The Upper and Lower Geodetic Numbers of Graphs *

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Abstract For every two vertices u and v in a graph G, a u-v geodesic is a shortest path between u and v. Let $I(u,v)$ denote the set of all vertices lying on a u-v geodesic. For a vertex subset S, let $I_G(S)$ denote the union of all $I_G(u,v)$ for $u,v \in S$. The geodetic number $g(G)$ of a graph G is the minimum cardinality of a set S with $I_G(S) = V(G)$. For a digraph D, there is analogous terminology for the geodetic number $g(D)$. The geodetic spectrum of a graph G, denote by $S(G)$, is the set of geodetic numbers over all orientations of graph G. The lower geodetic number is $g^-(G) = \min S(G)$ and the upper geodetic number is $g^+(G) = \max S(G)$. The main purpose of this paper is to investigate lower and upper geodetic numbers of graphs. Our main results in this paper are:

(i) For every spanning tree T of a connected graph G, $g^-(G) \leq \ell(T)$, where $\ell(T)$ is the number of leaves of T.

(ii) The conjecture $g^+(G) \geq g(G)$ is true for chordal graphs, triangle-free graphs and 4-colorable graphs.

Keywords: Convex set; Digraph; Distance; Geodesic; Geodetic number

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph. If $xy \in E(G)$, we say that y is a *neighbor* of x, and denote by $N(x)$ the set of neighbors of x. $\deg(x) = |N(x)|$ is called the *degree* of x. Let $S \subseteq V(G)$. Denote by $G - S$ the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. When $S = \{x\}$, we simplify this notation to $G - x$. A set S of vertices of G is called a *cut set* of G if $G - S$ has more components than G; if S consists of a single vertex, we simply call it a *cut-vertex*. A subgraph H of G is said to be *induced* by S if $V(H) = S$ and every edge of G contained in S belongs to $E(H)xy \in E(H)$; we also use $G[S]$ to denote the graph induced by S. If every two vertices of S are adjacent, we say $G[S]$ is a *clique*. S is an *independent set* if no two vertices of S are adjacent in G. A subgraph H is called a *spanning subgraph* of G if $V(H) = V(G)$ and $xy \in E(H)$ implies $xy \in E(G)$. A *tree* T is a connected graph with no cycle. A vertex x is called a *leaf* of tree T if the degree of x is one. T is a *spanning tree* of a connected graph G if T is a spanning subgraph of G and T is tree. The readers are referred to [15] for other basic definitions.

For every two vertices u and v in a graph G (digraph D, respectively), a u-v geodesic of graph G (digraph D, respectively) is a shortest path between u and v (from u to v, respectively). Let $I_G(u,v)$ ($I_D(u,v)$, respectively) denote the set of all vertices lying on a u-v geodesic. For a vertex subset S of a graph G(digraph D, respectively), let $I_G(S)$ ($I_D(S)$, respectively) denote the union of all $I_G(u,v)$ ($I_D(u,v)$, respectively) for $u, v \in S$. A *geodetic set* of G (D, respectively) is a set S with $I_G(S) = V(G)$ ($I_D(S) = V(D)$, respectively). The *geodetic number* $g(G)$ ($g(D)$, respectively) of a graph G (digraph D, respectively) is the minimum cardinality of a geodetic set of G (D, respectively), and we call such geodetic set *minimum geodetic set.*

An *orientation* of a graph G, denoted by $\overrightarrow{G}$, is a digraph obtained from G by assigning to each edge of G a direction. The *geodetic spectrum* of G is the set

$$S(G) = \{g(\overrightarrow{G}) : \overrightarrow{G} \text{ is an orientation of } G\}. \tag{1}$$

The *lower geodetic number* of G is $g^-(G) = \min S(G)$, and the *upper geodetic number* is $g^+(G) = \max S(G)$. The orientation $\overrightarrow{G}$ is called a *minimum spectrum orientation* when $g(\overrightarrow{G}) = g^-(G)$ and a *maximum spectrum orientation* when $g(\overrightarrow{G}) = g^+(G)$. The concepts of the geodetic number and geodetic spectrum of a graph are introduced in [2] and [3] and investigated further in ([3]; [4]; [6]-[8]).

