ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATORS FROM THE BERGMAN SPACE TO WEIGHTED-TYPE SPACES ON THE UNIT BALL

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Abstract

We estimate the essential norm of the weighted composition operator uC_φ from the weighted Bergman space $A^p_\alpha(\mathbb{B})$ to the weighted space $H^\infty_\mu(\mathbb{B})$ on the unit ball $\mathbb{B}$, when $p > 1$ and $\alpha \geq -1$ (for $\alpha = -1$, A^p_α is the Hardy space $H^p(\mathbb{B})$). We also give a necessary and sufficient condition for the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ to be compact, and for the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ to be bounded or compact, when $p > 0$, $\alpha \geq -1$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$, $S = \partial\mathbb{B}$ its boundary, $dV(z)$ the Lebesgue measure on $\mathbb{B}$, $dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z)$ where c_α is a constant chosen such that $V_\alpha(\mathbb{B}) = 1$, $d\sigma$ the rotation invariant measure on S such that $\sigma(S) = 1$, $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$ and $H^\infty(\mathbb{B})$ the space of all bounded holomorphic functions on $\mathbb{B}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

For $p > 0$ the Hardy space $H^p = H^p(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$

It is well known that for every $f \in H^p$ the radial limit $\lim_{r\to 1} f(r\zeta)$ exists for almost all $\zeta \in S$. The limit function is denoted by $f^*(\zeta)$.

The Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{B})$, $p > 0$, $\alpha > -1$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) < \infty.$$

When $p \geq 1$, the Bergman space with the norm $\|\cdot\|_{A^p_\alpha}$ becomes a Banach space. If $p \in (0,1)$, it is a Frechet space with the translation invariant metric

$$d(f,g) = \|f - g\|^p_{A^p_\alpha}.$$

Since for every $f \in H^p$

$$\lim_{\alpha \to -1+0} \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) = \int_S |f^*(\zeta)|^p d\sigma(\zeta)$$

we will also use the notation A^p_{-1} for the Hardy space H^p.

A positive continuous function ϕ on $[0,1)$ is called normal ([25]) if there is $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0;$$
$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = \infty.$$

If we say that a function $\phi : \mathbb{B} \to [0,\infty)$ is normal we will also assume that $\phi(z) = \phi(|z|)$, $z \in \mathbb{B}$.

The weighted space $H^\infty_\mu = H^\infty_\mu(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^\infty_\mu} := \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < \infty,$$

where μ is normal. For $\mu(z) = (1-|z|^2)^\beta$, $\beta > 0$ we obtain the weighted space $H^\infty_\beta = H^\infty_\beta(\mathbb{B})$ (for the weighted Bolch space see, e.g., [35] and [36]).

The little weighted space $H^\infty_{\mu,0} = H^\infty_{\mu,0}(\mathbb{B})$ is a subspace of H^∞_μ consisting of all $f \in H(\mathbb{B})$ such that $\lim_{|z|\to 1} \mu(z)|f(z)| = 0$.

Let $u \in H(\mathbb{B})$ and φ be a holomorphic self-map of $\mathbb{B}$. For $f \in H(\mathbb{B})$ the weighted composition operator is defined by

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)).$$

It is of interest to provide function theoretic characterizations when u and φ induce bounded or compact weighted composition operators on spaces of holomorphic functions. For some classical results in the topic, see [6]. For some recent results see, e.g., [5], [10]-[22], [24], [29], [32], [34], [35], [37]-[39], [41]-[43] and the references therein.

In [32], among others, we give some necessary and sufficient conditions for the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_\beta(\mathbb{B})$ to be bounded or compact. Motivated by [32] and [34], in [35], among other results, we calculated the operator norm of $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$. More precisely, we have proved the following result:

Theorem A. *Assume $p > 0$, $\alpha \geq -1$, $u \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Then*

$$\|uC_\varphi\|_{A^p_\alpha \to H^\infty_\mu} = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} =: M.$$

Moreover, $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded if and only if M is finite.

