Circumferences of 2-connected quasi-claw-free graphs

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Abstract

A graph G is quasi-claw-free if it satisfies the property: $d(x, y) = 2 \Rightarrow$ there exists $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. In this paper, we prove that the circumference of a 2-connected quasi-claw-free graph G on n vertices is at least $\min\{3\delta + 2, n\}$ or $G \in \mathcal{F}$, where $\mathcal{F}$ is a class of nonhamiltonian graphs of connectivity 2. Moreover, we prove that if $n \leq 4\delta$, then G is hamiltonian or $G \in \mathcal{F}$.

Keywords: Circumference; claw-free graph; quasi-claw-free graph

1 Introduction

Throughout this paper, we consider only finite, undirected and simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Throughout we use n for $|V|$ and we use δ for the minimum degree of vertices of G. The open and closed neighborhoods of a vertex u are denoted by $N(u) = \{x \in V : xu \in E\}$ and $N[u] = \{u\} \cup N(u)$, respectively. For each pair of vertices a, b at distance 2 we set $J(a, b) = \{u \in N(a) \cap N(b) : N[u] \subseteq N[a] \cup N[b]\}$. A graph is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. In [1], Ainouche introduced the concept of quasi-claw-free graphs, and extended many known results on claw-free graphs to quasi-claw-free graphs. A graph is *quasi-claw-free*, if every pair of vertices x, y at distance 2 satisfies the condition $J(x, y) \neq \emptyset$. Note that a claw-free graph is quasi-claw-free, but a quasi-claw-free graph is not necessarily $K_{1,r}$-free for $r \geq 3$. There exist infinite classes of quasi claw-free graphs which contain many induced $K_{1,r}$ for $r \geq 3$ (see [1]).

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The circumference $c(G)$ of a graph G is the length of a longest cycle in G. For terminology and notation not defined here see [2].

To state our result, we introduce a class of graphs. Let $\mathcal{K}_3$ denote the set of all graphs consisting of three disjoint complete graphs, where each of the components has order at least 3. The class $\mathcal{F}$ is the set of all spanning subgraphs that can be obtained from a graph G in $\mathcal{K}_3$ by adding the edges of two triangles between two disjoint triple of vertices, each containing one vertex of each component of G. A graph is called 3-cyclable, if any three vertices lie on a common cycle. Note that $\mathcal{F}$ is one of the three classes of 2-connected and not 3-cyclable graphs, which were first characterized by Watkins and Mensner [9].

In [5], M. Li proved the following result for 2-connected claw-free graphs.

Theorem 1 ([5]) *Every* 2*-connected claw-free graph* G *on* n *vertices contains a cycle of length at least* $\min\{3\delta+2, n\}$ *or* $G \in \mathcal{F}$. *Moreover, if* $n \leq 4\delta$, *then* G *is hamiltonian or* $G \in \mathcal{F}$.

Note that the second part of Theorem 1 was also obtained independently by H. Li [4] . In [7] R. Li obtained the following result for 2-connected quasi-claw-free graphs.

Theorem 2 ([7]) *Let* G *be a* 2*-connected quasi-claw-free graph on* n *vertices. If* $n \leq 4\delta$, *then* G *is hamiltonian or* $G \in \mathcal{F}$.

In this paper, we consider the circumferences of 2-connected quasi claw-free graphs and obtain the following Theorem 3, the proof of it will be given in Section 3.

Theorem 3 *Every* 2*-connected quasi-claw-free graph* G *on* n *vertices contains a cycle of length at least* $\min\{3\delta+2, n\}$ *or* $G \in \mathcal{F}$. *Moreover, if* $n \leq 4\delta$, *then* G *is hamiltonian or* $G \in \mathcal{F}$.