The main purpose of this paper is to investigate lower and upper geodetic numbers of graphs.

2 Lower geodetic number of graphs

First we give a useful result on geodetic sets of graphs.

Theorem 2.1 *Let S be a vertex cut set of connected graph G and C be a component of $G - S$. If $G[S]$ is a clique, then*
(i) for every geodetic set N of graph G, $V(C) \cap N \neq \phi$;
(ii) for every geodetic set N of an orientation $\vec{G}$ of graph G, if no arc of $\vec{G}\,[S]$ is contained in a directed cycle, then $V(C) \cap N \neq \phi$.

Proof (i) Suppose there exists a component C of $G - S$ with $V(C) \cap N = \phi$. Then for every vertex $v \in V(C)$, there must exist two distinct vertices $v_s, v_t \in N$ (note that $v_s, v_t \notin V(C)$) such that $v \in I_G(v_s, v_t)$. Since S is a cut set, $I_G(v, v_s) \cap S \neq \phi$ and $I_G(v, v_t) \cap S \neq \phi$. Assume that $v_a \in I_G(v, v_s) \cap S$ and $v_b \in I_G(v, v_t) \cap S$. Obviously, $v_a \neq v_b$, otherwise, $v \notin I_G(v_s, v_t)$. Since the subgraph induced by S is a clique, $v_a v_b \in E(G)$ and the path $v_s - \cdots - v_a - v_b - \cdots - v_t$ is shorter than the shortest path $v_s - \cdots v_a - \cdots - v - \cdots - v_b - \cdots - v_t$, a contradiction.

(ii) Suppose there exists a component C of $G - S$ with $V(C) \cap N = \phi$. Then for every vertex $v \in V(C)$, there also exist two distinct vertices $v_s, v_t \in N$ (note that $v_s, v_t \notin V(C)$) with $v \in I_{\vec{G}}(v_s, v_t)$ and $I_{\vec{G}}(v_s, v) \cap S \neq \phi$, $I_G(v, v_t) \cap S \neq \phi$. Assume that $v_a \in I_{\vec{G}}(v_s, v) \cap S$ and $v_b \in I_{\vec{G}}(v, v_t) \cap S$. Obviously, $v_a \neq v_b$. Since no arc of $\vec{G}\,[S]$ is contained in a directed cycle, $v_a v_b \in E(\vec{G})$. Similarly, we can find a shorter directed path from v_s to v_t passing through the edge $v_a v_b$. This is a contradiction. □

A vertex v is a *simplicial vertex* of G if the subgraph induced by its neighbors is a clique. In [6], Chartrand, Harary and Zhang have shown that every geodetic set of a graph contains its simplicial vertices. From Theorem 2.1, we immediately obtain:

Corollary 2.1.1 *Every simplicial vertex is in every geodetic set of G. Every simplicial vertex of a graph G is also in every geodetic set of all orientation of G without directed cycles.*

Another result gained from Theorem 2.1 is

Corollary 2.1.2 *(i) Any minimum geodetic set of a connected graph G does not contain any cut-vertex;*

(ii) If G-v has at least three components, then v is not contained in any minimum geodetic set of any minimum spectrum orientations of G.

Proof (i) Let G_1 and G_2 be two components of $G - v$. Assume that S is a minimum geodetic set of G and $v \in S$. According to Theorem 2.1,

$S \cap V(G_1) \neq \phi$ and $S \cap V(G_2) \neq \phi$. Assume that $v_s \in S \cap V(G_1)$ and $v_t \in S \cap V(G_2)$. Obviously, $I_G(v_s, v) \subset I_G(v_s, v_t)$ and $I_G(v, v_t) \subset I_G(v_s, v_t)$. Hence $I_G(S - v) = I_G(S) = V(G)$, a contradiction to the minimality of S.

