Constructing regular graphs with smallest defining number

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Abstract

In a given graph G, a set S of vertices with an assignment of colors is a defining set of the vertex coloring of G, if there exists a unique extension of the colors of S to a $\chi(G)$-coloring of the vertices of G. A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality, the defining number, is denoted by $d(G, \chi)$. Let $d(n, r, \chi = k)$ be the smallest defining number of all r-regular k-chromatic graphs with n vertices. Mahmoodian et. al [7] proved that, for a given k and for all $n \geq 3k$, if $r \geq 2(k-1)$ then $d(n, r, \chi = k) = k - 1$. In this paper we show that for a given k and for all $n < 3k$ and $r \geq 2(k-1)$, $d(n, r, \chi = k) = k - 1$.

Keywords: regular graphs, colorings, defining sets, uniquely extendible colorings.

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1 Introduction

A k-coloring of a graph G is an assignment of k different colors to the vertices of G such that no two adjacent vertices receive the same color. The (vertex) chromatic number, $\chi(G)$, of a graph G is the minimum number k for which there exists a k-coloring for G. A graph G with $\chi(G) = k$ is called a k–chromatic graph. In a given graph G, a set of vertices S with an assignment of colors is called a defining set of vertex coloring, if there exists a unique extension of the colors of S to a $\chi(G)$-coloring of the vertices of G. A defining set with minimum cardinality is called a smallest defining set (of a vertex coloring) and its cardinality is the defining number, denoted by $d(G, \chi)$.

There are some results on defining numbers in [6] (see also [3], and [4]). Here we study the following concept. Let $d(n, r, \chi = k)$ be the smallest value of $d(G, \chi)$ for all r-regular k-chromatic graphs with n vertices. Note that for any graph G, we have $d(G, \chi) \geq \chi(G) - 1$, therefore $d(n, r, \chi = k) \geq k - 1$. By Brooks' Theorem [2], if G is a connected r-regular k-chromatic graph which is not a complete graph or an odd cycle, then $k \leq r$. For the case of $r = k$, Mahmoodian and Mendelsohn [5] determined the value of $d(n, k, \chi = k)$ for all $k \leq 5$. Mahmoodian and Soltankhah [8] determined this value for $k = 6$ and $k = 7$. Also in [8], for each k, the value of $d(n, k, \chi = k)$ is determined for some congruence classes of n. For the case of $k < r$, it is proved in [5] that, for each n and each $r \geq 4$, we have $d(n, r, \chi = 3) = 2$. The following question is raised in [5]:

Question. *Is it true that for every k, there exist $n_0(k)$ and $r_0(k)$, such that for all $n \geq n_0(k)$ and $r \geq r_0(k)$ we have $d(n, r, \chi = k) = k - 1$?*

Mahmoodian et. al. [7] proved that the answer to this question is positive and that, for a given k and all $n \geq 3k$, if $r \geq 2(k-1)$ then $d(n, r, \chi = k) = k - 1$.

We show the above statement for $n < 3k$. In fact we prove that:

Theorem. *Let k be a positive integer. For each $n < 3k$, if $r \geq 2(k-1)$ then $d(n, r, \chi = k) = k - 1$.*

2 Preliminaries

In this section, we state some known results and definitions which will be used in the sequel.

Definition 1 [5]. Let G and H be two graphs, each with a given proper k-coloring say c_G and c_H, (respectively) with k colors. Then the chromatic join of G and H, denoted by $G \overset{\chi}{\vee} H$ is a graph where $V(G \overset{\chi}{\vee} H)$ is $V(G) \cup V(H)$, and $E(G \overset{\chi}{\vee} H)$ is $E(G) \cup E(H)$, together with the set $\{xy \mid x \in V(G),\ y \in V(H)$ such that $c_G(x) \neq c_H(y)\}$.

Theorem A [5]. *Let n be a multiple of k, say $n = kl$ $(l \geq 2)$; then* $d(kl, 2(k-1),\ \chi = k) = k - 1$.

To prove this theorem Mahmoodian and Mendelsohn constructed a $2(k-1)$-regular k-chromatic graph with $n = kl$ vertices as follows. Let $G_1, G_2, \ldots, G_l$ be vertex disjoint graphs such that G_1 and G_l are two copies of K_k and if $l \geq 3$, $G_2, \cdots, G_{l-1}$ are copies of $\overline{K}_k$. Color each G_i with k colors $1, 2, \cdots, k$. Then construct a graph G with lk vertices by taking the union of $G_1 \cup G_2 \cup \ldots \cup G_l$, and by making a chromatic join between G_i and G_{i+1}; for $i = 1, 2, \cdots, l-1$. This is the desired graph. We denote such a graph by $G_{l(k)}$ and use this construction in Section 3.

Definition 2 [8]. Let G be a k-chromatic graph and let S be a defining set for G. Then a set $F(S)$ of edges is called nonessential edges, if the chromatic number of $G - F(S)$, the graph obtained from G by removing the edges in $F(S)$, is still k, and S is also a defining set for $G - F(S)$.

