*

Theorem 2.8 *Let m, n be positive integers, $m \equiv 2 \ (mod\ 4)$ with $m \geq 6$ and m is square-free. There exists a cyclic m-cycle system of $K_{n,n}$ if and only if $n \equiv 0 \ (mod\ 2m)$.*

3 Cyclic m-Cycle Systems of $K_{n,n}$ for $m \leq 30$

In this section, we first prove the nonexistence of cyclic m-cycle systems of $K_{n,n}$ for $m \equiv 2 \ (mod\ 4)$ and $n \equiv 2 \ (mod\ 4)$.

Theorem 3.1 *Let m, n be positive integers, $m \equiv 2 \ (mod\ 4)$. There is no cyclic m-cycle system of $K_{n,n}$ for $n \equiv 2 \ (mod\ 4)$.*

Proof. Suppose $\mathscr{B}$ is a cyclic m-cycle system of $K_{n,n}$, suppose $B = (c_0, c_1, \cdots, c_{m/d}, \cdots, c_{m-1})$ is an m-cycle in $\mathscr{B}$ of type d, then $c_{m/d} - c_0 = x \cdot 2n/d$ with $\gcd(x, d) = 1$. We show that d is odd. If d is even, since d divides m, we have $d \equiv 2 \ (mod\ 4)$ and m/d is odd. It follows by the oscillation of even and odd parities of labels along the cycle that $c_{m/d} - c_0$ is odd. It is a contradiction since $2n/d$ is even. Therefore d is odd.

Now, suppose $\mathcal{F}$ is a $(K_{n,n}, C_m)$-DS for $\mathscr{B}$ and the m-cycles in $\mathcal{F}$ have distinct types $d_1, d_2, \cdots, d_\tau$. Since d_i is odd for $i = 1, 2, \cdots, \tau$, we have $2m/d_i \equiv 0 \ (mod\ 4)$. Let x_i denote the number of base m-cycles of type d_i

for $i = 1, 2, \cdots, \tau$. For an m-cycle B of type d, ∂B contains $2m/d$ distinct differences. Then we have

$$\sum_{i=1}^{\tau} x_i \cdot 2m/d_i = n.$$

It is a contradiction since $n \equiv 2$ $(mod\ 4)$. Hence, for $m \equiv 2$ $(mod\ 4)$, there is no cyclic m-cycle system of $K_{n,n}$ for $n \equiv 2$ $(mod\ 4)$. □

Theorem 3.2 *For $m = 4, 8, 12, 20, 24, 28$, there exists a cyclic m-cycle systems of $K_{n,n}$ if and only if $n \equiv 0, m/2, m$ or $3m/2$ $(mod\ 2m)$.*

Proof. Since $m \equiv 0$ $(mod\ 4)$ and $m/4$ is square-free for $m = 4, 8, 12, 20, 24, 28$, the conclusion follows from Theorem 2.7. □

Theorem 3.3 *For $m = 6, 10, 14, 22, 26, 30$, there exists a cyclic m-cycle systems of $K_{n,n}$ if and only if $n \equiv 0$ $(mod\ 2m)$.*

Proof. Since $m \equiv 2$ $(mod\ 4)$ and m is square-free for $m = 6, 10, 14, 22, 26, 30$, the conclusion follows from Theorem 2.8. □

The existence question for cyclic 16-cycle systems of $K_{n,n}$ was settled in [8].

Theorem 3.4 *There exists a cyclic 16-cycle systems of $K_{n,n}$ if and only if $n \equiv 0, 8, 16$ or 24 $(mod\ 32)$.*

Lemma 3.5 *If there exists a cyclic 18-cycle system of $K_{n,n}$, then $n \equiv 0, 12$ or 24 $(mod\ 36)$.*

Proof. It follows from Theorem 1.1 that if there exists a 18-cycle system of $K_{n,n}$, then $n \equiv 0, 6, 12, 18, 24$ *or* 30 $(mod\ 36)$. And from Theorem 3.1 we have that there is no cyclic 18-cycle system of $K_{n,n}$ for $n \equiv 6, 18$ *or* 30 $(mod\ 2m)$. □

Lemma 3.6 *There exists a cyclic 18-cycle systems of $K_{n,n}$ for $n \equiv 12$ $(mod\ 36)$.*