(ii) Let $\vec{G}$ be a minimum spectrum orientation of G and S is a minimum geodetic set of $\vec{G}$. Suppose $v \in S$ and $G_1, \cdots, G_k (k \geq 3)$ is connected components of $G - v$. According to Theorem 2.1, $S \cap V(G_i) \neq \phi$ for $1 \leq i \leq k$. We first claim that, for each component G_i $(1 \leq i \leq k)$, there exists some vertex $v_i \in S \cap V(G_i)$ such that there is directed path connected v_i with v. If not, there exists some $j \in \{1, 2, \cdots, k\}$ such that there does not exist any directed path connecting v with any vertices in $S \cap V(G_j)$. Hence, $I_{\vec{G}}(S \cap V(G_j)) = V(G_j)$. Let x be a vertex in $N_G(v) \cap V(G_j)$. Obviously $x \notin S$ and x lies on a shortest directed path between some two vertices in $S \cap V(G_j)$. No matter what the direction of the edge xv is, there must exist a directed path connecting v and one of the vertices in $S \cap V(G_j)$.

Since $k \geq 3$, by reversing the direction of all the edges of some components (if needed), we can obtain another orientation $\vec{G'}$ for which the following hold: for every $v_i \in S \cap V(G_i)$ $(1 \leq i \leq k)$, if there exists a directed path from v_i to v in $\vec{G'}$ (from v to v_i in $\vec{G'}$, respectively), there exist $j \neq i$ and $u_j \in S \cap V(G_j)$ such that there is a directed path from v to u_j in $\vec{G'}$ (from u_j to v in $\vec{G'}$, respectively). Therefore, $I_{\vec{G'}}(S - v) = I_{\vec{G}}(S) = V(G)$. This contradicts the fact that $\vec{G}$ is a minimum spectrum orientation. Hence $v \notin S$. □

Remark 1 *If $G - v$ has two components, the results in Corollary 2.1.2 may be false. See Figure 1, the cut-vertex v belongs to any minimum geodetic set of $\vec{G}$, where $\vec{G}$ is a minimum spectrum orientation of G.*

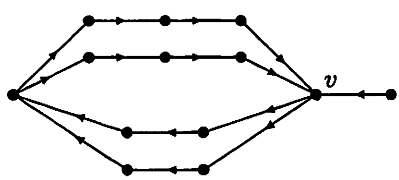

Figure 1: a minimum spectrum orientation of G.

Let f be a mapping from $V(G)$ to the set N of positive integers. An orientation $\vec{G}$ of G is *compatible* with f if the following hold:

(i) if uv is an edge and $f(u) - f(v) = 1$, then the edge is oriented from v to u.

(ii) if uv is an edge and $f(u) - f(v) \geq 2$, then the edge is oriented from u to v.

(iii) if uv is an edge and $f(u) = f(v)$, then the orientation of the edge is arbitrary.

Theorem 2.2 *Let $\ell(T)$ denote the number of leaves in a tree T. For every nontrivial connected graph,*

$$g^-(G) \leq \min\{\ell(T) : T \text{ is a spanning tree of } G\}.$$

Proof Let T be a spanning tree of a connected graph G and suppose that u is a leaf of T. For $i \geq 0$, let V_i be the set of vertices at distance i from u in tree T. Obviously, u is the unique vertex in V_0. First we define the mapping f on $V(G)$ by $f(v) = i + 1$ if $v \in V_i$. As above, a compatible orientation $\vec{G}$ of G with f can be obtained. For any leaf $v \in V(T)$, the unique path between u and v in T is a directed path from u to v in $\vec{G}$ and the values of f on the vertices of this path are continuously increased by one from u to v. Suppose that $P : u = v_1 \to v_2 \to \cdots \to v_k = v$ is this directed path. If P is not a u-v geodesic in $\vec{G}$, we assume that $P' : u = u_1 \to u_2 \to \cdots \to u_t = v$ with $t < k$ is a u-v geodesic of $\vec{G}$. By the definition of f, we know $f(u_i) \leq f(u_{i-1}) + 1$ for $2 \leq i \leq t$. Hence, $f(v) \leq f(u) + t - 1$. Since $f(v) = f(u) + k - 1$, it follows that $k \leq t$, producing a contradiction. Therefore, P is a u-v geodesic of $\vec{G}$. Since T is a spanning tree of G and $g(T) = \ell(T)$ (see [6]), so $g(\vec{G}) \leq \ell(T)$. Hence, $g^-(G) \leq \ell(T)$. □

