On Balance Index Sets of Rooted Trees

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Abstract

Any vertex labeling $f : V \to \{0,1\}$ of the graph $G = (V, E)$ induces a partial edge labeling $f^* : E \to \{0,1\}$ defined by $f^*(uv) = f(u)$ if and only if $f(u) = f(v)$. The balance index set of G is defined as $\{|f^{*-1}(0) - f^{*-1}(1)| : |f^{-1}(0) - f^{-1}(1)| \le 1\}$. In this paper, we first determine the balance index sets of rooted trees of height not exceeding two, thereby completely settling the problem for trees with diameter at most four. Next we show how to extend the technique to rooted trees of any height, which allows us to derive a method for determining the balance index set of any tree.

1 Introduction

Lee, Liu and Tan [8] considered a new labeling problem in graph theory. Given any vertex labeling $f : V \to \{0,1\}$ of a simple graph $G = (V, E)$, define a partial edge labeling f^* of G as follows. For each edge uv in E, define

$$f^*(u,v) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that the edge uv is unlabeled if $f(u) \neq f(v)$.

Denote by $v_f(0)$ and $v_f(1)$ the number of vertices of G that are labeled 0 and 1, respectively, under the mapping f. In a similar fashion, let $e_f(0)$ and $e_f(1)$ denote, respectively, the number of edges of G that are labeled 0 and 1 by the induced partial function f^*. For brevity, when the context is clear, we will simply write $v(0)$, $v(1)$, $e(0)$, and $e(1)$ without any subscript.

Definition 1.1. A vertex labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \le 1$, and ***balanced*** if f is friendly *and* $|e_f(0) - e_f(1)| \le 1$.

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We call a graph *balanced* if it admits a balanced labeling. See [2, 3, 10] for further results in balanced graphs. It is clear that not all graphs are balanced. Lee, Lee and Ng [7] introduced the following notion as an extension of their study of balanced graphs.

Definition 1.2. The ***balance index set*** of the graph G is defined as

$$\mathrm{BI}(G) = \{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}.$$

Example 1. It is not difficult to verify that the balance index set of the graph G displayed in Figure 1 is $\{0, 1, 2\}$. □

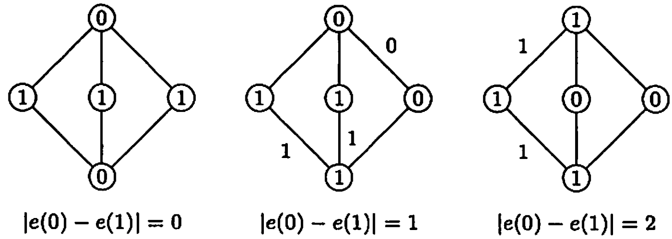

Figure 1: The friendly labelings of a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$.

In general, determining the balance index set of a given graph is a difficult task. Most of existing research on this problem focus on special families of graphs with simple structures [1, 2, 7, 9]. Examples include

$$\mathrm{BI}(\mathrm{St}(n)) = \begin{cases} \{k\} & \text{if } n = 2k+1, \\ \{k-1, k\} & \text{if } n = 2k, \end{cases}$$

and

$$\mathrm{BI}(C_n(t)) = \begin{cases} \{0, 1\} & \text{if } n \text{ is even}, \\ \{0, 1, 2\} & \text{if } n \text{ is odd}, \end{cases}$$

where $\mathrm{St}(n)$ is the star with n pendant vertices, and $C_n(t)$ denotes an n-cycle with a chord connecting two nonadjacent vertices at distance $t-1$ apart on the cycle.

Graphs with more complicated structure such as those formed by the amalgamation of complete graphs, stars, and generalized theta graphs, and L-products with cycles and complete graphs were studied in [4, 5, 6].

In [11], Zhang, Ho, Lee and Wen investigated and found the balance index sets of selective families of trees of diameter at most four. Their algorithmic approach limits its extension. In this paper, we propose an algebraic method to tackle this problem. Using this unified approach, we are able to obtain a complete solution.

Note that any tree with a diameter at most four can be viewed as a rooted tree with height not exceeding two. This is the graph that we shall study in the following section. The technique can be extended to a rooted tree of any height. Consequently, we are able to describe a method for determining the balance index set of any given tree.

