A novel characterization of n-extendable bipartite graphs *

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Abstract

Let G be a simple connected graph. For a subset S of $V(G)$ with $|S| = 2n+1$, let $\alpha_{(2n+1)}(G,S)$ denote the graph obtained from G by contracting S to a single vertex. The graph $\alpha_{(2n+1)}(G,S)$ is also said to be obtained from G by an $\alpha_{(2n+1)}$-*contraction*. For pairwise disjoint subsets $S_1, S_2, \cdots, S_{2n}$ of $V(G)$, let $\beta_{2n}(G, S_1, S_2, \cdots, S_{2n})$ denote the graph obtained from G by contracting each S_i ($i = 1, 2, \cdots, 2n$) to a single vertex respectively. The graph $\beta_{2n}(G, S_1, S_2, \cdots, S_{2n})$ is also said to be obtained from G by a β_{2n}-*contraction*. In the present paper, based on $\alpha_{(2n+1)}$-contraction and β_2-contraction, some new characterizations for n-extendable bipartite graphs are given.

Key words: $\alpha_{(2n+1)}$-contraction; β_{2n}-contraction; n-extendable bipartite graphs

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1 Introduction and terminology

Let G be a graph. For any subset S of $V(G)$, we denote by $N_G(S)$ the set of the vertices in $V(G) \setminus S$ adjacent to a vertex in S. The vertex connectivity of G is denoted by $\kappa(G)$, and the number of odd components of G is denoted by $c_o(G)$. Let n be a positive integer and G a connected graph with $|V(G)| \geq 2n + 2$. G is said to be n-extendable if G has n independent edges and any n independent edges are contained in a perfect matching of G. Other terminologies and notations not defined here can be found in [4] or [6].

In the investigation of the graphs with perfect matchings, various classes of graphs have been introduced, such as elementary graphs, saturated graphs, bicritical graphs and n-extendable graphs [6]. As early as in 1960, Hetyei first studied the 1-extendable bipartite graphs with the term "elementary bipartite graphs" and pointed out these graphs have simple "ear structure" [6]. Lovász and Plummer showed that any elementary graphs could be constructed by using only 1-extendable bipartite graphs and bicritical graphs as building blocks [6]. The concept of n-extendable graphs was introduced by Plummer [7] in 1980. Since then, many investigations on this topic have been made (see, e.g., [1-3, 5-12]).

For n-extendable bipartite graphs, Plummer [8] gave some characterizations which are similar to the characterizations for 1-extendable bipartite graphs given by Hetyei [6, Chapter 4]. A Menger type characterization for n-extendable bipartite graphs was obtained by Aldred, Holton, Lou and Saito [1]. F.Zhang and H. Zhang gave a recursive method to construct all n-extendable bipartite graphs [12].

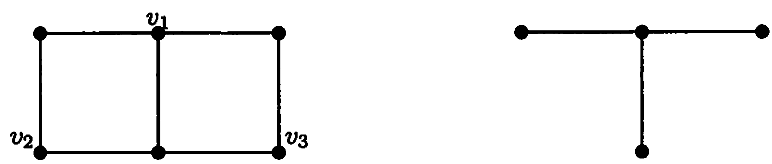
Note some properties of n-extendable bipartite graphs, see Theorem 1(3) and Theorem 3(3) below, focus attention on the property of having a perfect matching if under some special vertex deletions, the resulting subgraph continues to have a perfect matching. The purpose of this paper is to reveal some novel properties of n-extendable bipartite graphs under some special vertex contractions. In this work, based on two type vertex contractions called $\alpha_{(2n+1)}$-contraction and β_2-contraction defined in the following, some new necessary and sufficient conditions for a graph to be an n-extendable bipartite graph are established. Particularly, several new equivalent propositions for 1-extendable bipartite graphs are given.

In order to state our results precisely, we need to define certain types vertex contractions.

Definition. Let G be a simple connected graph. For a subset S of $V(G)$ with $|S| = 2n+1$, let $\alpha_{(2n+1)}(G,S)$ denote the graph obtained from G by contracting S to a single vertex. The graph $\alpha_{(2n+1)}(G,S)$ is also said to be obtained from G by an $\alpha_{(2n+1)}$-*contraction*. $\alpha_{(2n+1)}(G,S)$ may contain multiple edges and loops due to the contraction of S, the underlying simple graph of $\alpha_{(2n+1)}(G,S)$ is denoted by $\alpha^*_{(2n+1)}(G,S)$.