From Theorem 2.2, we can directly obtain the following result in [9].

Corollary 2.2.1 *([9]) If G with order $n \geq 2$ contains a Hamiltonian path, then $g^-(G) = 2$.*

3 On the conjecture $g^+(G) \geq g(G)$

It was conjectured in [13] that $g^+(G) \geq g(G)$ for every graph G. Results in ([3],[6],[9]) establish this conjecture for several classes of graphs including complete graphs, complete multipartite graphs, cycles and trees. It was shown in [13] that the conjecture is also true for all graphs G with $g(G) \leq 4$ and for all those connected graphs G of order n with $\text{diam}(G) \geq \frac{n-1}{2}$. In this section, we will show that this conjecture is true for triangle-free graphs, 4-colorable graphs and chordal graphs.

Theorem 3.1 *If G is a triangle-free graph, then $g^+(G) \geq g(G)$.*

Proof Let V_1 be a maximum independent set in G and let V_2 be a maximum independent set in $G - V_1$. We claim that $V_1 \cup V_2$ is a geodetic set of G. For each $v \in V_i$ with $i \neq 1, 2$, v must be adjacent to some vertex (say v_1) in V_1 since V_1 is a maximum independent set of G. Similarly, v must be adjacent to some vertex (say v_2) in V_2 since V_2 is a maximum independent set of $G - V_1$. Since G has no 3-cycle, v_1 is not adjacent to v_2. So, the distance between v_1 and v_2 in G is 2. Hence v lies on a v_1-v_2 geodesic. Therefore, $V_1 \cup V_2$ is a geodetic set of G.

Now we orient $E(G)$ as follows. Let $uv \in E(G)$.
(i) If $u \in V_1$, then the direction of uv is from u to v;
(ii) If $u \in V_2$, then the direction of uv is from v to u;
(iii) For the cases not covered in (i) and (ii), oriented uv arbitrarily.

Obviously, all vertices in V_1 are sources and all vertices in V_2 are sinks. Thus, $g(\vec{G}) \geq |V_1| + |V_2|$. Hence, $g^+(G) \geq g(\vec{G}) \geq |V_1| + |V_2| \geq g(G)$. □

A graph G is *k-colorable* if there exists an assignment of k colors $1, 2, ...k$ to $V(G)$ such that no two distinct adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of graph G is the minimum k for which G is $k-$colorable. The distance from u to v in an orientation $\vec{G}$, denoted by $d_{\vec{G}}(u, v)$, is the length of u-v geodesic. If there are no directed path from u to v, we define $d_{\vec{G}}(u, v) = \infty$. Let $d(\vec{G}) = \max\{d_{\vec{G}}(u, v) < \infty : \forall u, v \in V(G)\}$.

Lemma 3.1.1 *Let $\vec{G}$ be an orientation of a graph G. Then $g(G) \leq g^+(G)$ if one of the following statements holds:*

(i) For all $u, v \in V(G)$, $I_{\vec{G}}(u, v) \cup I_{\vec{G}}(v, u) \subseteq I_G(u, v)$;

(ii) The orientation $\vec{G}$ has no directed cycle and $d(\vec{G}) \leq 2$.