2 Rooted Trees of Height At Most Two

Consider a rooted tree of height at most two with root r. Let $u_1, u_2, \ldots, u_k$ be the children of r. Let n_i be the number of children u_i has, and call its children $u_{i,j}$, where $1 \le j \le n_i$, if $n_i \ne 0$. Under these assumptions, we shall denote the tree $T(k; n_1, n_2, \ldots, n_k)$. Then

$$p = |V(T(k; n_1, n_2, \ldots, n_k))| = 1 + k + \sum_{i=1}^{k} n_i.$$

Due to symmetry, we further assume that $n_1 \ge n_2 \ge \cdots \ge n_k \ge 0$. If t of these n_i's are equal to n, we shall abbreviate them as n^t. For example, $T(k; n^k) = T(k; \underbrace{n, n, \ldots, n}_{k})$.

For each i, let $\epsilon_i = f(u_i)$, and for $1 \le j \le n_i$, let $\epsilon_{i,j} = f(u_{i,j})$. Observe that, given any friendly labeling, switching the vertex labels from 0 to 1, and 1 to 0, produces another friendly labeling with the same value in $e(0) - e(1)$. Hence we may assume $f(r) = 0$. We find

$$v(0) - v(1) = p - 2v(1) = p - 2\left(\sum_{i} \epsilon_i + \sum_{i,j} \epsilon_{i,j}\right).$$

Since f is friendly, we deduce that

$$2\left(\sum_{i} \epsilon_i + \sum_{i,j} \epsilon_{i,j}\right) = \begin{cases} p & \text{if } p \text{ is even} \\ p \pm 1 & \text{if } p \text{ is odd} \end{cases} \tag{1}$$

We also find

$$e(0) = k - \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (1 - \epsilon_i)(1 - \epsilon_{i,j}),$$

$$e(1) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \epsilon_i \epsilon_{i,j}.$$

Hence

$$
\begin{aligned}
e(0)-e(1) &= k-\sum_i \epsilon_i+\sum_{i=1}^k n_i-\sum_{i,j}\epsilon_{i,j}-\sum_{i=1}^k n_i\epsilon_i \\
&= p-1-\left(\sum_i \epsilon_i+\sum_{i,j}\epsilon_{i,j}\right)-\sum_{i=1}^k n_i\epsilon_i.
\end{aligned}
$$

Together with (1), we obtain the following main result.

Theorem 2.1 *Let p be the number of vertices in $T(k;n_1,n_2,\dots,n_k)$, then*

$$
BI(T(k;n_1,n_2,\dots,n_k)) = \begin{cases} S_1 & \text{if } p \text{ is even,} \\ S_2\cup S_3 & \text{if } p \text{ is odd,} \end{cases}
$$

where

$$
S_1=\left\{\left|\frac{p-2}{2}-\sum_{i=1}^k n_i\epsilon_i\right| : 0\le \epsilon_i,\epsilon_{i,j}\le 1, \sum_i \epsilon_i+\sum_{i,j}\epsilon_{i,j}=\frac{p}{2}\right\},
$$

$$
S_2=\left\{\left|\frac{p-3}{2}-\sum_{i=1}^k n_i\epsilon_i\right| : 0\le \epsilon_i,\epsilon_{i,j}\le 1, \sum_i \epsilon_i+\sum_{i,j}\epsilon_{i,j}=\frac{p+1}{2}\right\},
$$

$$
S_3=\left\{\left|\frac{p-1}{2}-\sum_{i=1}^k n_i\epsilon_i\right| : 0\le \epsilon_i,\epsilon_{i,j}\le 1, \sum_i \epsilon_i+\sum_{i,j}\epsilon_{i,j}=\frac{p-1}{2}\right\}.
$$

In brief, the sets S_1, S_2 and S_3 in Theorem 2.1 are taken over all ϵ_i and $\epsilon_{i,j}$ for which the condition in (1) is met, that is, such that they could yield a friendly labeling. Our first example is the star $\mathrm{St}(n) = K_{1,n}$

Example 2. We give below an easy proof of the following result from [7]:

$$
\mathrm{BI}(\mathrm{St}(n)) = \begin{cases} \{\frac{n-1}{2}\} & \text{if } n \text{ is odd,} \\ \{\frac{n}{2},\frac{n-2}{2}\} & \text{if } n \text{ is even.} \end{cases}
$$

Since $\mathrm{St}(n) = T(n;0^n)$, we have $p = 1+n$, and $n_i = 0$ for each i. It is obvious that (1) can be met. The result follows from $S_1 = \{\frac{n-1}{2}\}$, $S_2 = \{\frac{n-2}{2}\}$, and $S_3 = \{\frac{n}{2}\}$. □

If condition (1) is satisfied for any combination of the ϵ_i's, the description of the balance index sets could be simplified.