2 Notation and lemmas

For subgraphs H, K of G, let $G-H$ denote the subgraph of G which is induced by $V(G) \setminus V(H)$, and let $N_K(H)$ denote the set of vertices in K that are adjacent to some vertex in H. Moreover, we abbreviate $N(K) := N_{G-K}(K)$. In particular, if K consists of one vertex v, we omit the brackets, and we use $d_H(v) = |N_H(v)|$ and $d(v) = |N(v)|$. If also H and K are edge-disjoint, we use $e(H; K)$ to denote the set of edges between H and K. In case when $V(H) = \{v_1, \ldots, v_s\}$ we write $e(v_1, \ldots, v_s; K)$ instead of $e(\{v_1, \ldots, v_s\}; K)$.

Given a cycle C with a fixed cyclic orientation and vertices $x, y \in V(C)$, we use $C[x, y]$, $C[x, y)$, $C(x, y]$ and $C(x, y)$ to denote the corresponding subpaths between x and y of C, respectively including both x and y (with possibly $x = y$), only x or only y (if x and y are distinct), and none of x and y (if there is at least a vertex between x and y on C). For a vertex $x \in V(C)$ we use x^+ and x^- to denote the successor and the predecessor of x on C, respectively. Moreover, $x^{++} := (x^+)^+$ and $x^{--} := (x^-)^-$ etc. If $Z \subseteq V(C)$, then $Z^+ = \{u^+ \mid u \in Z\}$ and $Z^- = \{u^- \mid u \in Z\}$. A path Q, which has its end vertices on C and openly disjoint from C, is called a C-chord. We use $Q_H[x, y]$ to denote a C-chord with end vertices x, y on C and all inner vertices in a component H of $G - C$.

For vertices a and b in a connected graph G, let $L_G(a, b)$ be the length of a longest (a, b)-path P in G, i.e., $L_G(a, b) = |P| - 1$. If G is connected with $n \geq 2$, we set $D(G) = \min\{L_G(a, b) : a, b \in V(G), a \neq b\}$. Obviously, for each pair of distinct vertices x, y of a connected graph G with $n \geq 2$, we have a path $P[x, y]$ such that $|P[x, y]| \geq D(G) + 1$. For $|G| = 1$ set $D(G) = 0$.

Next we present some lemmas.

Lemma 1 *Let G be a 2-connected quasi-claw-free graph and C a longest cycle in G with a fixed cyclic orientation. Let H be a component of $G - C$ and $N(H) = \{x_1, \ldots, x_s\}$ in order around C.*
(a) $x_i^- x_i^+ \in E$ for $i = 1, \ldots, s$.
(b) $e(x_i^+, x_i^{++}; x_j, x_j^+, x_j^{++}) = e(x_i^-, x_i^{--}; x_j, x_j^-, x_j^{--}) = \emptyset$ for any pair of distinct elements x_i, x_j of $N(H)$.
(c) $N(H) \cap N(K) = \emptyset$ for any pair of distinct components H and K of $G - C$.
(d) Let $u \in N_H(x_j)$, $v \in N_H(x_{j+1})$ for some $x_j \in N(H)$. Then $|C(x_j, x_{j+1}]| \geq 4 + L_H(u, v)$.
(e) Let K be a component of $G - C$ other than H and let $Q = Q_K[z_j, z_k]$ be a C-chord joining $C(x_j^+, x_{j+1}^-)$ and $C(x_k^+, x_{k+1}^-)$, where x_j, x_k are distinct elements of $N(H)$, $z_j \in C(x_j^+, x_{j+1}^+)$, $z_k \in C(x_k^+, x_{k+1}^+)$, and all the internal vertices of Q are in K. Let $u_i \in N_H(x_i)$ and $v_i \in N_H(x_{i+1})$, $i = j, k$. Then

$$|C(x_j, z_j)| + |C(x_k, z_k)| \geq 3 + L_H(u_j, u_k) + (|Q| - 2)$$

$$|C(z_j, x_{j+1})| + |C(z_k, x_{k+1})| \geq 3 + L_H(v_j, v_k) + (|Q| - 2).$$

Proof. Claim (a) is Lemma 3 in Section 4 in [1]. The Claims $(b) - (e)$ follow from the fact that C is a longest cycle in G. □

The following lemma is derived from Lemma 2 and the proof of Theorem 3 in Section 4 in [1].