Proof (i) Suppose S is a minimum geodetic set of $\vec{G}$. Since $I_{\vec{G}}(u, v) \cup I_{\vec{G}}(v, u) \subseteq I_G(u, v)$ for any $u, v \in S$, $I_G(S) = V(G)$, i.e., S is also a geodetic set of G. Hence, $g(G) \leq |S| = g(\vec{G}) \leq g^+(G)$.

(ii) Since $\vec{G}$ has no directed cycles, $d_{\vec{G}}(u, v) < \infty$ implies that $d_{\vec{G}}(v, u) = \infty$ for any $u, v \in V(G)$. If $d_{\vec{G}}(u, v) = d_{\vec{G}}(v, u) = \infty$, then $I_{\vec{G}}(u, v) \cup I_{\vec{G}}(v, u) = \{u, v\} \subseteq I_G(u, v)$; If $d_{\vec{G}}(u, v) = 1$, then $I_{\vec{G}}(u, v) \cup I_{\vec{G}}(v, u) = \{u, v\} \subseteq I_G(u, v) = \{u, v\}$ and if $d_{\vec{G}}(u, v) = 2$, then $I_{\vec{G}}(u, v) \cup I_{\vec{G}}(v, u) = I_{\vec{G}}(u, v) \subseteq I_G(u, v)$. Hence, $g(G) \leq g^+(G)$. □

Theorem 3.2 *If G is a 4-colorable graph, then $g(G) \leq g^+(G)$.*

Proof Let V_i be the set of vertices colored by i ($i = 1, 2, \cdots, k$ when $k \leq 4$). Every V_i is an independent set. Now we orient $E(G)$ as follows: For any edge $uv \in E(G)$, where $u \in V_i$ and $v \in V_j$, the direction of uv is from u to v if and only if $i < j$.

Observe that $\vec{G}$, oriented as above, has no directed cycles. If G is 3-colorable, then $d(\vec{G}) \leq 2$. By Lemma 3.1.1, we know $g(G) \leq g^+(G)$. Now we assume that $\chi(G) = 4$. We will show that $I_{\vec{G}}(u,v) \cup I_{\vec{G}}(v,u) \subseteq I_G(u,v)$ for any $u, v \in V(G)$. Hence, $g(G) \leq g^+(G)$ by Lemma 3.1.1.

For any $u, v \in V(G)$, if $d_{\vec{G}}(u,v) = d_{\vec{G}}(v,u) = \infty$ or $d_{\vec{G}}(u,v) \leq 2$, we know $I_{\vec{G}}(u,v) \cup I_{\vec{G}}(v,u) = I_{\vec{G}}(u,v) \subseteq I_G(u,v)$. For the case $d_{\vec{G}}(u,v) = 3$, we have $u \in V_1$ and $v \in V_4$. Obviously, $d_G(u,v) \leq 3$. Note that $d_G(u,v) = i$ implies that $d_{\vec{G}}(u,v) = i$ for $i = 1, 2$. Hence, the distance between u and v in G must be 3, i.e., $d_G(u,v) = 3$. So, a u-v geodesic of $\vec{G}$ is also a u-v geodesic of G. Hence, $I_{\vec{G}}(u,v) \cup I_{\vec{G}}(v,u) = I_{\vec{G}}(u,v) \subseteq I_G(u,v)$. □

A graph G is *chordal* if it is simple and no chordless cycle. A *simplicial elimination ordering* is an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ for deletion of vertices so that each vertex v_i is a simplicial vertex of the subgraph induced by $\{v_i, \ldots, v_n\}$. A graph has a simplicial elimination ordering if and only if it is a chordal graph(see [15]).