Corollary 2.2 *Let $G = T(k; , n_1, n_2, \ldots, n_k)$, where $n_1 \geq n_2 \geq \cdots \geq n_k \geq 0$. Let α be the largest integer such that $n_\alpha \neq 0$. If (i) $\alpha < k$, or (ii) $\alpha = k$, and $n_1 \geq 2$, then*

$$BI(G) = \begin{cases} \left\{\left|\frac{p-2}{2} - \sum_{i=1}^{\alpha} n_i\epsilon_i\right| : 0 \leq \epsilon_i \leq 1\right\} & \text{if } p \text{ is even,} \\ \left\{\left|\frac{p-3}{2} - \sum_{i=1}^{\alpha} n_i\epsilon_i\right| : 0 \leq \epsilon_i \leq 1\right\} & \\ \quad \cup \left\{\left|\frac{p-1}{2} - \sum_{i=1}^{\alpha} n_i\epsilon_i\right| : 0 \leq \epsilon_i \leq 1\right\} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. The definition of α implies that $\sum_{i=1}^{k} n_i\epsilon_i = \sum_{i=1}^{\alpha} n_i\epsilon_i$. Therefore we need to study what values each ϵ_i can assume, such that (1) is satisfied.

If $\alpha < k$, set $\epsilon_{i,1} = 1 - \epsilon_i$ for $1 \leq i \leq \alpha$, and $\epsilon_{\alpha+1} = 1$. If $\alpha = k$ and $n_1 \geq 2$, set $\epsilon_{i,1} = 1 - \epsilon_i$ for $1 \leq i \leq k$, and $\epsilon_{1,2} = 1$. In both cases, we have labeled, thus far, $\alpha + 1$ vertices with 0, and another $\alpha + 1$ vertices with 1. It is clear that we can label the remaining vertices evenly with 0 and 1 to obtain a friendly labeling that satisfies (1). Thus for $1 \leq i \leq \alpha$, each ϵ_i can be either 0 or 1. □

Example 3. The double star $D(m, n)$, where $1 \leq m \leq n$, consists of m pendant vertices appended to one end of P_2, and n pendant vertices appended to the other end. Thus $D(m, n) = T(n + 1; m, 0^n)$. We have $p = m+n+2$, $n_1 = m$, and $n_i = \epsilon_i = 0$ if $i \neq 1$. We find $S_1 = \left\{\frac{n+m}{2}, \frac{n-m}{2}\right\}$ when $n + m$ is even; $S_2 = \left\{\frac{n+m-1}{2}, \frac{n-m-1}{2}\right\}$, and $S_3 = \left\{\frac{n+m+1}{2}, \frac{n-m+1}{2}\right\}$ when $n + m$ is odd. Therefore, we obtain

$$\mathrm{BI}(D(m, n)) = \begin{cases} \left\{\frac{n+m}{2}, \frac{n-m}{2}\right\} & \text{if } n + m \text{ is even,} \\ \left\{\frac{n+m\pm1}{2}, \frac{n-m\pm1}{2}\right\} & \text{if } n + m \text{ is odd,} \end{cases}$$

a result that was first reported in [7]. □

The only tree that Corollary 2.2 does not cover is the ***spider*** $\mathrm{Sp}(2^n)$, which is the amalgamation (that is, one-point union) of n copies of paths of length two identified at one of the two pendant vertices of each path.