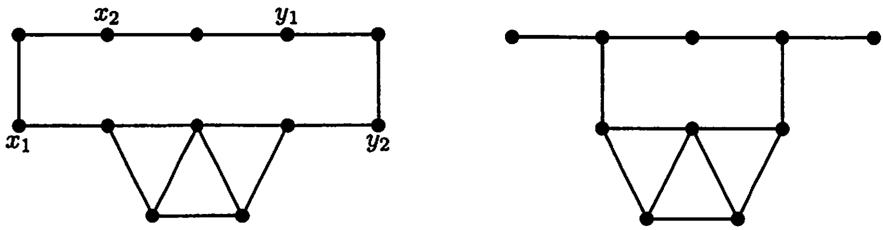
Let G be a simple connected graph. For pairwise disjoint subsets $S_1, S_2, \cdots, S_{2n}$ of $V(G)$, let $\beta_{2n}(G, S_1, S_2, \cdots, S_{2n})$ denote the graph obtained from G by contracting each S_i, $i = 1, 2, \cdots, 2n$, to a single vertex respectively. The graph $\beta_{2n}(G, S_1, S_2, \cdots, S_{2n})$ is said to be obtained from G by a β_{2n}-*contraction*. The underlying simple graph of $\beta_{2n}(G, S_1, S_2, \cdots, S_{2n})$ is denoted by $\beta^*_{2n}(G, S_1, S_2, \cdots, S_{2n})$.

The following Figure 1 and Figure 2 show examples of an α_3-*contraction* and a β_2-*contraction* respectively.



H, $S = \{v_1, v_2, v_3\}$ $\qquad$ $\alpha^*_3(H,S)$

Figure 1



G, $S_1 = \{x_1, x_2\}$, $S_2 = \{y_1, y_2\}$ $\qquad$ $\beta_2^*(G, S_1, S_2)$

Figure 2

2 Preliminary results

In this section, we review some known results which will help to prove our main results.

Theorem 1 (Hetyei [6]). Suppose G is a bipartite graph with bipartition (U, W). Then the following are equivalent:
(1) G is 1-extendable;
(2) $|U| = |W|$ and for every non-empty proper subset X of U, $|N_G(X)| \geq |X| + 1$;
(3) for any $u \in U$, $w \in W$, $G - u - w$ has a perfect matching;
(4) G has exactly two minimum vertex covers, namely U and W.

Theorem 2 (Plummer [7]). If G is n- extendable, then $\kappa(G) \geq n + 1$.

Theorem 3 (Plummer [8]). Let G be a connected bipartite graph with bipartition (U, W). Suppose n is a positive integer such that $n \leq (|V(G)| - 2)/2$. Then the following are equivalent:
(1) G is n-extendable;
(2) $|U| = |W|$ and for each non-empty subset X of U such that $|X| \leq |U| - n$, $|N_G(X)| \geq |X| + n$;
(3) For all $u_1, u_2, ..., u_n \in U$ and $w_1, w_2, ..., w_n \in W$, $G - u_1 - u_2 - ... - u_n - w_1 - w_2 - ... - w_n$ has a perfect matching.

3 Main results

In this section, we give some new characterizations for n-extendable (or 1-extendable) bipartite graphs based on $\alpha_{(2n+1)}$-contraction and β_2-contraction respectively.

In the following Theorem, we only consider the bipartite graph G with $|V(G)| \geq 2n+4$. Since for $|V(G)| = 2n+2$, by Theorem 2, there are no n-extendable bipartite graphs other than the complete bipartite graph $K_{n+1,n+1}$.

Theorem 4. Let G be a bipartite graph with bipartition (U, W) such that $|U| = |W| \geq n+2$. Then the following are equivalent:
(1) G is n-extendable;
(2) For any subset $S \subseteq V(G)$ such that $|S| = 2n+1$, and $|S \cap U| = n$ or $|S \cap W| = n$, the graph $\alpha_{(2n+1)}(G, S)$ has a perfect matching;
(3) G is $(n+1)$-connected, and, for any two subsets $S_1 \subseteq U$, $S_2 \subseteq W$ such that $|S_1| = |S_2| = n+1$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching.

Proof. (1)⇔(2). Suppose G is n-extendable. For any subset $S \subseteq V(G)$ such that $|S| = 2n+1$, and $|S \cap U| = n$ or $|S \cap W| = n$. Let $u_1, u_2, ..., u_n \in (S \cap U)$ and $w_1, w_2, ..., w_n \in (S \cap W)$. By Theorem 3(3), the graph $G - u_1 - u_2 - ... - u_n - w_1 - w_2 - ... - w_n$ has a perfect matching. It is easy to verify that the graph $G - u_1 - u_2 - ... - u_n - w_1 - w_2 - ... - w_n$ is a spanning subgraph of $\alpha_{(2n+1)}(G, S)$, hence $\alpha_{(2n+1)}(G, S)$ has a perfect matching.