Remark 1. A necessary condition for the existence of an r-regular k-chromatic graph is $\frac{r}{k-1} \leq \frac{n}{k}$. For, if G is an r-regular k-chromatic graph with n vertices, then each chromatic class in G has at most $n - r$ vertices. Therefore $n \leq k(n-r)$. This implies $\frac{r}{k-1} \leq \frac{n}{k}$. Thus, for $r \geq 2(k-1)$ there are not any graph of order $n < 2k$. Hence when $r \geq 2(k-1)$, it is sufficient to investigate $d(n, r,\ \chi = k)$ only for $n \geq 2k$. Also it is obvious that n and r cannot be both odd.

For the definitions and notations not defined here we refer the reader to texts, such as [9].

3 Main results

In this section in the following four theorems we prove our main result, which was mentioned at the end of Section 1.

Theorem 1. *For each $k \geq 3$ and each $r \geq 2(k-1)$, we have* $d(3k-1, r,\ \chi = k) = k - 1$.

Proof. Let $n = 3k - 1$ and $r = 2(k-1) + t$. By Remark 1 it is obvious that $t \leq k-2$. First for $t = 0$, we construct a $2(k-1)$-regular k-chromatic graph H with n vertices and $d(H, \chi) = k - 1$ as follows. By Theorem A we have $d(3k, 2(k-1), \chi = k) = k - 1$. In graph $G_{3(l)}$ which was constructed to prove Theorem A, let $V(G_1) = \{u_1, u_2, ..., u_k\}$, $V(G_2) = \{v_1, v_2, ..., v_k\}$, and $V(G_3) = \{w_1, w_2, ..., w_k\}$. Also assume that $c(u_i) = c(v_i) = c(w_i) = i$, for $i = 1, 2, ..., k$. Note that the set of vertices adjacent to v_k is $N_{G_{3(l)}}(v_k) = \{u_1, ..., u_{k-1}\} \cup \{w_1, ..., w_{k-1}\}$. We delete the vertex v_k and join its neighbors in the following manner: we join u_i to w_{i+1} for $i = 1, 2, ..., k-2$ and u_{k-1} to w_1. It can be easily seen that the new graph, say H, is $2(k-1)$-regular k-chromatic with $n = 3k - 1$ vertices with a defining set $S = \{u_1, u_2, ..., u_{k-1}\}$.

Now for $1 \leq t \leq k-3$, to construct an r-regular k-chromatic graph, we consider the graph H, and we add the edges $u_i w_{i+j+2} \pmod{k}$, for $i = 1, ..., k$ and $j = 1, ..., t$, to H. Also, in the case of k odd, we add the edges of t mutually disjoint 1-factors of K_{k-1}, and in the case of k even, the edges of $\frac{t}{2}$ mutually disjoint 2-factors of K_{k-1}, on vertex set $\{v_1, ..., v_{k-1}\}$.

Note that if $t = k-2$ then such a graph does not exist. For, if G is a graph satisfying such conditions then we know that each chromatic class in G has at most 3 vertices. Since $n = 3k - 1$, G must have $k - 1$ chromatic classes of size 3 and one chromatic class of size 2. And each vertex in a chromatic class of size 3 must be adjacent to all other vertices. This implies that the degree of each vertex in the chromatic class of size 2 is $3(k-1) = r + 1$, which contradicts the r-regularity of the graph. ∎

Example 1. *In Figure 1 we show the graph H when $k = 5$ and $r = 8$. The vertices of the defining set are shown by the filled circles.*

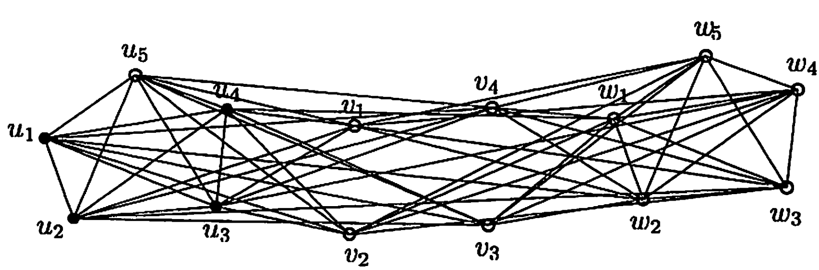

Figure 1: $d(H, \chi = 5) = 4$.