Lemma 2 ([1]) *Let C be a longest cycle in a 2-connected quasi-claw-free graph G and H a component of $G - C$. There exists an independent set I in G with cardinality $|N(H)| + 1$ such that $N(x) \cap N(y) = \emptyset$ for any pair of distinct elements x, y of I.* □

Let K_4^- denote the graph obtained from K_4 by deleting one edge. The following lemma is due to Jung.

Lemma 3 ([3]) *Let H be a 2-connected graph. There exist distinct vertices v_1, v_2 and v_3 in $V(H)$ such that*
(i) $D(H) \geq d_H(v_i)$ for $i = 1, 2$ and $L_H(v_1, v_2) \geq d_H(v_3)$;
(ii) $D(H) \geq d_H(v_3) - 1$ with strict inequality unless $H = K_4^-$. □

3 Proof of Theorem 3

In this section let G be a 2-connected nonhamitonian quasi claw-free graph and let C be a longest cycle in G with a given orientation.

In the following we first distinguish the cases pertaining to the first part of Theorem 3, and then using the obtained lower bound of $c(G)$ we prove the second part of Theorem 3.

Case 1. There is a component, say H, of $G - C$ with at most two vertices.

We label $N(H) = \{x_1, \ldots, x_s\}$ in cyclic order aroud C, where the subscripts are taken modulo s. By Lemma 1 we have $|C| \geq 4s$, and consequently $|C| \geq 4\delta \geq 3\delta + 2$ if $|H| = 1$.

Suppose $H = \{v_1, v_2\}$. If $|N_C(v_i)| < s$ for some $v_i \in \{v_1, v_2\}$, then also $|C| \geq 4s \geq 4\delta$. Thus assume $|N_C(v_i)| = s$ for $i = 1, 2$, and therefore $|C(x_k, x_{k+1}]| \geq 5$ for each $x_k \in N(H)$. Then $|C| \geq 5s = 3(s+1) + 2s - 3$, and $|C| \geq 3\delta + 2$ unless $s = 2$ and $|C(x_k, x_{k+1}]| = 5$ for $k = 1, 2$, in this final case it is not difficult to verify $G \in \mathcal{F}$.

Case 2. There is a component, say H, of $G - C$ such that H has a cut vertex.

Let $B_1, \ldots, B_t$ be all end blocks of H with the unique cut vertices c_i of H in $V(B_i)$, $i = 1, \ldots, t$.

Case 2.1. There exist distinct end blocks B_i, B_j such that $|N_C(B_i - c_i) \cup N_C(B_j - c_j)| \geq 2$.

It suffices to consider $i = 1 = j - 1$. Label $X = N_C(B_1 - c_1) \cup N_C(B_2 - c_2) = \{z_1, z_2, ... z_r\}$ in cyclic order around C, where the subscripts are taken modulo r. By the assumption of this subcase, there exist distinct $z_p, z_q \in X$ such that $z_k \in N_C(B_1 - c_1)$ and $z_{k+1} \in N_C(B_2 - c_2)$ or vice versa ($k = p, q$). For $k \in \{p, q\}$, we have $|C(x_k, x_{k+1}]| \geq (D(B_1) + L_H(c_1, c_2) + D(B_2) + 1) + 3 \geq D(B_1) + D(B_2) + 4$. Note that $|C(x_j, x_{j+1}]| \geq 4$ for $x_j \in X \setminus \{x_p, x_q\}$. By Lemma 3, there exist $v_i \in V(B_i - c_i)$ such that $D(B_i) \geq d_{B_i}(v_i) = d_H(v_i)$ for $i = 1, 2$. Now we have $|C| \geq 4r + 2D(B_1) + 2D(B_2) \geq 2d(v_1) + 2d(v_2) \geq 4\delta$.

Case 2.2. $\bigcup_{i=1}^{t} N_C(B_i - c_i) = \{x\}$.