Theorem 3.3 *For every chordal graph G, $g(G) \leq g^+(G)$.*

Proof Suppose G is a chordal graph and $v_1, \ldots, v_n$ is a simplicial elimination ordering of $V(G)$. Let $S = \{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\}$ be a maximum independent set of G with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $i_1 + i_2 + \cdots + i_k$ maximum. For $v \in V(G) - S$, $N(v) \cap S \neq \emptyset$ since S is a maximum independent set. Let

$$U = \{v \in V(G) - S : |N(v) \cap S| \geq 2\}$$

and

$$Q(v_{i_j}) = \{v \in V(G) - S : N(v) \cap S = \{v_{i_j}\}\} \cup \{v_{i_j}\}$$

for $v_{i_j} \in S$. Note that the subgraph induced by $Q(v_{i_j})$ is clique for $1 \leq j \leq k$. If not, there exists some j such that $x \in Q(v_{i_j})$ and $y \in Q(v_{i_j})$ with $xy \notin E(G)$. Obviously, $(S - \{v_{i_j}\}) \cup \{x, y\}$ is also an independent set of G. This is a contradiction to the maximum of S. For any $v_t \in Q(v_{i_j})$, we have $i_j > t$. If not, let $S' = (S - \{v_{i_j}\}) \cup \{v_t\}$. The set S' is also a maximum independent set of G since $N(v_t) \cap S = \{v_{i_j}\}$. It is a contradiction that $i_1 + i_2 + \cdots + i_k$ is maximum. Now we claim that $uv \notin E(G)$ for $u \in Q(v_{i_j})$ and $v \in Q(v_{i_l})$ if $j \neq l$. Without loss of generality, we assume $v = v_t$

and $u = v_m$ with $t > m$. If $uv \in E(G)$, then $vv_{i_j} \in E(G)$ since $i_j > m$ and $t > m$. Then, $|N(v) \cap S| \geq 2$ and hence $v \in U$, a contradiction to $v \in Q(v_{i_l})$.

Let $W = V(G) - (S \cup U)$. We direct the edges of $E(G)$ as follows: Every edge between S and $V(G) - S$ is directed from S to $V(G) - S$, every edge between U and W is directed from U to W, and every edge $v_i v_j$ in $G[U]$ or $G[W]$, is directed from v_i to v_j if and only if $i < j$.

Let $M = S \cup U^* \cup W^*$ be a minimum geodetic set of the orientation $\overrightarrow{G}$, where $U^* \subseteq U$ and $W^* \subseteq W$. In what follows, we will show that M is also a geodetic set of the underlying graph G, and hence $g^+(G) \geq g(\overrightarrow{G}) \geq g(G)$.

For every vertex $v \in U - U^*$, since $|N(v) \cap S| \geq 2$, we know $v \in I_G(S)$. If $W - W^* = \emptyset$, then the proof is completed. So we assume $W - W^* \neq \emptyset$. For a vertex $v \in W - W^*$, since the subgraph induced by $Q(v_{i_j})$ is a clique for $v_{i_j} \in S$ and there does not exist any edge between $Q(v_{i_j})$ and $Q(v_{i_l})$ for $j \neq l$, v does not lie on any x-y geodesic of $\overrightarrow{G}$ with $x \in W^*$ and $y \in W^*$. Note that the direction of any edge between $S \cup U$ and W is from $S \cup U$ to W. Hence, v must lie on some x-y geodesic of $\overrightarrow{G}$ with $x \in S \cup U^*$ and $y \in W^*$. Assume that $v \in Q(v_{i_t})$ for some $1 \leq t \leq k$. Note that there does not exist any edge between $Q(v_{i_j})$ and $Q(v_{i_l})$ for $j \neq l$. We know that y also belongs to $Q(v_{i_t})$. Hence, if v lies on some x-y geodesic of $\overrightarrow{G}$, then vy is the last arc on x-y geodesic. Let $P = x \rightarrow \cdots \rightarrow u \rightarrow \cdots \rightarrow v \rightarrow y$ be an x-y geodesic of $\overrightarrow{G}$, where $x \in S \cup U^*$, $\{v, y\} \subset Q(v_{i_t})$ and u is the last vertex on $V(P) \cap U$. $u \in U$ implies that $|N(u) \cap S| \geq 2$. Let $x' \neq v_{i_t}$ be a vertex on $N(u) \cap S$. We know that $P' = x' \rightarrow u \rightarrow \cdots \rightarrow v \rightarrow y$ is also a geodesic of $\overrightarrow{G}$ passing through v. Note that all vertices on P' after u must be in $Q(v_{i_t})$ and the direction $v_i v_j \in G[W]$ is from v_i to v_j when $i < j$. Hence, we know the length of P' is 3, and hence $P' = x' \rightarrow u \rightarrow v \rightarrow y$, where $x' \in S$, $u \in U$, $v \in (W - W^*) \cap Q(v_{i_t})$ and $y \in W^* \cap Q(v_{i_t})$. Now we show that $x' \rightarrow u \rightarrow v \rightarrow y$ is also an x'-y geodesic in the underlying graph G. Note that x' is not adjacent to y in G. If the distance between x' and y in G is 2, let $x' - w - y$ be an x'-y geodesic in G, we know that $w \notin W$ since $x' \notin Q(v_{i_t})$ and $y \in Q(v_{i_t})$. So, $w \in U$. This implies that $x' \rightarrow w \rightarrow y$ is also an x'-y geodesic in $\overrightarrow{G}$, a contradiction to $x' \rightarrow u \rightarrow v \rightarrow y$ is an x'-y geodesic of $\overrightarrow{G}$. Hence, the distance between x' and y in underlying graph G is at least 3. This implies that the x'-y geodesic $x' \rightarrow u \rightarrow v \rightarrow y$ in $\overrightarrow{G}$ is also an x'-y geodesic of underlying graph G.