Theorem 2. *For each odd number $k \geq 3$, and each $2k \leq n \leq 3k-2$, we have $d(n, 2(k-1),\ \chi = k) = k-1$.*

Proof. By Theorem A we have $d(2k, 2(k-1),\ \chi = k) = k-1$. Let $n = 2k+s$, $s = 1, 2, ..., k-2$. We construct a $2(k-1)$-regular k-chromatic graph H_s with n vertices and defining number equals to $k-1$. For this, we consider graph $G_{2(l)}$ and add s new vertices to it, delete some suitable edges as follows and join the new vertices to the end vertices of deleted edges. In graph $G_{2(l)}$, for convenience let $V(G_1) = \{u_1, ..., u_i, ..., u_{\frac{k-1}{2}}, u_{1'}, ..., u_{i'}, ..., u_{(\frac{k-1}{2})'}, u_k\}$ and $V(G_2) = \{v_1, ..., v_i, ..., v_{\frac{k-1}{2}}, v_{1'}, ..., v_{i'}, ..., v_{(\frac{k-1}{2})'}, v_k\}$, where $i' = i + \frac{k-1}{2}$, $i = 1, 2, ..., \frac{k-1}{2}$; and $c(u_j) = c(v_j) = j$, for $j = 1, 2, ..., k$.

If $1 \leq s \leq \frac{k-1}{2}$ then denote new vertices by $x_1, ..., x_s$. Let $M_1, M_2, ..., M_{\frac{k-1}{2}}$ be mutually disjoint 1-factors of subgraph $< u_1, ..., u_i, ..., u_{\frac{k-1}{2}}, u_{1'}, ..., u_{i'}, ..., u_{(\frac{k-1}{2})'} >$ in $G_{2(l)}$ such that each edge in M_i has one end in $\{u_1, u_2, ..., u_{\frac{k-1}{2}}\}$ and the other end in $\{u_{1'}, ..., u_{(\frac{k-1}{2})'}\}$. For each i $(1 \leq i \leq s)$ we join x_i to each of the vertices of M_i, and delete all of the edges of M_i. Also with respect to each $u_a u_b \in M_i$, we delete the edge $v_a v_b$ and join x_i to the vertices v_a and v_b. Now it can be easily seen that $deg(x_i) = 2(k-1)$. Note that the new graph contains a complete subgraph say, $< u_1, u_2, ..., u_{\frac{k-1}{2}}, v_{1'}, ..., v_{(\frac{k-1}{2})'}, x_1 >= K_k$ and a defining set $S = \{u_1, ..., u_{k-1}\}$. Also the colors of vertices of $G_{2(l)}$ force all new vertices to be colored k.

If $\frac{k-1}{2} < s \leq k-2$ then we denote the new vertices by $x_1, x_2, ..., x_{\frac{k-1}{2}}, y_1, y_2, ..., y_{s-\frac{k-1}{2}}$. For x_i $(1 \leq i \leq \frac{k-1}{2})$ we proceed as before. For y_t $(1 \leq t \leq s - \frac{k-1}{2})$, first we recognize some nonessential edges in $H_{\frac{k-1}{2}}$. If for each i, we let z_i be either u_i or v_i and, for each j, we let w_j be either u_j or v_j, then the following edges form a nonessential set in $H_{\frac{k-1}{2}}$:

$$\begin{aligned} F = \quad & \{v_i v_j \mid 1 \leq i < j \leq \tfrac{k-1}{2}\} \cup \{u_{i'} u_{j'} \mid 1' \leq i' < j' \leq (\tfrac{k-1}{2})'\} \cup \\ & \{x_1 u_{i'} \textit{ or } x_1 v_i \mid 1 \leq i \leq \tfrac{k-1}{2}\} \cup \{x_i w_j \mid 2 \leq i \leq \tfrac{k-1}{2}, 1 \leq j \leq k-1\} \cup \\ & \{z_i v_k \mid 1 \leq i \leq k-1\}. \end{aligned}$$

There are two cases to be considered.

Case 1. $k = 4l + 1$.

In this case the induced subgraphs $A =< u_{1'}, u_{2'}, ..., u_{(\frac{k-1}{2})'} >$ and $B =< v_1, v_2, ..., v_{\frac{k-1}{2}} >$ are complete graphs $K_{\frac{k-1}{2}}$. So they are 1-factorable. Let $F_1, F_2, ..., F_{\frac{k-3}{2}}$ and $F'_1, F'_2, ..., F'_{\frac{k-3}{2}}$ be 1-factorizations of A and B, respectively, such that the edge $u_{t'}u_{(\frac{k-1}{2})'} \in F_t$ and $v_tv_{\frac{k-1}{2}} \in F'_t$. Now for each t $(1 \le t \le s - \frac{k-1}{2} \le \frac{k-3}{2})$ we delete all of the edges of $F_t \backslash \{u_{t'}u_{(\frac{k-1}{2})'}\}$ and $F'_t \backslash \{v_tv_{\frac{k-1}{2}}\}$. Also we delete the edges $u_tv_{\frac{k-1}{2}}$ and $u_{t'}v_k$. Finally we delete all the edges x_1v_t, $x_2u_{t+1}, \ldots, x_{\frac{k-1}{2}}u_{t+\frac{k-3}{2}}$ (mod $\frac{k-1}{2}$). We join y_t to the ends of all deleted edges. It can be easily seen that $deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $k-1$.

Case 2. $k = 4l + 3$.