For any $y_p \in N(x) \cap V(B_p - c_p)$ and $y_q \in N(x) \cap V(B_q - c_q)$ with $p \neq q$, we have $d(y_p, y_q) = 2$. Since $x^+ \notin N_C(B_i - c_i)$ ($i = 1, \ldots, t$), we deduce that $x \notin J(y_p, y_q)$, and this in turn implies that $c_1 = c_2 = \cdots = c_t = c$ and $J(y_p, y_q) = \{c\}$. Since G is 2-connected and C is a longest cycle, c has a neighbor z in $V(C) \setminus \{x^-, x, x^+\}$. But then, by definition of $J(y_p, y_q)$, $z(\neq x)$ has a neighbor y_p or y_q in $V(B_p - c_p) \cup V(B_q - c_q)$, a contradiction.

Case 3. All components of $G - C$ are 2-connected.

Let H be a 2-connected component of $G-C$. We label $N(H) = \{x_1, \ldots, x_s\}$ in cyclic order around C, where the subscripts are taken modulo s. For convenience, we abbreviate $d_{C_i}(v) := |N(v) \cap C(x_i, x_{i+1}]|$ for a vertex v of G and for $i = 1, \ldots, s$, and $D := D(H)$

We call a segment $C[x_i, x_{i+1}]$ *good* if $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$. Let $q := q(H)$ denote the number of good segments on C with respect to H. Note that $q \geq 2$ since G is 2-connected. For $x_i \in N(H)$, by Lemma $1(d)$, $|C(x_i, x_{i+1}]| \geq 4 + \varepsilon_i D$, where $\varepsilon_i = 1$ or 0 according to whether $C[x_i, x_{i+1}]$ is good or not. Thus, by Lemma $3(i)$, $|C| \geq 4s + qD = 3(D+s) + s + (q-3)D \geq 3\delta + 2$ unless $q = 2$. Also $|C| \geq 12 > 3\delta + 2$ if $\delta \leq 3$. Hence in the rest of Case 3 we may assume $q = 2$ and $\delta \geq 4$.

Case 3.1. $s \geq 3$.

Without loss of generality, we may assume that $C[x_1, x_2]$ and $C[x_t, x_{t+1}]$ are the only good segments on C with respect to H, where $3 \leq t \leq s$. Then there exists a vertex $y \in V(H)$ such that $N_H(x_2) \cup N_H(x_3) \cup \cdots \cup N_H(x_t) = \{y\}$, and moreover, if $t \neq s$, then $N_H(x_{t+1}) \cup \cdots \cup N_H(x_s) \cup N_H(x_1) = \{y'\}$ for some $y' \in V(H - y)$.

We pick x_p, x_q with $2 \leq p < q \leq t$. By Lemma $1(c)$, we have $d(x_p) = d_C(x_p) + 1$ and $d(x_q) = d_C(x_q) + 1$. For $x_i \in N(H) \setminus \{x_p, x_q\}$ set $t_i = 1$, if $N_H(x_i) \neq \{y\}$, and $t_i = 0$ otherwise.

The following two claims can be obtained by constructing appropriate cycles.

Claim 1 *Let* $x_i \in N(H)$ *and* $z \in N(x_p) \cap C[x_i^{++}, x_{i+1}^{--}]$, $u \in N(x_q) \cap C[x_i^{++}, x_{i+1}^{--}]$. *If* $z \neq u$, *then* $|C[z^+, u^-]| \geq 1$ *when* $z \in C[x_i, u]$ *and* $|C[u^+, z^-]| \geq 1$ *when* $u \in C[x_i, z]$. □

Claim 2 *Let* $x_i \in N(H) \setminus \{x_p, x_q\}$ *and let* z_i *and* u_i *be the first and last elements of* $N(x_p) \cup N(x_q)$ *on* $C[x_i^{++}, x_{i+1}^{--}]$. *Then* $|C(x_i, z_i)| \geq 2 + t_i D$, *and moreover* $|C(u_i, x_{i+1}]| \geq 3 + t_{i+1} D$, *if* $x_{i+1} \notin \{x_p, x_q\}$. □