Example 4. The following result from [11]

$$\mathrm{BI}(\mathrm{Sp}(2^n)) = S_3 = \{0, 1, 2, \ldots, n\}$$

can be obtained in a very straightforward manner. Note that $\mathrm{Sp}(2^n) = T(n; 1^n)$, hence $p = 1 + 2n$ is always odd, and $n_i = 1$ for each i. Setting $\epsilon_{i,1} = 1 - \epsilon_i$ yields a friendly labeling with $\sum_i \epsilon_i + \sum_{i,j} \epsilon_{i,j} = (p - 1)/2$. Since each ϵ_i can be either 0 or 1, we see that $\sum_{i=1}^{k} n_i\epsilon_i \in \{0, 1, 2, \ldots, n\}$. Hence $S_3 = \{0, 1, 2, \ldots, n\}$.

To ensure that $\sum_i \epsilon_i + \sum_{i,j} \epsilon_{i,j} = (p+1)/2$, we need, without loss of generality, $\epsilon_1 = \epsilon_{1,1} = 1$, and $\epsilon_{i,1} = 1 - \epsilon_i$ if $i \neq 1$. This time, $\sum_{i=1}^{k} n_i \epsilon_i \in \{1, 2, 3, \ldots, n\}$. Hence $S_2 = \{|n-1-j| : 1 \leq j \leq n\} \subseteq S_3$. Therefore $\mathrm{BI}(\mathrm{Sp}(2^n)) = S_3$. □

The last example confirms that Corollary 2.2 is also valid for $\mathrm{Sp}(2^n)$. Hence, we have obtained a rather simple result.

Theorem 2.3 *Let $G = T(k; , n_1, n_2, \ldots, n_k)$, where $n_1 \geq n_2 \geq \cdots \geq n_k \geq 0$. Let α be the largest integer such that $n_\alpha \neq 0$. Then*

$$BI(G) = \begin{cases} \left\{\left|\frac{p-2}{2} - \sum_{i=1}^{\alpha} n_i \epsilon_i\right| : 0 \leq \epsilon_i \leq 1\right\} & \text{if } p \text{ is even,} \\ \left\{\left|\frac{p-3}{2} - \sum_{i=1}^{\alpha} n_i \epsilon_i\right| : 0 \leq \epsilon_i \leq 1\right\} & \\ \quad \cup \left\{\left|\frac{p-1}{2} - \sum_{i=1}^{\alpha} n_i \epsilon_i\right| : 0 \leq \epsilon_i \leq 1\right\} & \text{if } p \text{ is odd.} \end{cases}$$

The ***spider*** $\mathrm{Sp}(\ell_1, \ell_2, \ldots, \ell_n)$ is the amalgamation of n paths of length $\ell_1, \ell_2, \ldots, \ell_n$, respectively. As a tree with diameter at most four, it could take on the form of $\mathrm{Sp}(1^m, 2^n)$, where $m, n \geq 1$. Only a few special cases were studied in [11]. The complete solution is listed below.

Example 5. For $m, n \geq 1$, note that $\mathrm{Sp}(1^m, 2^n) = T(n+m; 1^n, 0^m)$. We have $p = 1 + m + 2n$, and $n_i = 0$ for $n+1 \leq i \leq n+m$; thus $\sum_{i=1}^{n+m} n_i \epsilon_i = \sum_{i=1}^{n} \epsilon_i \in \{0, 1, 2, \ldots, n\}$. We find $S_1 = \{\frac{m-1}{2} + j \mid 0 \leq j \leq n\}$ when m is odd; $S_2 = \{\frac{m}{2} - 1 + j \mid 0 \leq j \leq n\}$, and $S_3 = \{\frac{m}{2} + j \mid 0 \leq j \leq n\}$ when m is even. Therefore

$$\mathrm{BI}(\mathrm{Sp}(1^m, 2^n)) = \begin{cases} \left\{\frac{m-1}{2}, \frac{m-1}{2} + 1, \ldots, \frac{m-1}{2} + n\right\} & \text{if } m \text{ is odd,} \\ \left\{\frac{m}{2} - 1, \frac{m}{2}, \ldots, \frac{m}{2} + n\right\} & \text{if } m \text{ is even.} \end{cases}$$

This general solution covers all the cases studied in [11]. □

A ***caterpillar*** is a tree formed by appending pendant vertices to a path. Denote by $\mathrm{Ct}(n; m_1, m_2, \ldots, m_n)$ the caterpillar which becomes $P_n = v_1 v_2 \ldots v_n$ when all its leaves are deleted, where m_i denotes the number of pendant vertices incident to v_i. For it to be a tree of diameter four, a caterpillar must be of the form $\mathrm{Ct}(3; m_1, m_2, m_3)$, where $m_1 \geq m_3 \geq 1$, and $m_2 \geq 0$. Several special cases were studied in [11]. Here is the complete solution.