In this case the induced subgraphs $A = < u_{1'}, u_{2'}, \ldots, u_{(\frac{k-1}{2})'}, u_k >$ and $B = < v_1, v_2, \ldots, v_{\frac{k-1}{2}}, v_k >$ are complete graphs $K_{\frac{k+1}{2}}$. Thus they are 1-factorable. Let $F_1, F_2, \ldots, F_{\frac{k-1}{2}}$ and $F'_1, F'_2, \ldots, F'_{\frac{k-1}{2}}$ be 1-factorizations of A and B, respectively, such that $u_{t'}u_k \in F_t$ and $v_tv_k \in F'_t$, for $1 \le t \le \frac{k-1}{2}$. Now for each t $(1 \le t \le s - \frac{k-1}{2} \le \frac{k-3}{2})$ we delete all of the edges of $F_t \backslash \{u_{t'}u_k\}$ and $F'_t \backslash \{v_tv_k\}$. Also we delete the edge v_ku_t. Finally we delete the edges x_1v_t, $x_2u_{t+1}, \ldots, x_{\frac{k-1}{2}}u_{t+\frac{k-3}{2}}$ (mod $\frac{k-1}{2}$). We join y_t to the ends of all deleted edges. It can be easily seen that $deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $t + \frac{k-1}{2}$. ∎

To illustrate the construction shown in the proof of Theorem 2, we provide the following example.

Example 2. Let $k = 7$. For $15 \le n \le 19$, we construct a 12-regular 7-chromatic graph of order n with a defining set of size 6. For $n = 14 + s$, $1 \le s \le 5$, we add s new vertices to the 12-regular 7-chromatic graph $G_{2(7)}$ of order 14 and delete some nonessential edges as explained in the proof of Theorem 2.

Table 1: New vertices and deleted edges.

New vertices	x_1	x_2	x_3	y_1	y_2
Deleted edges	$u_1u_{1'}$	$u_1u_{2'}$	$u_1u_{3'}$	$u_{2'}u_{3'}$	$u_{1'}u_{3'}$
	$u_2u_{2'}$	$u_2u_{3'}$	$u_2u_{1'}$	v_2v_3	v_1v_3
	$u_3u_{3'}$	$u_3u_{1'}$	$u_3u_{2'}$	x_1v_1	x_1v_2
	$v_1v_{1'}$	$v_1v_{2'}$	$v_1v_{3'}$	x_2u_2	x_2u_3
	$v_2v_{2'}$	$v_2v_{3'}$	$v_2v_{1'}$	x_3u_3	x_3u_1
	$v_3v_{3'}$	$v_3v_{1'}$	$v_3v_{2'}$	v_7u_1	v_7u_2

Table 1 gives all the deleted edges of $G_{2(7)}$ with respect to addition of new vertices. In Figure 2, we show the deleted edges and the added edges to construct a 12-regular 7-chromatic graph H_1 of order 15 ($s = 1$) with a defining set of size 6. The dotted lines are the deleted edges and the vertices of the defining set are shown by the filled circles.

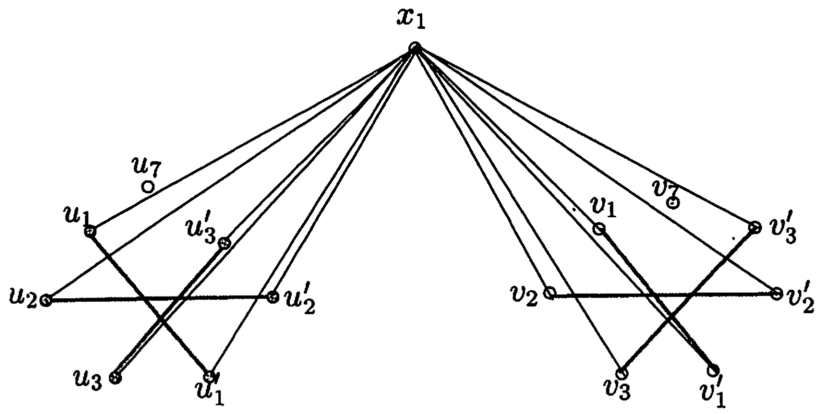

Figure 2: $d(H_1, \chi = 7) = 6$.

Theorem 3. *For each even number $k \geq 4$, and each $2k \leq n \leq 3k-2$, we have $d(n, 2(k-1),\ \chi = k) = k-1$.*

Proof. By Theorem A we have $d(2k, 2(k-1),\ \chi = k) = k-1$. For $n = 2k+s$, $s = 1, 2, ..., k-2$, we construct a $2(k-1)$-regular k-chromatic graph H_s with n vertices and defining number equal to $k-1$.
To construct H_s, we consider graph $G_{2(k)}$ and add s new vertices to it, delete some suitable edges and join the new vertices to the end vertices of the deleted edges as follows. In graph $G_{2(k)}$ for convenience let $V(G_1) = \{u_1, ..., u_i, ..., u_{\frac{k}{2}}, u_{1'}, ..., u_{i'}, ..., u_{(\frac{k}{2})'}\}$ and $V(G_2) = \{v_1, ..., v_i, ..., v_{\frac{k}{2}}, v_{1'}, ..., v_{i'}, ..., v_{(\frac{k}{2})'}\}$, where $i' = i + \frac{k}{2}$, $i = 1, 2, ..., \frac{k}{2}$; and $c(u_j) = c(v_j) = j$, for $j = 1, 2, ..., k$.