In this note we estimate the essential norm of the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$, when $p > 1$. For the completeness, we also give a necessary and sufficient condition for the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ to be compact, and for the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ to be bounded or compact, when $p > 0$ (these results are modifications of some in [32], see also [24, 37, 41]).

Throughout the paper C will denote a positive constant not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \le B \le CA$.

We need the following auxiliary results in the proofs of the main results.

Lemma 1. ([4, Corollary 3.5]) *Suppose $p \in (0, \infty)$ and $\alpha \ge -1$. Then for all $f \in A^p_\alpha(\mathbb{B})$ and $z \in \mathbb{B}$, the following inequality holds*

$$|f(z)| \le \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{n+1+\alpha}{p}}}. \tag{1}$$

The following criterion for the compactness follows by standard arguments (see, e.g., [6, 9, 28, 29, 30]). Hence, we omit its proof.

Lemma 2. *Suppose $0 < p < \infty$, $\alpha \ge -1$, $u \in H(\mathbb{B})$, μ is normal and φ is a holomorphic self-map of $\mathbb{B}$. Then the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact if and only if $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A^p_α converging to zero uniformly on compacts of $\mathbb{B}$, we have $\|uC_\varphi f_k\|_{H^\infty_\mu} \to 0$ as $k \to \infty$.*

The following result can be found in [23]. For closely related results see also [1, 2, 3, 7, 26, 27, 31, 33, 40] and the references therein.

Lemma 3. *Suppose $0 < p < \infty$, $\alpha > -1$, then*

$$\|f\|^p_{A^p_\alpha} \asymp |f(0)|^p + \int_{\mathbb{B}} |\nabla f(z)|^p (1-|z|^2)^{p+\alpha} dV(z),$$

for every $f \in A^p_\alpha$.

The next lemma can be proved similar to Lemma 1 in [19] (see also [20]).

Lemma 4. *Suppose μ is normal. A closed set K in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|f(z)| = 0.$$

2. The boundedness of the operator $uC_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$

Here we characterize the boundedness of the operator $uC_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$.

Theorem 1. *Assume $p > 0$, $\alpha \ge -1$, $u \in H(\mathbb{B})$, μ is normal and φ is a holomorphic self-map of $\mathbb{B}$. Then $uC_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded if and only if $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and $u \in H^\infty_{\mu,0}$.*

Proof. Assume that $uC_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded. Then clearly $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Taking the test function $f(z) = 1 \in A^p_\alpha$ we obtain $u \in H^\infty_{\mu,0}$.

Conversely, assume $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded and $u \in H^\infty_{\mu,0}$. Then, for each polynomial p, we have

$$\mu(z)|uC_\varphi p(z)| \le \mu(z)|u(z)p(\varphi(z))| \le \mu(z)|u(z)|\,\|p\|_\infty \to 0, \quad \text{as } |z| \to 1$$

from which it follows that $uC_\varphi p \in H^\infty_{\mu,0}$. Since the set of all polynomials is dense in A^p_α (see, for example, [40]), we have that for every $f \in A^p_\alpha$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{A^p_\alpha} \to 0$, as $k \to \infty$. From this and since the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded, it follows that

$$\|uC_\varphi f - uC_\varphi p_k\|_{H^\infty_\mu} \le \|uC_\varphi\|_{A^p_\alpha \to H^\infty_\mu}\|f - p_k\|_{A^p_\alpha} \to 0, \quad \text{as} \quad k \to \infty.$$

Hence $uC_\varphi(A^p_\alpha) \subset H^\infty_{\mu,0}$. Since $H^\infty_{\mu,0}$ is a closed subset of H^∞_μ the boundedness of $uC_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$ follows. □

3. Compactness of the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$

This section is devoted to studying of the compactness of the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$. We prove the following result.