Claim 3 $|C(x_1, x_2]| \geq d_{C_1}(x_p) + d_{C_1}(x_q) + D + 2$, $|C(x_t, x_{t+1}]| \geq d_{C_t}(x_p) + d_{C_t}(x_q) + D + 1$ *and* $|C(x_i, x_{i+1}]| \geq d_{C_i}(x_p) + d_{C_i}(x_q) + 1$ *for* $x_i \in N(H) \setminus \{x_1, x_t\}$.

Proof of Claim 3. We only prove the inequality for $C(x_1, x_2]$, the other two inequalities can be proved in the same manner.

If $C(x_1^+, x_2^-) \cap (N(x_p) \cup N(x_q)) = \emptyset$, then obviously $d_{C_1}(x_p) + d_{C_1}(x_q) \le 2$, and consequently $|C(x_1, x_2]| \ge D + 4 \ge d_{C_1}(x_p) + d_{C_1}(x_q) + D + 2$. Let z_1 and u_1 be the first and last elements of $N(x_p) \cup N(x_q)$ on $C(x_1^+, x_2^-)$, respectively. By Claim 2, $|C(x_1, z_1)| \ge D + 2$. If $x_2 \ne x_p$, then by Lemma 1(i) we have $x_2^-, x_2^{--} \notin N(x_p) \cup N(x_q)$, and by Claim 1 we have $|C[z_1, u_1]| \ge d_{C_1}(x_p) + d_{C_1}(x_q) - 3$. By Claim 2, $|C[z_1, x_2]| \ge d_{C_1}(x_p) + d_{C_1}(x_q)$, and the specified inequality follows readily. Let $x_2 = x_p$. Since $x_2^-, x_2^{--} \notin N(x_q)$ and $u_1 \in C[z_1, x_2^{--}]$, we have $|C[z_1, u_1]| \ge d_{C_1}(x_p) + d_{C_1}(x_q) - 2$. Hence again $|C[z_1, x_2]| \ge d_{C_1}(x_p) + d_{C_1}(x_q)$, and the specified inequality follows. □

Claim 4 *There exists a vertex $v \in V(H - y)$ such that $D + 1 \ge d(v)$.*

Proof of Claim 4. If $x_t \ne x_s$, then, as noted above, $N_H(x_{t+1}) \cup \cdots \cup N_H(x_s) \cup N_H(x_1) = \{y'\}$ for some $y' \in V(H - y)$. Since $|H| \ge 3$ and $q = 2$, we infer that $N_C(H - y - y') = \emptyset$. By Lemma 3 (ii) there exists a vertex $v \in V(H - y - y')$ such that $D \ge d_H(v) - 1 = d(v) - 1$. If $x_t = x_s$, then $N_C(H - y) \subseteq \{x_1\}$ since $q = 2$ and $t = s \ge 3$. In this case, by Lemma 3 (i), there exists a vertex $v \in V(H - y)$ such that $D \ge d_H(v) \ge d(v) - 1$. □

Now by the above claims, we have $|C| \ge d_C(x_p) + d_C(x_q) + 2D + 4 = d(x_p) + d(x_q) + 2D + 2 \ge d(x_p) + d(x_q) + 2d(v)$. This settles Case 3.1.

Case 3.2. $s = 2$.

Let $N(H) = \{x_1, x_2\}$ in order around C. By Lemma 3 there exist distinct vertices $v_1, v_2 \in V(H)$ such that $D \ge d_H(v_i)$ for $i = 1, 2$. Hence $D + 2 \ge d(v_1) \ge \delta$. If there is a component K of $G - C$ other than H, then by the previous results, we may assume that K is 2-connected with $|N_C(K)| = 2$. Applying Lemma 3 to K, we get $D(K) + 2 \ge \delta$. We prove seven claims to settle Case 3.2.