If $1 \leq s \leq \frac{k}{2} - 1$ then we denote the new vertices by $x_1, ..., x_s$. Let $M_1, M_2, ..., M_{\frac{k}{2}}$ be mutually disjoint 1-factors of the induced subgraph $G_1 = < u_1, ..., u_i, ..., u_{\frac{k}{2}}, u_{1'}, ..., u_{i'}, ..., u_{(\frac{k}{2})'} >$, where, for $i = 1, 2, ..., \frac{k}{2}$;

$$M_i = \{u_1 u_{i'}, u_2 u_{(i+1)'}, ..., u_t u_{(i+t-1)'}, ..., u_{\frac{k}{2}} u_{(i+\frac{k}{2}-1)'}\} \pmod{\frac{k}{2}}.$$

Also let $M'_1, M'_2, ..., M'_{\frac{k}{2}}$ be mutually disjoint 1-factors of the induced subgraph $G_2 =< v_1, ..., v_i, ..., v_{\frac{k}{2}}, v_{1'}, ..., v_{i'}, ..., v_{(\frac{k}{2})'} >$, where, for $i = 1, 2, ..., \frac{k}{2}$;

$$M'_i = \{v_1 v_{i'}, v_2 v_{(i+1)'}, ..., v_t v_{(i+t-1)'}, ..., v_{\frac{k}{2}} v_{(i+\frac{k}{2}-1)'}\} \pmod{\frac{k}{2}}.$$

Now for each i ($i = 1, 2, ..., s$) we delete all of the edges of $M_{i+1} \backslash \{u_{\frac{k}{2}-i} u_{(\frac{k}{2})'}\}$, and all of the edges of $M'_i \backslash \{v_{\frac{k}{2}-i+1} v_{(\frac{k}{2})'}\}$. Finally we delete the edge $u_{\frac{k}{2}-i} v_{\frac{k}{2}-i+1}$. We join x_i to the ends of all deleted edges. Now it can be easily seen that $deg(x_i) = 2(k-1)$. Note that the new graph contains a complete subgraph say $< u_1, u_2, ..., u_{\frac{k}{2}}, u_{(\frac{k}{2})'}, v_{1'}, ..., v_{(\frac{k}{2}-1)'} >= K_k$ and a defining set $S = \{u_1, ..., u_{k-1}\}$. Also the colors of vertices of $G_{2(k)}$ force the colors of all new vertices to be k.

If $\frac{k}{2} \leq s \leq k-2$ then we denote the new vertices by $x_1, x_2, ..., x_{\frac{k}{2}-1}, y_1, y_2, ..., y_{s-\frac{k}{2}+1}$. For x_i ($1 \leq i \leq \frac{k}{2} - 1$) we treat as before. For y_t ($1 \leq t \leq s - \frac{k}{2} + 1$) first we recognize some nonessential edges in $H_{\frac{k}{2}-1}$. If for each j, we let w_j be either u_j or v_j, then the following edges form a nonessential set in $H_{\frac{k}{2}-1}$:

$$\begin{aligned} F = \quad & \{v_i v_j \mid 1 \leq i < j \leq \tfrac{k}{2}, j \neq i+1\} \cup \{u_{i'} u_{j'} \mid 1' \leq i' < j' \leq (\tfrac{k}{2})' - 1\} \cup \\ & \{x_i w_j \mid 1 \leq i \leq \tfrac{k}{2} - 1, 1 \leq j \leq k-1\} \cup \{v_i v_{(\frac{k}{2})'} \mid 1 \leq i \leq (\tfrac{k}{2})' - 1\} \cup \\ & M_1 \backslash \{u_{\frac{k}{2}} u_{(\frac{k}{2})'}\} \cup M'_{\frac{k}{2}}. \end{aligned}$$

There are two cases to be considered.

Case 1. $k = 4l$.

In this case the induced subgraphs $A =< u_{1'}, u_{2'}, ..., u_{(\frac{k}{2})'} >$ and $B =< v_1, v_2, ..., v_{\frac{k}{2}} >$ are complete graphs $K_{\frac{k}{2}}$. So they are 1-factorable. Let $F_1, F_2, ..., F_{\frac{k}{2}-1}$ and $F'_1, F'_2, ..., F'_{\frac{k}{2}-1}$ be standard 1-factorizations (see [1], page 166) of A and B, respectively, such that the edges $u_{t'} u_{(\frac{k}{2})'} \in F_t$ and $v_t v_{\frac{k}{2}} \in F'_t$. Now for each t ($1 \leq t \leq s - \frac{k}{2} + 1 \leq \frac{k}{2} - 1$) we delete all of the edges of $F_t \backslash \{u_{t'} u_{(\frac{k}{2})'}\}$ and F'_t. Also we delete the edge $v_{(t+1)'} v_{(\frac{k}{2})'}$ (mod $(\frac{k}{2} - 1)$). If there exist some edges such as $v_i v_{i+1} \in F'_t$, then instead of these edges we delete the edges $v_{i'} v_{i+1} \in M'_{\frac{k}{2}}$.