Conversely, suppose for any subset $S \subseteq V(G)$ such that $|S| = 2n+1$, and $|S \cap U| = n$ or $|S \cap W| = n$, the graph $\alpha_{(2n+1)}(G, S)$ has a perfect matching. Assume that G is not n-extendable. By Theorem 3(2), there exists a non-empty subset X of U such that

$$|X| \leq |U| - n, \ |N_G(X)| \leq |X| + (n-1).$$

We shall derive a contradiction.

Case 1. $|N_G(X)| \geq n+1$.

We can choose $S_1 \subseteq U \setminus X$ such that $|S_1| = n$, and $S_2 \subseteq N_G(X)$ such that $|S_2| = n+1$. Let $S = S_1 \cup S_2$ and $G' = \alpha_{(2n+1)}(G, S)$. By the

$\alpha_{(2n+1)}$-contraction, it is clear that $|N_{G'}(X)| \leq |X| + (n-1) - n = |X| - 1$.

Notice that any two vertices of X are not adjacent in G', thus $c_o(G' - N_{G'}(X)) \geq |X| \geq |N_{G'}(X)| + 1$. By Tutte's theorem, G' has not a perfect matching, a contradiction.

Case 2. $|N_G(X)| \leq n$.

Subcase 2.1. $|U \setminus X| \leq n + 1$.

We can choose $T_1 \subseteq U$, $T_2 \subseteq W$ such that $|T_1| = n + 1$, $T_1 \supseteq U \setminus X$, $|T_2| = n$ and $T_2 \supseteq N_G(X)$. Let $S = T_1 \cup T_2$ and $G' = \alpha_{(2n+1)}(G, S)$. Let the set S become a vertex v in G' under the $\alpha_{(2n+1)}$-contraction.

Notice that any two vertices of $V(G') \setminus \{v\}$ are not adjacent in G' and $|V(G)| \geq 2n + 4$, so $c_o(G' - v) \geq 3$. By Tutte's theorem, G' has not a perfect matching, a contradiction.

Subcase 2.2. $|U \setminus X| \geq n + 2$.

We may choose $T_3 \subseteq U \setminus X$, $T_4 \subseteq W$ such that $|T_3| = n + 1$, $|T_4| = n$ and $T_4 \supseteq N_G(X)$.

Let $S = T_3 \cup T_4$ and $G' = \alpha_{(2n+1)}(G, S)$. Let the set S become a vertex v in G' under the $\alpha_{(2n+1)}$-contraction. Set $R = [U \setminus (X \cup T_3)] \cup \{v\}$. Thus, $N_{G'}(R) = X \cup (W \setminus T_4)$ and $|R| = |U| - |X| - |T_3| + 1 = |U| - |X| - n$.

Notice that any two vertices of $N_{G'}(R)$ are not adjacent in G'. Hence $c_o(G' - R) = |W| - n + |X| = |U| - n + |X| > |R|$. By Tutte's theorem, G' has not a perfect matching, again a contradiction.

(1)$\Leftrightarrow$(3). By Theorem 2 and analogous argument as those in the proof of (1)$\Rightarrow$(2), the necessity is easy to be checked.

To prove the sufficiency, suppose G is $(n + 1)$-connected, and, for any two subsets $S_1 \subseteq U$, $S_2 \subseteq W$ with $|S_1| = |S_2| = n + 1$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching. Assume that G is not n-extendable. By Theorem 3(2), there is a non-empty proper subset X of U such that

$$|X| \leq |U| - n, \quad |N_G(X)| \leq |X| + (n - 1).$$

Note that G is $(n + 1)$-connected, this implies

$$|N_G(X)| \geq (n+1), \ |U \setminus X| \geq n+1.$$

So we can choose $S_1 \subseteq U \setminus X$, $S_2 \subseteq N_G(X)$ such that $|S_1| = |S_2| = n+1$. Let $G' = \beta_2(G, S_1, S_2)$. By the β_2-contraction, it is clear that $|N_{G'}(X)| \leq |X| + (n-1) - n = |X| - 1$. Since G' is also a bipartite graph, by Hall's marriage theorem, G' has not a perfect matching, a contradiction. □

For 1-extendable bipartite graphs, we can give more equivalent statements.

Theorem 5. Let G be a bipartite graph with bipartition (U, W) such that $|U| = |W|$. Then the following statements are equivalent:
(1) G is 1-extendable;
(2) For any subset $S \subseteq V(G)$ such that $|S| = 3$, and $|S \cap U| = 1$ or $|S \cap W| = 1$, the graph $\alpha_3(G, S)$ has a perfect matching;
(3) G is 2-connected, and, for any two subsets $S_1 \subseteq U$, $S_2 \subseteq W$ such that $|S_1| = |S_2| = 2$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching;
(4) For any two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1 \cap U| = |S_1 \cap W| = 1$, $|S_2| = 2$, and $S_2 \subseteq U$ or $S_2 \subseteq W$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching;
(5) G is 2-connected, and, for any two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1 \cap U| = |S_1 \cap W| = |S_2 \cap U| = |S_2 \cap W| = 1$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching.