Example 6. For $m_1 \geq m_3 \geq 1$,

$$\mathrm{BI}(\mathrm{Ct}(3; m_1, m_2, m_3)) = \begin{cases} S_1 & \text{if } m_1 + m_2 + m_3 \text{ is odd,} \\ S_2 \cup S_3 & \text{if } m_1 + m_2 + m_3 \text{ is even,} \end{cases}$$

where

$$\begin{aligned} S_1 &= \left\{\frac{m_2+m_1\pm m_3+1}{2}, \frac{|m_2-m_1\pm m_3+1|}{2}\right\}, \\ S_2 &= \left\{\frac{m_2+m_1\pm m_3}{2}, \frac{|m_2-m_1\pm m_3|}{2}\right\}, \\ S_3 &= \left\{\frac{m_2+m_1\pm m_3+2}{2}, \frac{|m_2-m_1\pm m_3+2|}{2}\right\}, \end{aligned}$$

because $\mathrm{Ct}(3; m_1, m_2, m_3) = T(m_2+2; m_1, m_3, 0^{m_2})$. □

Example 7. For $T(k; n^k)$, where $n, k \geq 1$, we find

$$\mathrm{BI}(T(k;n^k) = \begin{cases} \left\{\left|\frac{(n+1)k-1}{2} - nj\right| : 0 \leq j \leq k\right\} & \text{if } (n+1)k \text{ is odd,} \\ \left\{\left|\frac{(n+1)k-2}{2} - nj\right| : 0 \leq j \leq k\right\} & \\ \quad \cup \left\{\left|\frac{(n+1)k}{2} - nj\right| : 0 \leq j \leq k\right\} & \text{if } (n+1)k \text{ is even,} \end{cases}$$

because $p = 1 + (n+1)k$, and $n_i = n$ for each i. □

3 An Open Problem

Unlike many examples in existing results, the entries in $\mathrm{BI}(T(k;n^k))$ usually do not form an arithmetic progression, as illustrated in the cases of

$$\begin{aligned} \mathrm{BI}(T(3;5^3)) &= \{1,2,3,4,6,7,8,9\}, \\ \mathrm{BI}(T(3;6^3)) &= \{2,4,8,10\}, \\ \mathrm{BI}(T(3;7^3)) &= \{2,3,4,5,9,10,11,12\}, \\ \mathrm{BI}(T(3;8^3)) &= \{3,5,11,13\}, \\ \mathrm{BI}(T(4;3^4)) &= \{1,2,4,5,7,8\}, \\ \mathrm{BI}(T(4;4^4)) &= \{1,2,3,5,6,7,9,10\}, \\ \mathrm{BI}(T(4;5^4)) &= \{1,2,3,4,6,7,8,9,11,12\}, \\ \mathrm{BI}(T(4;6^4)) &= \{1,2,4,5,7,8,10,11,13,14\}. \end{aligned}$$

However, in some other instances, the entries do form an arithmetic progression:

$$\begin{aligned} \mathrm{BI}(T(3;2^3)) &= \{0,2,4\}, \\ \mathrm{BI}(T(3;3^3)) &= \{0,1,2,3,4,5,6\}, \\ \mathrm{BI}(T(3;4^3)) &= \{1,3,5,7\}. \end{aligned}$$

We invite the readers to investigate when will $\mathrm{BI}(T(k;n^k))$ consist of an arithmetic progression.