Also for an arbitrary index i of such as edges $v_i v_{i+1}$ we delete the edge $v_i v_{(\frac{k}{2})'}$ instead of the edge $v_{(t+1)'} v_{(\frac{k}{2})'}$. Finally we delete the edges $x_1 u_{t+1}$, $x_2 u_{t+2}, ..., x_{\frac{k}{2}-1} u_{t+\frac{k}{2}-1}$ (mod $\frac{k}{2}$).

We join y_t to the ends of all deleted edges. It can be easily seen that $deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $t+\frac{k}{2}$, for $t \neq \frac{k}{2}-1$ and the color of $y_{\frac{k}{2}-1}$ to be $\frac{k}{2}-1$.

Case 2. $k = 4l+2$.

In this case the induced subgraphs $A =< u_{1'}, u_{2'}, ..., u_{(\frac{k}{2})'}, u_1 >$ and $B =< v_1, v_2, ..., v_{\frac{k}{2}}, v_{(\frac{k}{2})'} >$ are complete graphs $K_{\frac{k}{2}+1}$. So they are 1-factorable. Let $F_1, F_2, ..., F_{\frac{k}{2}}$ and $F'_1, F'_2, ..., F'_{\frac{k}{2}}$ be 1-factorizations of A and B, respectively, such that $u_1u_{t'} \in F_t$ and $v_tv_{(\frac{k}{2})'} \in F'_t$. Now for each t $(1 \leq t \leq s-\frac{k}{2}+1 \leq \frac{k}{2}-1)$ we delete all of the edges of $F_t \backslash \{u_1u_{t'}, u_{j'}u_{(\frac{k}{2})'}\}$ and F'_t. Also we delete the edge $u_ju_{j'} \in M_1$. If there exist some edges such as $v_iv_{i+1} \in F'_t$ then instead of the edges v_iv_{i+1} we delete the edges $v_{i'}v_{i+1} \in M'_{\frac{k}{2}}$. Finally we delete the edges $x_1u_{j+1}, x_2u_{j+2}, ..., x_{\frac{k}{2}-1}u_{j+\frac{k}{2}-1}$ (mod $\frac{k}{2}$). We join y_t to the ends of all deleted edges. It can be easily seen that $deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $t+\frac{k}{2}$. ∎

To illustrate the construction shown in the proof of Theorem 3, we provide the following example.

Example 3. Let $k = 8$. For $17 \leq n \leq 22$, we construct a 14-regular 8-chromatic graph of order n with a defining set of size 7. For $n = 16+s$, $1 \leq s \leq 6$, we add s new vertices to the 14-regular 8-chromatic graph $G_{2(8)}$ of order 16 and delete some nonessential edges as explained in the proof of Theorem 3.

Table 2: New vertices and deleted edges.

New vertices	x_1	x_2	x_3	y_1	y_2	y_3
Deleted edges	$u_1u_{2'}$	$u_1u_{3'}$	$u_2u_{1'}$	$u_{2'}u_{3'}$	$u_{1'}u_{3'}$	$u_{1'}u_{2'}$
	$u_2u_{3'}$	$u_3u_{1'}$	$u_3u_{2'}$	v_1v_4	v_2v_4	$v_{3'}v_4$
	$u_4u_{1'}$	$u_4u_{2'}$	$u_4u_{3'}$	$v_{2'}v_3$	v_1v_3	$v_{1'}v_2$
	$v_1v_{1'}$	$v_1v_{2'}$	$v_1v_{3'}$	$v_2v_{4'}$	$v_{3'}v_{4'}$	$v_1v_{4'}$
	$v_2v_{2'}$	$v_2v_{3'}$	$v_3v_{1'}$	x_1u_2	x_1u_3	x_1u_4
	$v_3v_{3'}$	$v_4v_{1'}$	$v_4v_{2'}$	x_2u_3	x_2u_4	x_2u_1
	u_3v_4	u_2v_3	u_1v_2	x_3u_4	x_3u_1	x_3u_2

Table 2 gives all the deleted edges of $G_{2(8)}$ with respect to addition of new vertices. In Figure 3, we show the deleted edges and the added edges to construct a 14-regular 8-chromatic graph H_1 of order 17 ($s = 1$) with a defining set of size 7. The dotted lines are the deleted edges and the vertices of the defining set are shown by the filled circles.