Proof. **(1)⇔(2)** and **(1)⇔(3).** These conclusions are special cases of Theorem 4.

(1)⇔(4). Suppose G is 1-extendable. For any two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1 \cap U| = |S_1 \cap W| = 1$, $|S_2| = 2$, and $S_2 \subseteq U$ or $S_2 \subseteq W$. We can choose $u \in (S_1 \cup S_2) \cap U$ and $w \in (S_1 \cup S_2) \cap W$ such that $\{u, w\} \neq S_1$. By Theorem 1(3), $G - u - w$ has a perfect matching. It is easy to see that $G - u - w$ is a spanning subgraph of $\beta_2(G, S_1, S_2)$, hence $\beta_2(G, S_1, S_2)$ has a perfect matching.

Conversely, suppose for any two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1 \cap U| = |S_1 \cap W| = 1$, $|S_2| = 2$, and $S_2 \subseteq U$ or $S_2 \subseteq W$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching. Obviously, G is connected. Assume that G is not 1-extendable. By Theorem 1(2), there exists a non-empty proper subset X of U such that

$$|X| \leq |U| - 1, \ |N_G(X)| \leq |X|.$$

We shall derive a contradiction.

Case 1. $|N_G(X)| \geq 2$.

We can choose two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1| = |S_2| = 2$, $|S_1 \cap (U \setminus X)| = 1$, $|S_1 \cap (W \setminus N_G(X))| = 1$ and $S_2 \subseteq N_G(X)$. Let $G' = \beta_2(G, S_1, S_2)$. By the β_2-contraction, it is clear that $|N_{G'}(X)| \leq |X| - 1$.

Notice that any two vertices of X are not adjacent in G', so $c_o(G' - N_{G'}(X)) \geq |X| \geq |N_{G'}(X)| + 1$. By Tutte's theorem, G' has not a perfect matching, a contradiction.

Case 2. $|N_G(X)| = 1$.

In this case, we claim that $|X| = 1$. Otherwise, we can choose two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1| = |S_2| = 2$, $S_1 \supseteq N_G(X)$, $|S_1 \cap (U \setminus X)| = 1$ and $S_2 \subseteq W \setminus N_G(X)$. Clearly, $\beta_2(G, S_1, S_2)$ has not a perfect matching, a contradiction.

So we can choose two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1| = |S_2| = 2$, $S_1 = X \cup N_G(X)$ and $S_2 \subseteq U \setminus X$. Let $G' = \beta_2(G, S_1, S_2)$ and the set S_2 become a vertex v in G' under the β_2-contraction. Set $R = [U \setminus (X \cup S_2)] \cup \{v\}$. It is obvious that $c_o(G' - R) = |R| + 2$. By Tutte's theorem, G' has not a perfect matching, again a contradiction.

(1)$\Leftrightarrow$(5). By Theorem 2 and analogous argument as those in the proof of (1)$\Rightarrow$(4), the necessity is easy to be checked.

To prove the sufficiency, suppose G is 2-connected, and, for any two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1 \cap U| = |S_1 \cap W| = |S_2 \cap U| = |S_2 \cap W| = 1$, the graph $\beta_2(G, S_1, S_2)$ has a perfect matching. Assume that G is not 1-extendable. By Theorem 1(2), there exists a non-empty proper subset X of U such that

$$|X| \leq |U| - 1, \ |N_G(X)| \leq |X|.$$

Since G is 2-connected, hence

$$|N_G(X)| \geq 2, \ |U - X| \geq 2.$$

So we can choose two disjoint subsets $S_1, S_2 \subseteq V(G)$ such that $|S_1 \cap N_G(X)| = |S_2 \cap N_G(X)| = |S_1 \cap (U \setminus X)| = |S_2 \cap (U \setminus X)| = 1$. Let $G' = \beta_2(G, S_1, S_2)$ and $T = N_G(X) \cup (U \setminus X)$. Let the set T become the set T' in G' under the β_2-contraction. Obviously,

$$|T'| = |U| - |X| + |N_G(X)| - 2.$$

Notice that $N_{G'}(T') = X \cup [W \setminus N_G(X)]$ and any two vertices of $N_{G'}(T')$ are not adjacent in G'. So

$$\begin{aligned}
&c_o(G' - T') = |W \setminus N_G(X)| + |X| \\
&=|W| - |N_G(X)| + |X| \\
&=|U| - |N_G(X)| + |X| \\
&=|T'| + 2(|X| - |N_G(X)|) + 2 \\
&\geq |T'| + 2.
\end{aligned}$$

By Tutte's theorem, G' has no perfect matching, a contradiction. □

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