Claim 5 *If there exists a component K of $G - C$ other than H such that K has neighbors on both $C(x_1, x_2)$ and $C(x_2, x_1)$, then $|C| \ge 4\delta + 12$.*

Proof of Claim 5. As noted above, we may assume K is 2-connected with $|N_C(K)| = 2$. Let $N_C(K) = \{y_1, y_2\}$ such that $y_1 \in C(x_1, x_2)$ and $y_2 \in C(x_2, x_1)$. By Lemma 1(e), $|C(x_1, y_1) \cup C(x_2, y_2)| \ge D + D(K) + 6$ and $C(y_1, x_2) \cup C(y_2, x_1)| \ge D + D(K) + 6$, and consequently $|C| \ge 2D + 2D(K) + 16 \ge 4\delta + 12$. □

Claim 6 *If $x_i^+ \in N_C(K)$ for some $i \in \{1, 2\}$ and $K \subseteq G - C - H$, then $|C| \ge 4\delta + 3$.*

Proof of Claim 6. Suppose $x_1^+ y \in E(G)$ for some $y \in V(K)$. In view of Claim 5, we may assume $N_C(K) \subseteq C(x_1, x_2)$, and then suppose that the another neighbor of K on C is $z \in C(x_1^+, x_2)$. By Lemma 1 (a), $x_1 x_1^{++} \in E$, and using the edges $x_1 x_1^{++}, x_1^+ y$ and $x_1^- x_1^+$ we can construct a cycle that contains all vertices in $C - C(z, x_2^-)$ and at least $D + D(K) + 2$ vertices outside C. Hence $|C(z, x_2^-)| \geq D + D(K) + 2$, and then $|C| \geq 2D + 2D(K) + 11 \geq 4\delta + 3$. □

Assuming $N(x_i^+) \subseteq V(C)$ and $N(x_i^-) \subseteq V(C)$ for $i = 1, 2$, we prove the following claim.

Claim 7 *(i) If $x_i^{++} \in N(x_i) \cup N(x_i^-) \cup N(x_i^{--})$ for some $i \in \{1, 2\}$, then $|C| \geq 3\delta + 2$;*

(ii) if $x_i^{++} \in N(K)$ for some $i \in \{1, 2\}$ and for some $K \subseteq G - C - H$, then $|C| \geq 3\delta + 3$.

Proof of Claim 7. We only prove (i) for the case $x_i^{++} \in N(x_i)$, the other cases can be proved in the same manner.

Suppose $x_1 x_1^{++} \in E$. In view of Claim 6, we may assume $N(x_1^+) \subseteq V(C)$. If x_1^+ has neighbor on $C(x_1^{++}, x_2)$, then let z be such a neighbor closest to x_2. Then by constructing an appropriate cycle, we can obtain $|C(z, x_2]| \geq D + 3$, and this implies $|C(x_1, x_2]| \geq d_{C_1}(x_1^+) + D + 4$. If x_1^+ has no neighbor on $C(x_1^{++}, x_2)$, then the latter inequality holds trivially. Since $|C(x_2, x_1]| \geq d_{C_2}(x_1^+) + D + 2$, we have $|C| \geq d(x_1^+) + 2D + 6 \geq 3\delta + 2$. □

Now in view of Claims 5, 6 and 7, in the rest of the proof of Case 3.2, we assume the following:

All components K of $G - C - H$ (if there is any) have neighbors on only one of $C(x_1, x_2)$ and $C(x_2, x_1)$, moreover $(N(x_i^+) \cup N(x_i^{++}) \cup N(x_i^-) \cup N(x_i^{--})) \subseteq V(C)$ for $i = 1, 2$ and $x_i x_j^{++}, x_i x_j^{--} \notin E$ for $i, j \in \{1, 2\}$. (1)

