An upper bound on the restrained domination number of graphs*

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ABSTRACT. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called a restrained dominating set of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. In this paper we establish an upper bound on $\gamma_r(G)$ for a connected graph G by the probabilistic method.

Keywords: Restrained domination number; Probabilistic method; Expectation

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1 Introduction

In this paper we consider only finite undirected graphs without isolated vertices and multiple edges. For notation and graph theory terminology not given here, the reader is referred to [12]. Let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E. The *order* of G is given by $n = |V|$. Denote by $\delta(G)$ and $\Delta(G)$ the *minimum* and *maximum* degrees of vertices

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of G, respectively. Put $\Delta = \Delta(G)$ and $\delta = \delta(G)$. For $v \in V$, the *open neighborhood* of v in G is $N(v) = \{u \mid uv \in E\}$ and the *closed neighborhood* of v in G is $N[v] = N(v) \cup \{v\}$. The *degree* of v in G is $d(v) = |N(v)|$. For $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$.

A subset S of V is a *restrained dominating set* (RDS) of G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The minimum cardinality of a restrained dominating set in G is the *restrained domination number* of G, denoted by $\gamma_r(G)$. Restrained domination in graphs was introduced by Telle and Proskurowski [16].

Let G be a connected graph of order n. If G is not a star, then obviously $\gamma_r(G) \leq n - 2$. Domke et al. [6] characterized those graphs achieving the bound. Moreover, Domke et al. [7] showed that if T is a tree of order n, then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$ and characterized the extremal trees T attaining the lower bound. In [8] Domke et al. showed that if G is not one of eight exceptional graphs and $\delta \geq 2$, then $\gamma_r(G) \leq \frac{n-2}{2}$. Recently, Dankelmann et al. [4] showed that if $\delta \geq 2$, then $\gamma_r(G) \leq n - \Delta$. More results on the restrained domination number can be found in, for example, [2-5, 9-11, 13-15, 17].

The purpose of this paper is to establish new upper bound on the restrained domination number by the probabilistic method which be mentioned in [1].

2 Main results

For an event A and for a random variable Z of an arbitrary probability space, $P[A]$ and $E[Z]$ denote the probability of A, the expectation of Z, respectively.

Theorem 1. *If G is a connected graph of order n with minimum degree δ, then $\gamma_r(G) \leq 36\frac{n}{\delta+1} + n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}$.*

Proof. The theorem clearly holds for $\delta < 36$, so we may assume $\delta \geq 36$. Let us pick, randomly and independently, each vertex of V with probability p and let $p = \frac{\ln(\delta+1)}{\delta+1}$. Then $0 < p < 1$. Let X be the set of vertices such picked and $x = |X|$, $P\{v \in X\} = p$. Let Y be the random set of all the vertices that are not picked and have no neighbors in X, i.e., $Y = \{v \in V - X \mid N[v] \cap X = \emptyset\}$, and let $y = |Y|$. By the choice of Y, we can see that $X \cup Y$ is a dominating set of G. Let Z be the set of all isolate vertices in $V - X - Y$ and let $z = |Z|$. Obviously, the set $X \cup Y \cup Z$ is a restrained dominating set of G, so $\gamma_r(G) \le x+y+z$. It therefore suffices to show that the following inequality

$$x+y+z \le 36\frac{n}{\delta+1} + n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}$$

holds with positive probability.

Claim 1.

$$P[x > n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}] < 0.8920.$$

Proof of Claim 1. The expectation of x is $E[x] = np = n\frac{\ln(\delta+1)}{\delta+1}$. We use an inequality attributed to [2], that is, for any $a \ge 0$,

$$P[\alpha > E[\alpha] + a] \le \exp\Big\{\frac{-a^2}{2[E[\alpha]+\frac{a}{3}]}\Big\}$$

Take $a = n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}$ to this inequality, we have

$$\begin{aligned}
P\Big[x &> n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}\Big] \\
&\le \exp\Big\{-\frac{n\ln(\delta+1)}{8(\delta+1)(\ln(\delta+1)+\frac{1}{6}\sqrt{\ln(\delta+1)})}\Big\} \\
&= \exp\Big\{-\frac{n}{8(\delta+1)(1+\frac{1}{6\sqrt{\ln(\delta+1)}})}\Big\} \\
&\le \exp\Big\{-\frac{1}{8+\frac{4}{3}\frac{1}{\sqrt{\ln(\delta+1)}}}\Big\} \\
&\le \exp\Big\{-\frac{1}{8+\frac{4}{3}\frac{1}{\sqrt{\ln 37}}}\Big\} \\
&< 0.8920,
\end{aligned}$$

where $n \ge \delta+1$. This establishes Claim 1. □

Claim 2.

$$P\Big[y > 21\frac{n}{\delta+1}\Big] < 0.04761.$$

Proof of Claim 2. For each vertex $v \in Y$, the probability is exactly $(1-p)^{d(v)+1}$. Since $d(v) \ge \delta$, it follows that the expectation of y satisfies $E[y] \le$