Proof. We obtain a cyclic 18-cycle system of $K_{n,n}$ for $n \equiv 12$ $(mod\ 36)$ by constructing a $(K_{n,n}, C_{18})$-DS. Let $n = 36t + 12$, $t \geq 0$. Let

$$L = \pm\{1 + 12t, 3 + 12t, 5 + 12t, 7 + 12t, 9 + 12t, 11 + 12t\}$$

and $D = \pm\{1, 3, 5, \cdots, n-1\} \backslash L$. We first form t base cycles of type 1 for $t \geq 1$. Let $Y = (y_{i,h})$ be a $t \times 18$ matrix, and $B_i = (b_{i,1}, b_{i,2}, \cdots, b_{i,18})$

be defined by $b_{i,j} = \sum_{h=1}^{j} y_{i,h}$ for $i = 1, 2, \cdots, t$. Let H_1 and H_2 be $t \times 6$ matrices and

$$H_1 = \begin{pmatrix} -14t-13 & 26t+13 & -12t-13 & 24t+13 & -16t-13 & 28t+13 \\ -14t-15 & 26t+15 & -12t-15 & 24t+15 & -16t-15 & 28t+15 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -16t-11 & 28t+11 & -14t-11 & 26t+11 & -18t-11 & 30t+11 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} -18t-13 & 30t+13 & -20t-13 & 32t+13 & -22t-13 & 34t+13 \\ -18t-15 & 30t+15 & -20t-15 & 32t+15 & -22t-15 & 34t+15 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -20t-11 & 32t+11 & -22t-11 & 34t+11 & -24t-11 & 36t+11 \end{pmatrix}.$$

Case 1. *Suppose* $t \equiv 0 \ (mod\ 2)$. In this case, let

$$\begin{aligned} p_0 &= (-1, -7, -25, -31, \cdots, -1-12(t-2), -7-12(t-2))^T, \\ q_0 &= (13, 19, 37, 43, \cdots, 13+12(t-2), 19+12(t-2))^T, \\ p_1 &= (-3, -15, -27, -39, \cdots, -3-12(t-2), -15-12(t-2))^T, \\ q_1 &= (9, 21, 33, 45, \cdots, 9+12(t-2), 21+12(t-2))^T, \\ p_2 &= (-5, -17, -29, -41, \cdots, -5-12(t-2), -17-12(t-2))^T, \\ q_2 &= (11, 23, 35, 47, \cdots, 11+12(t-2), 23+12(t-2))^T, \end{aligned}$$

$$Y = \begin{pmatrix} p_0 & q_0 & p_1 & q_1 & p_2 & q_2 & H_1 & H_2 \end{pmatrix}.$$

It is not difficult to find that if r, s are even integers, then $b_{i,r} \not\equiv b_{i,s} \ (mod\ v)$ for $1 \le r < s \le 18$. Let S_i denote the multiset $\{b_{i,r} \mid 1 \le r \le 18 \ and\ r\ is\ odd\}$, then for $i \equiv 0 \ (mod\ 2)$ we have

$$\begin{aligned} S_i \quad = \quad & \{-7-12(i-2), -3-12(i-2), 1-12(i-2), \\ & 13-2i-14t, 13-2i, 13-2i+8t, \\ & 13-2i+18t, 13-2i+28t, 13-2i+38t\}, \end{aligned}$$

and for $i \equiv 1 \ (mod\ 2)$ we have

$$\begin{aligned} S_i \quad = \quad & \{-1-12(i-1), 9-12(i-1), 13-12(i-1), \\ & 13-2i-14t, 13-2i, 13-2i+8t, \\ & 13-2i+18t, 13-2i+28t, 13-2i+38t\}. \end{aligned}$$

It can be easily checked that S_i contains no repeated elements for $i = 1, 2, \cdots, t$. Hence, $\mathcal{F}(Y) = \{B_1, B_2, \cdots, B_t\}$ is a set of 18-cycles of type 1. Clearly, each element in D appears exactly once in $\partial\mathcal{F}(Y)$.