¿From the above discussion, if M is a minimum geodetic set of $\overrightarrow{G}$, we know $v \in I_G(M)$ for any $v \in V(G) - M$. Hence, M is also a geodetic set of the underlying graph G. This completes the proof. □

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References

[1] T. Bonnesens and W. Fenchel, *Theorie der Konvexen Körper*, Springer, Berlin, 1934 (Translated by L.Boron, C.Christenson, B. Smith, Theory of convex Bodies, BCS Associates, Moscow, ID, 1987).

[2] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, 1990.

[3] G. J. Chang, L. D. Tong and H. T. Wang, Geodetic spectra of graphs, European J. Combin.25(2004), 383-391.

[4] G. Chartrand, F.Harary and P. Zhang, Extremal problems in geodetic graph theory, Congr. Number.130(1998), 157-168.

[5] G. Chartrand, F.Harary and P. Zhang, On hull number of a graph, Ars Combin. 20(2000), 181-189.

[6] G. Chartrand, F.Harary and P. Zhang, The geodetic number of a graph, Networks 39(2002), 1-6.

[7] G. Chartrand and P. Zhang, The forcing geodetic number of a graph, Discuss. Math. Graph Theory 19(1999), 45-48.

[8] G. Chartrand and P. Zhang, Realizable ratios in graph theory: geodesic parameters, Bull. Inst. Combin. Appl.27(1999), 69-80.

[9] G. Chartrand and P. Zhang, The geodetic number of an oriented graph, European J. Combin. 21 (2000), 181-189.

[10] G. Chartrand and P. Zhang, The forcing hull number of a graph, J. Combin. Math. Combin.Comput. (in press).

[11] M. G. Everett and S. B. Seidman, The hull number of a graph, DiscreteMath. 57(1985), 185-190.

[12] F. Harary and J. Nieminen, Convexity in graph, J. Differential Geom. 16 (1981), 185-190.

[13] C. Lu, The geodetic numbers of graphs and digraphs, Preprint of Department of Mathematics, ECNU (2004).

[14] H. M. Mulder, The expansion procedure for graphs, in R. Bodendiek(Ed.), *Contemporary Methods in Graph Theory*, Wissenschaftsverlag, Mannheim, 1990, 459-477.

[15] D. B. West, Introduction to Graph Theory. 2nd ed., Prentice Hall, Inc., NJ, 2001.