$n(1-p)^{\delta+1}$. By using Taylor's formula, we have

$$\begin{aligned}(1-p)^{\delta+1} &< \exp\{-p(\delta+1)\} = \exp\Big\{-\frac{\ln(\delta+1)}{\delta+1}(\delta+1)\Big\} \\ &= \exp\{-\ln(\delta+1)\} = \frac{1}{\delta+1}.\end{aligned}$$

Thus $E[y] \leq \frac{n}{\delta+1}$. By Markov's inequality, for any $a > 0$, we have $P[y > a] < \frac{E[y]}{a}$, so

$$P\Big[y > 21\frac{n}{\delta+1}\Big] < \frac{1}{21} = 0.04761.$$

The Claim 2 follows. □

It is not easy to bound z directly. Instead, we say that a vertex $v \in V$ is weakly contained if v has fewer than $0.1(\delta - \ln(\delta+1))$ neighbors in $V-X-Y$. We now bound the probability that a vertex is weakly contained in $V-X-Y$. Let Z_v denote the number of neighbors of v in $V-X-Y$. Let D denote the set of vertices weakly contained in $V-X-Y$. Clearly,

$$\begin{aligned}E[Z_v] &= (1-p-(1-p)^{d(v)+1})\cdot d(v) \\ &\geq \Big(1-p-\frac{1}{\delta+1}\Big)\cdot\delta \\ &= \Big(1-\frac{\ln(\delta+1)}{\delta+1}-\frac{1}{\delta+1}\Big)\cdot\delta \\ &= \frac{\delta}{\delta+1}\cdot(\delta-\ln(\delta+1)).\end{aligned}$$

Indeed, when $\delta \geq 36$, we have

$$E[Z_v] \geq \frac{36}{37}(\delta - \ln(\delta+1)).$$

Claim 3.

$$P\Big[|D| > 15n\frac{1}{\delta+1}\Big] < 0.04.$$

Proof of Claim 3. Now we need to bound the lower tail, so we use the inequality in [1]

$$Prob[Z_v - E[Z_v] < -a] < \exp(-\frac{a^2}{2E[Z_v]}),$$

which is valid for every $a > 0$. Using $a = (1 - 37/360)E[Z_v]$, we obtain

$$\begin{aligned}
P[Z_v < 0.1(\delta - \ln(\delta+1))] &\le P\Big[Z_v < \frac{37E[Z_v]}{360}\Big] \\
&= P\Big[Z_v - E[Z_v] < -(1 - \frac{37}{360})E[Z_v]\Big] \\
&< \exp\Big\{-\frac{(1-\frac{37}{360})^2 E[Z_v]^2}{2E[Z_v]}\Big\} \\
&= \exp\Big\{-\frac{(1-\frac{37}{360})^2}{2}E[Z_v]\Big\} \\
&< \exp\Big\{-(\delta - \ln(\delta+1))\frac{36}{37}\frac{(1-\frac{37}{360})^2}{2}\Big\} \\
&< \frac{2}{5\cdot(\delta - \ln(\delta+1))}.
\end{aligned}$$

Hence, the probability that a vertex is weakly contained in $V - X - Y$ is

$$\begin{aligned}
P[v \in V - X - Y \quad &\wedge \quad Z_v \le 0.1(\delta - \ln(\delta+1))] \\
&< (1 - p - (1-p)^{d(v)+1}) \cdot \frac{2}{5(\delta - \ln(\delta+1))} \\
&< (1-p) \cdot \frac{2}{\delta - \ln(\delta+1)} \\
&< \frac{\delta + 1 - \ln(\delta+1)}{\delta+1} \cdot \frac{2}{5\cdot(\delta - \ln(\delta+1))} \\
&< \frac{2}{5\cdot(\delta+1)} \cdot (1 + \frac{1}{\delta - \ln(\delta+1)}) \\
&< \frac{3}{5\cdot(\delta+1)}.
\end{aligned}$$

Therefore, the expected number of weakly contained vertices in Z is at most $\frac{3\cdot n}{5\cdot(\delta+1)}$. We have

$$E[|D|] \le \frac{3\cdot n}{5\cdot(\delta+1)}.$$

From Markov's inequality, it follows that

$$P\Big[|D| > 15n\frac{1}{\delta+1}\Big] < \frac{6}{150} < 0.04.$$

This establishes Claim 3. □

From Claims 1–3, we find that all of these events that

$$x < n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}, \quad y < 21\frac{n}{\delta+1} \text{ and } |D| < 15\frac{n}{\delta+1}$$

could happen simultaneously with positive probability, that is,

$$1 - 0.8920 - 0.04761 - 0.04 = 0.021 > 0.$$

Now we choose a set X satisfying all of these events simultaneously. Since $z \le |D|$, we have

$$\begin{aligned} x + y + z &< 21\frac{n}{\delta+1} + n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1} + 15\frac{n}{\delta+1} \\ &< 36\frac{n}{\delta+1} + n\frac{\ln(\delta+1)}{\delta+1} + n\frac{0.5\sqrt{\ln(\delta+1)}}{\delta+1}. \end{aligned}$$

This completes the proof of Theorem 1. □

The result in Theorem 1 easily leads to the following conclusion.

Theorem 2. *If G is a connected graph of order n with minimum degree δ, then $\gamma_r(G) \le n \cdot \frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))$.*

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