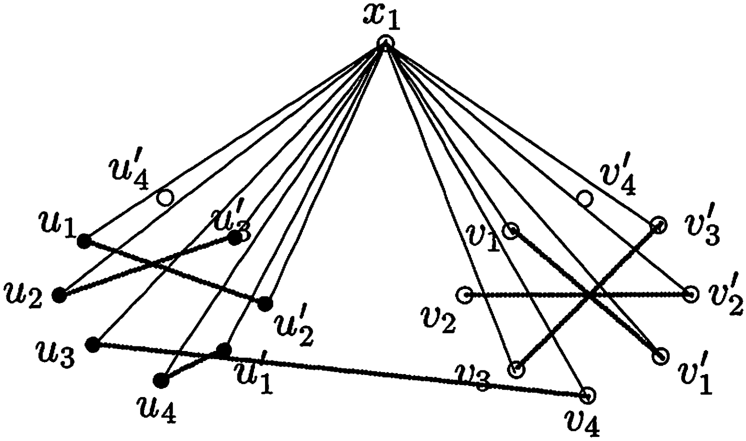

Figure 3: $d(H_1, \chi = 8) = 7$.

Theorem 4. *For each $k \geq 4$, $2k \leq n \leq 3k-2$, and $r > 2(k-1)$, we have*

$$d(n, r, \ \chi = k) = k - 1.$$

Proof. Let $n = 2k+s$, $0 \leq s \leq k-2$, and $r = 2(k-1)+t$. By Remark 1, if there exists an r-regular k-chromatic graph with n vertices then it is obvious that $t < s$. We construct an r-regular k-chromatic graph H with n vertices in the following manner.
Consider graph $G_{2(k)}$, let $V(G_1) = \{u_1, ..., u_k\}$ and $V(G_2) = \{v_1, ..., v_k\}$, and $c(u_i) = c(v_i) = i$, for $i = 1, 2, ..., k$. We add s new vertices say $x_1, ..., x_s$ to $G_{2(k)}$. For each x_i $(1 \leq i \leq s)$ we join x_i to each vertex of $V(G_1) \cup V(G_2) \backslash \{u_i, v_i\}$. Also, in the case of s even, we add the edges of t mutually disjoint 1-factors of K_s, and in the case of s odd, the edges of $\frac{t}{2}$ mutually disjoint 2-factors of K_s, to $x_1, ..., x_s$. The graph obtained in this way, say H', is a k-chromatic graph with n vertices and a defining set $S = \{x_2, ..., x_s, v_{s+1}, ..., v_k\}$ such that $deg(x_i) = 2(k-1)+t$ $(1 \leq i \leq s)$, $deg(u_i) = deg(v_i) = 2(k-1)+s-1$ $(1 \leq i \leq s)$, and $deg(u_i) = deg(v_i) = 2(k-1)+s$ $(s+1 \leq i \leq k)$. Now we show that by deleting some suitable nonessential edges of H' the desired r-regular graph H can be obtained.
In the graph H', for convenience let $A = \{u_1, ..., u_{\lfloor \frac{s}{2} \rfloor}\}$, $C = \{u_{\lfloor \frac{s}{2} \rfloor+1}, ..., u_s\}$,

$D = \{u_{s+1}, \ldots, u_{s+\lfloor \frac{k-s}{2} \rfloor}\}$, and $B = \{u_{s+\lfloor \frac{k-s}{2} \rfloor+1}, \ldots, u_k\}$. Also let $A' = \{v_1, \ldots, v_{\lfloor \frac{s}{2} \rfloor}\}$, $C' = \{v_{\lfloor \frac{s}{2} \rfloor+1}, \ldots, v_s\}$, $D' = \{v_{s+1}, \ldots, v_{s+\lfloor \frac{k-s}{2} \rfloor}\}$, and $B' = \{v_{s+\lfloor \frac{k-s}{2} \rfloor+1}, \ldots, v_k\}$. Let $i' = i + \lfloor \frac{k-s}{2} \rfloor$ for $s+1 \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor$.