Case 2. *Suppose* $t \equiv 1 \ (mod\ 2)$ *with* $t \geq 3$. In this case, let

$$\begin{aligned}
p_0 &= (3, -13, -19, -37, -43, \cdots, -13 - 12(t-3), -19 - 12(t-3))^T, \\
q_0 &= (7, 25, 31, 49, 55, \cdots, 25 + 12(t-3), 31 + 12(t-3))^T, \\
p_1 &= (-5, -15, -27, -39, -51, \cdots, -15 - 12(t-3), -27 - 12(t-2))^T, \\
q_1 &= (11, 21, 33, 45, 57, \cdots, 21 + 12(t-3), 33 + 12(t-3))^T, \\
p_2 &= (-1, -17, -29, -41, -53, \cdots, -17 - 12(t-3), -29 - 12(t-3))^T, \\
q_2 &= (9, 23, 35, 47, 59, \cdots, 23 + 12(t-3), 35 + 12(t-3))^T,
\end{aligned}$$

$$Y = \begin{pmatrix} p_0 & q_0 & p_1 & q_1 & p_2 & q_2 & H_1 & H_2 \end{pmatrix}.$$

It is not difficult to find that if r, s are even integers, then $b_{i,r} \not\equiv b_{i,s} \ (mod\ v)$ for $1 \leq r < s \leq 18$. Let S_i denote the multiset $\{b_{i,r} \mid 1 \leq r \leq 18 \ and\ r\ is\ odd\}$, then for $i \equiv 0 \ (mod\ 2)$ we have

$$\begin{aligned}
S_i \ = \ & \{-13 - 12(i-2), -3 - 12(i-2), 1 - 12(i-2), \\
& 13 - 2i - 14t, 13 - 2i, 13 - 2i + 8t, \\
& 13 - 2i + 18t, 13 - 2i + 28t, 13 - 2i + 38t\},
\end{aligned}$$

for $i \equiv 1 \ (mod\ 2)$ with $i \geq 3$ we have

$$\begin{aligned}
S_i \ = \ & \{-19 - 12(i-3), -15 - 12(i-3), -11 - 12(i-3), \\
& 13 - 2i - 14t, 13 - 2i, 13 - 2i + 8t, \\
& 13 - 2i + 18t, 13 - 2i + 28t, 13 - 2i + 38t\},
\end{aligned}$$

and for $i = 1$ we have

$$S_1 \ = \ \{3, 5, 15, 11 - 14t, 11, 11 + 8t, 11 + 18t, 11 + 28t, 11 + 38t\}.$$

It can be easily checked that S_i contains no repeated elements for $i = 1, 2, \cdots, t$. Hence, $\mathcal{F}(Y) = \{B_1, B_2, \cdots, B_t\}$ is a set of 18-cycles of type 1. Clearly, each element in D appears exactly once in $\partial\mathcal{F}(Y)$.

Case 3. *Suppose* $t = 1$. In this case, let

$$U = \begin{pmatrix} 3 & 7 & -5 & 11 & -1 & 9 \end{pmatrix}.$$

$$W = \begin{pmatrix} -25 & 37 & -27 & 39 & -29 & 41 & -31 & 43 & -33 & 45 & -35 & 47 \end{pmatrix}.$$

$$Y = \begin{pmatrix} U & W \end{pmatrix}.$$

It can be easily checked that B_1 is an 18-cycle of type 1. Clearly, each element in D appears exactly once in $\partial\mathcal{F}(Y)$.

Then we form one base cycle of type 3 for $t \geq 0$. Let

$$\begin{aligned}
C \ = \ & (0, 12t + 1, 72t + 22, 12t + 5, 24t + 10, 12t - 1, \\
& 24t + 8, 36t + 9, 24t + 6, 36t + 13, 48t + 18, 36t + 7, \\
& 48t + 16, 60t + 17, 48t + 14, 60t + 21, 2, 60t + 15).
\end{aligned}$$

It can be easily checked that C is an 18-cycle of type 3 and each element in L appears exactly once in ∂C. Finally, $\mathcal{F} = \mathcal{F}(Y) \bigcup \{C\}$ is a $(K_{n,n}, C_{18})$-DS.
□

Lemma 3.7 *There exists a cyclic 18-cycle systems of $K_{n,n}$ for $n \equiv 24 \pmod{36}$.*