Theorem 2. *Assume $p > 0$, $\alpha \ge -1$, $u \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Then the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} = 0. \tag{2}$$

Proof. First assume that the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is compact. If $\|\varphi\|_\infty < 1$ then condition (2) is vacuously satisfied. Hence, assume that $\|\varphi\|_\infty = 1$ and assume to the contrary that (2) does not hold. Then there is a sequence $(z_k)_{k\in\mathbb{N}}$ satisfying the condition $|\varphi(z_k)| \to 1$ as $k \to \infty$ and $\delta > 0$ such that

$$\frac{\mu(z_k)|u(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}} \ge \delta, \quad k \in \mathbb{N}. \tag{3}$$

For $w \in \mathbb{B}$ fixed, set

$$f_w(z) = \frac{(1 - |w|^2)^{\frac{n+1+\alpha}{p}}}{(1 - \langle z, w\rangle)^{\frac{2(n+1+\alpha)}{p}}}, \quad z \in \mathbb{B}. \tag{4}$$

It is known that $\|f_w\|_{A^p_\alpha} = 1$, for each $w \in \mathbb{B}$. Let $g_k(z) = f_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$. Then $\|g_k\|_{A^p_\alpha} = 1$, $k \in \mathbb{N}$ and it is easy to see that $g_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to \infty$. Hence, by Lemma 2, it follows that $\lim_{k\to\infty} \|uC_\varphi g_k\|_{H^\infty_\mu} = 0$.

On the other hand, we have

$$\|uC_\varphi g_k\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)|u(z)||g_k(\varphi(z))| \geq \frac{\mu(z_k)|u(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}} \geq \delta > 0,$$

for every $k \in \mathbb{N}$, which is a contradiction.

Now assume that (2) holds. Then for every $\varepsilon > 0$ there is an $r \in (0,1)$ such that when $r < |\varphi(z)| < 1$

$$\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} < \varepsilon. \tag{5}$$

On the other hand, since the operator $uC_\varphi : A_\alpha^p \to H_\mu^\infty$ is bounded, for $f(z) = 1 \in A_\alpha^p$, we obtain $\|u\|_{H_\mu^\infty} < \infty$.

Assume that $(h_k)_{k\in\mathbb{N}}$ is a bounded sequence in A_α^p, say by L, converging to zero uniformly on compacts of $\mathbb{B}$ as $k \to \infty$. Then by Lemma 1 and (5), for $r < |\varphi(z)| < 1$, we obtain

$$\mu(z)|u(z)||h_k(\varphi(z))| \leq \sup_{k\in\mathbb{N}} \|h_k\|_{A_\alpha^p} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} < L\varepsilon. \tag{6}$$

If $|\varphi(z)| \leq r$, we have

$$\mu(z)|u(z)||h_k(\varphi(z))| \leq \|u\|_{H_\mu^\infty} \sup_{|w|\leq r} |h_k(w)| \to 0, \quad \text{as} \quad k \to \infty. \tag{7}$$

From (6) and (7) it follows that $\|uC_\varphi h_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$, from which the compactness of the operator $uC_\varphi : A_\alpha^p \to H_\mu^\infty$ follows. □

3. Compactness of the operator $uC_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$

Here we study of the compactness of the operator $uC_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$.

Theorem 3. *Assume $p > 0$, $\alpha \geq -1$, $u \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and the operator $uC_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ is bounded. Then the operator $uC_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} = 0. \tag{8}$$

Proof. Assume $uC_\varphi : A_\alpha^p \to H_{\mu,0}^\infty$ is compact. Then $uC_\varphi(1) = u \in H_{\mu,0}^\infty$ (see the proof of Theorem 1).

Hence if $\|\varphi\|_\infty < 1$, then

$$\lim_{|z|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \leq \lim_{|z|\to 1} \frac{\mu(z)|u(z)|}{(1-\|\varphi\|_\infty^2)^{\frac{n+1+\alpha}{p}}} = 0,$$

from which the result follows in this case.

Now assume $\|\varphi\|_\infty = 1$. By using the test functions $g_k(z) = f_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$ (where f_w is defined in (4)), as in Theorem 2 we obtain that condition (2) holds, which implies that for every $\varepsilon > 0$, there is an $r \in (0,1)$ such that for $r < |\varphi(z)| < 1$ condition (5) holds.