Suppose $|C(x_i, x_{i+1}]| \geq 2D+4$ for some $i \in \{1, 2\}$. Then, using Lemma 1(d) and Lemma 3, we obtain $|C| \geq 3D + 8 = 3(D + 2) + 2 \geq 3\delta + 2$. Hence we assume:

$$|C(x_i, x_{i+1}]| < 2D + 4, \; i = 1, 2. \tag{2}$$

Suppose for some $i \in \{1, 2\}$, x_i^{++} and x_{i+1}^{--} have neighbor z and z' on $C(x_{i+1}, x_i)$, respectively. If $z \in C(x_{i+1}, z')$, then we can construct a cycle that contains all vertices in $C - C(z, z')$ and at least $D + 1$ vertices in H. Then $|C(z, z')| \geq D + 1$. Using a similar argument, we can get $|C(x_{i+1}, z)| \geq D + 1$, and this contradicts assumption (2). Hence assumption (2) yields the following claim.

Claim 8 *Let $N(x_i^{++}) \cap C(x_{i+1}, x_i) \neq \emptyset$ and $N(x_{i+1}^{--}) \cap C(x_{i+1}, x_i) \neq \emptyset$ for $i \in \{1, 2\}$. Let z and z' be the first and the last neighbors of x_i^{++} and x_{i+1}^{--} on $C(x_{i+1}, x_i)$, respectively. Then $z \in C(z', x_i)$. In particular, x_i^{++} and x_{i+1}^{--} have no common neighbor on $C(x_{i+1}, x_i)$.* □

Using similar arguments, we obtain, by Claim 8, the following claim.

Claim 9 *For* $i = 1, 2$, $|C(x_{i+1}, x_i]| \geq d_{C_{i+1}}(x_i^{++}) + d_{C_{i+1}}(x_{i+1}^{--}) + D + 1$. $\square$

Now suppose $|C(x_i, x_{i+1}]| \geq d_{C_i}(x_i^{++}) + d_{C_i}(x_{i+1}^{--}) + 1$ for some $i \in \{1, 2\}$. Then by Claim 9 we obtain $|C| \geq d(x_i^{++}) + d(x_{i+1}^{--}) + 2D + 2 \geq 4\delta - 2 \geq 3\delta + 2$. Therefore we assume the following:

$$|C(x_i, x_{i+1}]| \leq d_{C_i}(x_i^{++}) + d_{C_i}(x_{i+1}^{--}), \ i = 1, 2.$$

The last inequalities clearly yield:

There is no vertex $y \in C(x_i, x_{i+1})$ satisfying $N(x_i^{++}) \cap C(x_i, x_{i+1}) \subseteq C(x_i, y]$ and $N(x_{i+1}^{--}) \cap C(x_i, x_{i+1}) \subseteq C[y, x_{i+1})$ for $i = 1, 2$. (3)

Claim 10 (*i*) *If* $N(x_i^{++}) \cap C(x_{i+1}, x_i) \neq \emptyset$ *for some* $i \in \{1, 2\}$, *then* $|C| \geq 3\delta + 2$;

(*ii*) *if* $N(x_i^{+}) \cap C(x_{i+1}, x_i^{-}) \neq \emptyset$ *for some* $i \in \{1, 2\}$, *then* $|C| \geq 3\delta + 2$.

Proof of Claim 10. We only prove (*i*), the proof of (*ii*) is similar.

Suppose x_1^{++} has neighbor on $C(x_2, x_1)$. Let z and z' be the first and last neighbors of x_1^{++} on $C(x_2, x_1)$. By Claim 7, we may assume $z' \in C(x_2, x_1^{--})$. Using similar cycle constructions as before, the choice of C implies that $|C(x_2, z)| \geq D + 2$. Using this in combination with (2), we obtain that x_2^{++} has no neighbor on $C(z, x_1)$. By (3), there exist a vertex $u \in N(x_1^{--}) \cap C(x_2, z)$ and a vertex $u' \in N(x_2^{++}) \cap C(u, z]$ such that $C(u, u') \cap (N(x_1^{--}) \cup N(x_2^{++})) = \emptyset$. Then, with a slight abuse of notation, $C[x_1^{++}, x_2] \cup C[x_2^{++}, u] \cup C[u', z'] \cup C[x_1^{--}, x_1] \cup x_1^{++}z' \cup x_2^{++}u' \cup x_1^{--}u$ gives rise to a cycle which contains all vertices of $C - C(u, u') - C(z', x_1^{--})$ and at least $D + 1$ vertices in H. Hence $|C(u, u') \cup C(z', x_1^{--})| \geq D + 1$. Since $N_{C_2}(x_2^{++}) \subseteq C(x_2, z)$, we have $|C(x_2, x_1]| \geq d_{C_2}(x_1^{++}) + d_{C_2}(x_2^{++}) + D + 3$.