Proof. We obtain a cyclic 18-cycle system of $K_{n,n}$ for $n \equiv 24 \pmod{36}$ by constructing a $(K_{n,n}, C_{18})$-DS. Let $n = 36t + 24$, $t \geq 0$. Let

$$\begin{aligned} L \ = \pm \ & \{1+12t, 3+12t, 5+12t, 7+12t, 9+12t, 11+12t, \\ & 13+12t, 15+12t, 17+12t, 19+12t, 21+24t, 23+24t\} \end{aligned}$$

and $D = \pm\{1, 3, 5, \cdots, n-1\} \backslash L$. We first form t base cycles of type 1 for $t \geq 1$. Let $Y = (y_{i,h})$ be a $t \times 18$ matrix, and $B_i = (b_{i,1}, b_{i,2}, \cdots, b_{i,18})$ be defined by $b_{i,j} = \sum_{h=1}^{j} y_{i,h}$ for $i = 1, 2, \cdots, t$. Let H_1 and H_2 be $t \times 6$ matrices and

$$H_1 = \begin{pmatrix} -14t-21 & 26t+25 & -12t-21 & 24t+25 & -16t-21 & 28t+25 \\ -14t-23 & 26t+27 & -12t-23 & 24t+27 & -16t-23 & 28t+27 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -16t-19 & 28t+23 & -14t-19 & 26t+23 & -18t-19 & 30t+23 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} -18t-21 & 30t+25 & -20t-21 & 32t+25 & -22t-21 & 34t+25 \\ -18t-23 & 30t+17 & -20t-23 & 32t+27 & -22t-23 & 34t+27 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -20t-19 & 32t+23 & -22t-19 & 34t+23 & -24t-19 & 36t+23 \end{pmatrix}.$$

Case 1. *Suppose $t \equiv 0 \pmod 2$.* In this case, let

$$\begin{aligned} p_0 &= (-1, -7, -25, -31, \cdots, -1-12(t-2), -7-12(t-2))^T, \\ q_0 &= (13, 19, 37, 43, \cdots, 13+12(t-2), 19+12(t-2))^T, \\ p_1 &= (-3, -15, -27, -39, \cdots, -3-12(t-2), -15-12(t-2))^T, \\ q_1 &= (9, 21, 33, 45, \cdots, 9+12(t-2), 21+12(t-2))^T, \\ p_2 &= (-5, -17, -29, -41, \cdots, -5-12(t-2), -17-12(t-2))^T, \\ q_2 &= (11, 23, 35, 47, \cdots, 11+12(t-2), 23+12(t-2))^T, \end{aligned}$$

$$Y = \begin{pmatrix} p_0 & q_0 & p_1 & q_1 & p_2 & q_2 & H_1 & H_2 \end{pmatrix}.$$

It is not difficult to find that if r, s are even integers, then $b_{i,r} \not\equiv b_{i,s} \pmod v$ for $1 \leq r < s \leq 18$. Let S_i denote the multiset $\{b_{i,r} \mid 1 \leq r \leq 18 \text{ } and\, r \text{ } is \text{ } odd\}$, then for $i \equiv 0 \pmod 2$ we have

$$\begin{aligned} S_i \ = \ & \{-7-12(i-2), -3-12(i-2), 1-12(i-2), \\ & 5-2i-14t, 9-2i, 13-2i+8t, \\ & 17-2i+18t, 21-2i+28t, 25-2i+38t\}, \end{aligned}$$

and for $i \equiv 1 \pmod 2$ we have

$$\begin{aligned} S_i &= \{-1-12(i-1), 9-12(i-1), 13-12(i-1), \\ &\quad 5-2i-14t, 9-2i, 13-2i+8t, \\ &\quad 17-2i+18t, 21-2i+28t, 25-2i+38t\}. \end{aligned}$$

It can be easily checked that S_i contains no repeated elements for $i = 1, 2, \cdots, t$. Hence, $\mathcal{F}(Y) = \{B_1, B_2, \cdots, B_t\}$ is a set of 18-cycles of type 1. Clearly, each element in D appears exactly once in $\partial\mathcal{F}(Y)$.