Since $u \in H^\infty_{\mu,0}$, there is $\sigma \in (0,1)$ such that for $\sigma < |z| < 1$

$$\mu(z)|u(z)| < \varepsilon(1 - r^2)^{\frac{n+1+\alpha}{p}}. \tag{9}$$

Hence, if $|\varphi(z)| \le r$ and $\sigma < |z| < 1$, we have

$$\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \le \frac{\mu(z)|u(z)|}{(1-r^2)^{\frac{n+1+\alpha}{p}}} < \varepsilon. \tag{10}$$

From (5) and (10) condition (8) follows.

Now assume that condition (8) holds. Then the quantity M in Theorem A is finite. From this and the following inequality

$$\mu(z)|u(z)f(\varphi(z))| \le \|f\|_{A^p_\alpha} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}},$$

it follows that the set $uC_\varphi(\{f : \|f\|_{A^p_\alpha} \le 1\})$ is bounded in H^∞_μ, and moreover in $H^\infty_{\mu,0}$. Taking the supremum in the last inequality over the unit ball in A^p_α, then letting $|z| \to 1$, using condition (8) and employing Lemma 4, we obtain the compactness of the operator $uC_\varphi : A^p_\alpha \to H^\infty_{\mu,0}$, as desired. □

4. Essential norm of $uC_\varphi : A^p_\alpha \to H^\infty_\mu$

Let X and Y be Banach spaces, and $L : X \to Y$ be a bounded linear operator. The essential norm of the operator $L : X \to Y$, denoted by $\|L\|_{e,X\to Y}$, is defined as follows

$$\|L\|_{e,X\to Y} = \inf\{\|L+K\|_{X\to Y} \ : \ K \text{ is compact from } X \text{ to } Y\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm.

From this definition and since the set of all compact operators is a closed subset of the set of bounded operators it follows that operator L is compact if and only if $\|L\|_{e,X\to Y} = 0$.

In this section, we prove the main result in this paper, namely we find some lower and upper bounds for the essential norm of the operator $uC_\varphi : A^p_\alpha \to H^\infty_\mu$, when $p > 1$.

Theorem 4. *Assume $p \in (1,\infty)$, $\alpha \ge -1$, $u \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$ and $uC_\varphi : A^p_\alpha \to H^\infty_\mu$ is bounded. Then the following inequalities hold*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}} \le \|uC_\varphi\|_{e,A^p_\alpha\to H^\infty_\mu} \le 2\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}. \tag{11}$$

Proof. Assume that $(\varphi(z_k))_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Note that the sequence $(f_{\varphi(z_k)})_{k\in\mathbb{N}}$ (where f_w is defined in (4)) is such that $\|f_{\varphi(z_k)}\|_{A^p_\alpha} = 1$, for each $k \in \mathbb{N}$ and it converges to zero uniformly on compacts of $\mathbb{B}$. By Theorems 2.12 and 4.50 in [40] it follows that $f_{\varphi(z_k)}$ converges weakly to zero as $k \to \infty$ (here we use condition $p > 1$). Hence for every compact operator $K : A^p_\alpha \to H^\infty_\mu$ we have that $\|Kf_{\varphi(z_k)}\|_{H^\infty_\mu} \to 0$ as $k \to \infty$. Hence, for every such sequence and for every compact operator $K : A^p_\alpha \to H^\infty_\mu$ we have

$$\begin{aligned}
\|uC_\varphi + K\|_{A^p_\alpha \to H^\infty_\mu} &\geq \limsup_{k\to\infty} \frac{\|uC_\varphi f_{\varphi(z_k)}\|_{H^\infty_\mu} - \|Kf_{\varphi(z_k)}\|_{H^\infty_\mu}}{\|f_{\varphi(z_k)}\|_{A^p_\alpha}} \\
&= \limsup_{k\to\infty} \|uC_\varphi f_{\varphi(z_k)}\|_{H^\infty_\mu} \\
&\geq \limsup_{k\to\infty} \mu(z_k)|u(z_k) f_{\varphi(z_k)}(\varphi(z_k))| \\
&= \limsup_{n\to\infty} \frac{\mu(z_k)|u(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}}. \qquad (12)
\end{aligned}$$

Taking the infimum in (12) over the set of all compact operators $K : A^p_\alpha \to H^\infty_\mu$ we obtain

$$\|uC_\varphi\|_{e, A^p_\alpha \to H^\infty_\mu} \geq \limsup_{n\to\infty} \frac{\mu(z_k)|u(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}}},$$

from which the first inequality in (11) follows.