4 Rooted Trees of Height Above Two

It is easy to extend the technique to a rooted tree of height 3. Denote the labels of the leaves $\epsilon_{i,j,k}$. We find

$$v(0) - v(1) = p - 2v(1) = p - 2\sum{}',$$

where

$$\sum{}' = \sum_i \epsilon_i + \sum_{i,j} \epsilon_{i,j} + \sum_{i,j,k} \epsilon_{i,j,k}.$$

This allows us to derive a result analogous to (1):

$$2\sum{}' = \begin{cases} p & \text{if } p \text{ is even,} \\ p \pm 1 & \text{if } p \text{ is odd.} \end{cases}$$

Next, we turn our attention to the edges *between level 2 and 3*. Since $e(0) = \sum_{i,j,k}(1 - \epsilon_{i,j})(1 - \epsilon_{i,j,k})$, and $e(1) = \sum_{i,j,k} \epsilon_{i,j}\epsilon_{i,j,k}$,

$$\begin{aligned} e(0) - e(1) &= \sum_{i,j,k}(1 - \epsilon_{i,j} - \epsilon_{i,j,k}) \\ &= \sum_{i,j} n_{i,j} - \sum_{i,j} n_{i,j}\epsilon_{i,j} - \sum_{i,j,k} \epsilon_{i,j,k}, \end{aligned}$$

where $n_{i,j}$ denotes the number of children of $u_{i,j}$. Therefore, over the *entire* rooted tree (taken into consideration what we have already found up to level 2),

$$e(0) - e(1) = p - 1 - \sum{}' - \sum{}'',$$

where

$$\sum{}'' = \sum_i n_i\epsilon_i + \sum_{i,j} n_{i,j}\epsilon_{i,j}.$$

Theorem 4.1 *Let p be the number of vertices in a rooted tree RT of height 3, then*

$$BI(RT) = \begin{cases} S_1 & \text{if } p \text{ is even,} \\ S_2 \cup S_3 & \text{if } p \text{ is odd,} \end{cases}$$

where

$$S_1 = \left\{ \left| \frac{p-2}{2} - \sum{}'' \right| : \ 0 \le \epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k} \le 1, \sum{}' = \frac{p}{2} \right\},$$

$$S_2 = \left\{ \left| \frac{p-3}{2} - \sum{}'' \right| : \ 0 \le \epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k} \le 1, \sum{}' = \frac{p+1}{2} \right\},$$

$$S_3 = \left\{ \left| \frac{p-1}{2} - \sum{}'' \right| : \ 0 \le \epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k} \le 1, \sum{}' = \frac{p-1}{2} \right\},$$

where ϵ_i, $\epsilon_{i,j}$, $\epsilon_{i,j,k}$, $\sum'$ and $\sum''$ are defined above.

It is obvious that the result can be pushed to a rooted tree of any height.

Theorem 4.2 *Let p be the number of vertices in a rooted tree RT of height h, then*

$$BI(RT) = \begin{cases} S_1 & \text{if } p \text{ is even,} \\ S_2 \cup S_3 & \text{if } p \text{ is odd,} \end{cases}$$

where

$$S_1 = \left\{ \left| \frac{p-2}{2} - \sum{}'' \right| : 0 \le \epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k}, \dots \le 1, \sum{}' = \frac{p}{2} \right\},$$

$$S_2 = \left\{ \left| \frac{p-3}{2} - \sum{}'' \right| : 0 \le \epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k}, \dots \le 1, \sum{}' = \frac{p+1}{2} \right\},$$

$$S_3 = \left\{ \left| \frac{p-1}{2} - \sum{}'' \right| : 0 \le \epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k}, \dots \le 1, \sum{}' = \frac{p-1}{2} \right\},$$

where $\epsilon_i, \epsilon_{i,j}, \epsilon_{i,j,k}, \dots$ are the labels of the vertices at level 1, 2, 3, ..., respectively, and

$$\sum{}' = \underbrace{\sum_i \epsilon_i + \sum_{i,j} \epsilon_{i,j} + \sum_{i,j,k} \epsilon_{i,j,k} + \cdots}_{h},$$

$$\sum{}'' = \underbrace{\sum_i n_i \epsilon_i + \sum_{i,j} n_{i,j} \epsilon_{i,j} + \cdots}_{h-1}.$$

5 Another Open Problem

How about an unrooted tree? We can designate a center of it to be its root, and apply Theorem 4.2. Although it does provide a systematic approach to find the balance set of any given tree T, it is not clear whether it is possible to find a simpler description of BI(T). Judging from the theorems and the examples given above, it becomes evident that if a simple description exists, it has to be expressed in terms of the degree sequences of T. We invite the readers to find such a solution.

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