Case 2. *Suppose* $t \equiv 1 \pmod 2$ *with* $t \geq 3$. In this case, let

$$\begin{aligned} p_0 &= (3, -13, -19, -37, -43, \cdots, -13-12(t-3), -19-12(t-3))^T, \\ q_0 &= (7, 25, 31, 49, 55, \cdots, 25+12(t-3), 31+12(t-3))^T, \\ p_1 &= (-5, -15, -27, -39, -51, \cdots, -15-12(t-3), -27-12(t-2))^T, \\ q_1 &= (11, 21, 33, 45, 57, \cdots, 21+12(t-3), 33+12(t-3))^T, \\ p_2 &= (-1, -17, -29, -41, -53, \cdots, -17-12(t-3), -29-12(t-3))^T, \\ q_2 &= (9, 23, 35, 47, 59, \cdots, 23+12(t-3), 35+12(t-3))^T, \end{aligned}$$

$$Y = \begin{pmatrix} p_0 & q_0 & p_1 & q_1 & p_2 & q_2 & H_1 & H_2 \end{pmatrix}.$$

It is not difficult to find that if r, s are even integers, then $b_{i,r} \not\equiv b_{i,s} \pmod v$ for $1 \leq r < s \leq 18$. Let S_i denote the multiset $\{b_{i,r} \mid 1 \leq r \leq 18 \text{ and } r \text{ is odd}\}$, then for $i \equiv 0 \pmod 2$ we have

$$\begin{aligned} S_i &= \{-13-12(i-2), -3-12(i-2), 1-12(i-2), \\ &\quad 5-2i-14t, 9-2i, 13-2i+8t, \\ &\quad 17-2i+18t, 21-2i+28t, 25-2i+38t\}, \end{aligned}$$

for $i \equiv 1 \pmod 2$ with $i \geq 3$ we have

$$\begin{aligned} S_i &= \{-19-12(i-3), -15-12(i-3), -11-12(i-3), \\ &\quad 5-2i-14t, 9-2i, 13-2i+8t, \\ &\quad 17-2i+18t, 21-2i+28t, 25-2i+38t\}, \end{aligned}$$

and for $i = 1$ we have

$$S_1 = \{3, 5, 15, 3-14t, 7, 11+8t, 15+18t, 19+28t, 23+38t\}.$$

It can be easily checked that S_i contains no repeated elements for $i = 1, 2, \cdots, t$. Hence, $\mathcal{F}(Y) = \{B_1, B_2, \cdots, B_t\}$ is a set of 18-cycles of type 1. Clearly, each element in D appears exactly once in $\partial\mathcal{F}(Y)$.

Case 3. *Suppose* $t = 1$. In this case, let

$$U = \begin{pmatrix} 3 & 7 & -5 & 11 & -1 & 9 \end{pmatrix}.$$

$$W = \begin{pmatrix} -33 & 49 & -35 & 51 & -37 & 53 & -39 & 55 & -41 & 57 & -43 & 59 \end{pmatrix}.$$

$$Y = (\ U \quad W\ ).$$

It can be easily checked that B_1 is an 18-cycle of type 1. Clearly, each element in D appears exactly once in $\partial\mathcal{F}(Y)$.

Then we form two base cycles of type 3 for $t \geq 0$. Let

$$\begin{aligned} C_1 &= (0, 12t+5, 24t+12, 36t+15, 24t+14, 36t+29, \\ &\quad 24t+16, 36t+21, 48t+28, 60t+31, 48t+30, 60t+45, \\ &\quad 48t+32, 60t+37, 72t+44, 12t-1, 72t+46, 12t+13), \\ C_2 &= (0, 12t+9, 24t+20, 36t+37, 24t+18, 48t+39, \\ &\quad 24t+16, 36t+25, 48t+36, 60t+53, 48t+34, 7, \\ &\quad 48t+32, 60t+41, 4, 12t+21, 2, 24t+23). \end{aligned}$$

It can be easily checked that C_1 and C_2 are 18-cycles of type 3 and each element in L appears exactly once in $\partial C_1 \bigcup \partial C_2$. Finally, $\mathcal{F} = \mathcal{F}(Y) \bigcup \{C_1, C_2\}$ is a $(K_{n,n}, C_{18})$-DS. □

From Lemmas 3.5-3.7 and Theorem 2.6, we finally have the following theorem:

Theorem 3.8 *There exists a cyclic* 18*-cycle systems of* $K_{n,n}$ *if and only if* $n \equiv 0, 12$ *or* 24 (*mod* 36).

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