Similarly, $|C(x_1, x_2]| \geq d_{C_1}(x_1^{++}) + d_{C_1}(x_2^{++}) + 1$.

Combining the last two inequalities, we get $|C| \geq d(x_1^{++}) + d(x_2^{++}) + D + 4 \geq 3\delta + 2$. $\square$

Claim 11 *If there exists an edge* $e = z_i z_{i+1}$ *with* $z_i \in C(x_i^{++}, x_{i+1}^{--})$ *and* $z_{i+1} \in C(x_{i+1}^{++}, x_i^{--})$, *then* $|C| \geq 3\delta + 3$.

Proof of Claim 11. Say $i = 1$. By (3) we infer that either $N(x_1^{++}) \cap C(z_1, x_2) \neq \emptyset$ or $N(x_2^{--}) \cap C(x_1, z_1) \neq \emptyset$, say the former. Let u_1 be the first vertex of $N(x_1^{++})$ on $C(z_1, x_2)$. Again slightly abusing notation, we set $R = C[x_1^{+}, z_1] \cup x_1^{++}u_1 \cup C[u_1, x_2^{--}]$. We define a (z_2, x_1^{--})-path Q as follows. If $N(x_2^{++}) \cap$

$C(z_2, x_1) \neq \emptyset$, then let u_2 denote the first neighbor of $N(x_2^{++})$ on $C(z_2, x_1)$ and set $Q = C[x_2^{++}, z_2] \cup x_2^{++}u_2 \cup C[u_2, x_1]$. If $N(x_2^{++}) \cap C(z_2, x_1) = \emptyset$, then by (3) there exist vertices $u_2' \in N(x_1^{--}) \cap C(x_2^{++}, z_2)$ and $u_2'' \in N(x_2^{++}) \cap C(u_2', z_2]$ such that $(N(x_1^{++}) \cup N(x_2^{++})) \cap C(u_2', u_2'') = \emptyset$. In this event we set $Q = C[u_2'', z_2] \cup u_2''x_2^{++} \cup C[x_2^{++}, u_2'] \cup u_2'x_1^{--}$. Now we set $L = (C(x_1, x_2) - R) \cup (C(x_2, x_1) - Q)$. Note that the segment $C[x_1^{--}, x_1]$, $C[x_2^{--}, x_2]$ together with R and Q and the edge z_1z_2 give rise to a cycle which contains all vertices of $C - L$ and at least $D + 1$ vertices in H. Hence $|L| \geq D + 1$. Since $(\{x_1, x_2\} \cup L) \cap (N(x_1^{++}) \cup N(x_2^{++})) = \emptyset$, we have $|C| \geq d(x_1^{++}) + d(x_2^{++}) + D + 5 \geq 3\delta + 3$. □

So far we have shown that either $c(G) \geq \min\{3\delta + 2, n\}$ or G belongs to $\mathcal{F}$. Now we prove the second part of Theorem 3. Obviously, $s \geq 2$. By Lemma 2 we have $n \geq (s+1)\delta + (s+1)$ so $s \leq 2$. Thus $s = 2$. Let v be any vertex in H. Then $|H| \geq d_H(v) + 1 \geq d(v) - 1 \geq \delta - 1$. Thus $n \geq |C| + |H| \geq 4\delta + 1$, a contradiction.

The proof of Theorem 3 is now complete.

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