Now we prove the second inequality in (11). Assume that $(r_l)_{l\in\mathbb{N}}$ is a sequence which increasingly converges to 1. Consider the operators defined by

$$(uC_{r_l\varphi} f)(z) = u(z) f(r_l \varphi(z)), \quad l \in \mathbb{N}.$$

We prove that these operators are compact. Indeed, since $|r_l\varphi(z)| \leq r_l < 1$, it follows that condition (2) in Theorem 2 is vacuously satisfied, from which the claim follows.

Recall that $u \in H^\infty_\mu$. Let $\rho \in (0,1)$ be fixed for a moment. Employing Lemma 1, and using the fact

$$\|f - f_{r_l}\|_{A^p_\alpha} \leq 2\|f\|_{A^p_\alpha}, \quad l \in \mathbb{N},$$

which follows by using the triangle inequality for the norm, the monotonicity of the integral means

$$M^p_p(f,r) = \int_S |f(r\zeta)|^p d\sigma(\zeta)$$

and the polar coordinates, it follows that

$$
\begin{aligned}
\|uC_\varphi - uC_{r_l\varphi}\|_{A^p_\alpha\to H^\infty_\mu} &= \sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{z\in\mathbb{B}}\mu(z)|u(z)||f(\varphi(z))-f(r_l\varphi(z))| \\
&\le \sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}\mu(z)|u(z)||f(\varphi(z))-f(r_l\varphi(z))| \\
&\quad + \sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi(z)|>\rho}\mu(z)|u(z)||f(\varphi(z))-f(r_l\varphi(z))| \\
&\le \|u\|_{H^\infty_\mu}\sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}|f(\varphi(z))-f(r_l\varphi(z))| \quad (13)\\
&\quad +2\sup_{|\varphi(z)|>\rho}\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}}}. \quad (14)
\end{aligned}
$$

Now we estimate the quantity in (13). Let

$$
I_l := \sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}|f(\varphi(z))-f(r_l\varphi(z))|.
$$

By using the mean value theorem, the subharmonicity of the partial derivatives of f and Lemma 3, when $\alpha > -1$ we obtain

$$
\begin{aligned}
I_l &\le \sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi(z)|\le\rho}(1-r_l)|\varphi(z)|\sup_{|w|\le\rho}|\nabla f(w)| \quad (15)\\
&\le C_\rho(1-r_l)\sup_{\|f\|_{A^p_\alpha}\le 1}\left(\int_{|w|\le\frac{1+\rho}{2}}|\nabla f(w)|^p(1-|w|^2)^{p+\alpha}dV(w)\right)^{1/p} \\
&\le C_\rho(1-r_l)\sup_{\|f\|_{A^p_\alpha}\le 1}\left(\int_{\mathbb{B}}|f(w)|^p(1-|w|^2)^{\alpha}dV(w)\right)^{1/p}. \\
&\le C_\rho(1-r_l)\to 0 \quad \text{as } l\to\infty. \quad (16)
\end{aligned}
$$

If $\alpha = -1$, then applying in (15), the well known fact that for each compact $K \subset \mathbb{B}$ there is a positive constant C depending on K, p and n such that

$$
\sup_{w\in K}|\nabla f(w)| \le C\|f\|_p,
$$

(see, for example, [40]) we obtain that (16) also holds in this case.

Letting $l \to \infty$ in (13) and (14), using (16), and then letting $\rho \to 1$, the second inequality in (11) follows, finishing the proof of the theorem. □

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