First we delete a maximal matching of each complete bipartite subgraph with parts B and D of G_1 and parts B' and D' of G_2. For $k-s$ odd, we assume u_{k-1} and v_k to be vertices unsaturated by the maximal matchings. Then we delete the edge $u_{k-1}v_k$.
Secondly, we delete the edges of $s-t-1$ mutually disjoint maximal matchings of each complete bipartite subgraph with parts $A \cup B$ and $C \cup D$ of G_1 and parts $A' \cup B'$ and $C' \cup D'$ of G_2. For k odd, we assume that the following vertices are unsaturated by the maximal matchings: $\{u_1, \ldots, u_{\lfloor \frac{s}{2} \rfloor}, u_{(s+1)'}, \ldots, u_{(s+1)'+s-t-2-\lfloor \frac{s}{2} \rfloor}\}$ and $\{v_2, \ldots, v_{\lfloor \frac{s}{2} \rfloor}, v_1, v_{(s+1)'+1}, \ldots, v_{(s+1)'+s-t-1-\lfloor \frac{s}{2} \rfloor}\}$, in the case of s even, or $\{u_{\lfloor \frac{s}{2} \rfloor+1}, \ldots, u_{\lfloor \frac{s}{2} \rfloor+s-t-1}\}$ and $\{v_{\lfloor \frac{s}{2} \rfloor+2}, \ldots, v_s, v_{\lfloor \frac{s}{2} \rfloor+1}, v_{s+2}, \ldots, v_{2s-t-1-\lfloor \frac{s}{2} \rfloor}\}$, in the case of s odd. Then we delete the edges $u_1v_2, u_2v_3, \ldots, u_{\lfloor \frac{s}{2} \rfloor-1}v_{\lfloor \frac{s}{2} \rfloor}, u_{\lfloor \frac{s}{2} \rfloor}v_1, u_{(s+1)'}v_{(s+1)'+1}, \ldots, u_{(s+1)'+s-t-2-\lfloor \frac{s}{2} \rfloor}v_{(s+1)'+s-t-1-\lfloor \frac{s}{2} \rfloor}$, or the edges $u_{\lfloor \frac{s}{2} \rfloor+1}v_{\lfloor \frac{s}{2} \rfloor+2}, \ldots, u_sv_{\lfloor \frac{s}{2} \rfloor+1}, u_{s+1}v_{s+2}, \ldots, u_{\lfloor \frac{s}{2} \rfloor+s-t-1}v_{2s-t-1-\lfloor \frac{s}{2} \rfloor}$, depending on the parity of s, respectively.
If $s-t > \lfloor \frac{k}{2} \rfloor$ then in the second step we delete $\lfloor \frac{k}{2} \rfloor - 1$ maximal matchings. Finally we delete the edges of $s-t-\lfloor \frac{k}{2} \rfloor$ mutually disjoint 1-factors F_j ($1 \leq j \leq s-t-\lfloor \frac{k}{2} \rfloor$) of bipartite subgraph with parts $C \cup D$ and $C' \cup D'$, where

$$\begin{aligned} F_j &= \{u_iv_{i+j+1} \mid \lfloor \tfrac{s}{2} \rfloor + 1 \leq i \leq s + \lfloor \tfrac{k-s}{2} \rfloor - j - 1\} \cup \\ & \quad \{u_iv_{i-i_0+\lfloor \frac{s}{2} \rfloor+1} \mid i_0 = s + \lfloor \tfrac{k-s}{2} \rfloor - j \leq i \leq s + \lfloor \tfrac{k-s}{2} \rfloor\}. \end{aligned}$$

In fact if we consider the order $u_{\lfloor \frac{s}{2} \rfloor+1}, \ldots, u_s, u_{s+1}, \ldots, u_{s+\lfloor \frac{k-s}{2} \rfloor}$, and $v_{\lfloor \frac{s}{2} \rfloor+1}, \ldots, v_s, v_{s+1}, \ldots, v_{s+\lfloor \frac{k-s}{2} \rfloor}$, for the vertices in $C \cup D$ and $C' \cup D'$, respectively, then each 1-factor F_j contains the edges in which the ith vertex in $C \cup D$ is matched with $(i+j+1)$th vertex $\pmod{|C \cup D|}$ in $C' \cup D'$. (See Figure 4.)

Also for decreasing the degree of vertex sets $A \cup B$ and $A' \cup B'$, we delete the edges of $s-t-\lfloor \frac{k}{2} \rfloor$ mutually disjoint 1-factors F'_j ($1 \leq j \leq s-t-\lfloor \frac{k}{2} \rfloor$) of bipartite subgraph with parts $A \cup B$ and $A' \cup B'$ the same as above. Therefore the graph H obtained in this way contains a complete subgraph say $K_k = < A \cup B \cup C' \cup D' >$ and H is an r-regular graph.

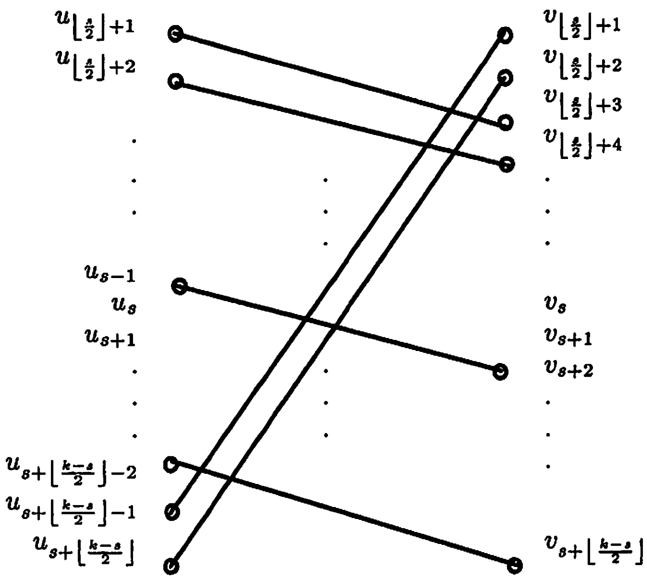

Figure 4: 1-factor F_1.

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Acknowledgments

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