Efficient domination in directed tori and the Vizing's conjecture for directed graphs

Ľudovít Niepel, Anton Černý, Bader AlBdaiwi
Department of Mathematics and Computer Science
Kuwait University
P.O. Box 5969, Safat , 13060, Kuwait
email: niepel@mcs.sci.kuniv.edu.kw

Abstract

Though the well-known Vizing's conjecture is not true for directed graphs in general, we show that it is true when the digraph and its reversal contain an efficient dominating set. In this paper we investigate the existence of such sets in directed tori and infinite grids. We give a complete characterization of efficient dominating sets in 3-dimensional case and show the nonexistence of efficient d-dominating sets in directed tori for any $d > 1$ and any dimension $n > 1$.

Keywords: product of digraphs, Vizing's conjecture, efficient domination, directed torus, infinite grid.

2000 Mathematics Subject Classification: 05C38

1 Introduction.

All graphs in this paper are directed without multiple arcs. Let $G = (V, E)$ be a directed graph. A set D of vertices is d-dominating, $d \geq 1$, if each vertex in $V - D$ can be reached by a directed path of length at most d from some vertex in D. A set D of vertices is d-independent if no vertex in D is d-dominated by some other vertex in D. A d-dominating set is perfect if each vertex in $V - D$ is d-dominated by exactly one vertex from D. A d-dominating set is efficient if it is both perfect and d-independent. We omit the prefix "d-" if d=1. The definition of efficient dominating sets was introduced in [5] and it was proved there, that the problem of the existence of an efficient dominating set in a general digraph is NP-complete. In [4] it was shown that every undirected graph has an orientation with efficient dominating set. Efficient dominating sets were used in [11] for finding bounds of bondage numbers of vertex-transitive graphs and digraphs.

There are many results for efficient dominance in undirected graphs. Efficient d-dominating sets are often referred as perfect d-correcting codes ([6], [13]) or perfect distance-d placements [1]. Graphs with regular structure such as Cayley graphs or Cartesian products of graphs, allow constructions of symmetric dominating sets that are efficient ([14], [15], [7]).

Several results have been reached concerning the existence of efficient dominating sets in undirected tori ([1], [3], [15]). Relaxing the condition of the unique domination, one can define quasi-perfect distance-d placements. Existence and properties of these placements are discussed in [2].

The conjecture of Vizing ([16]) is one of basic problems on the domination of Cartesian products of undirected graphs. The conjecture is still open (see [10], [9] for references). In the present paper, we first show in Section 3, that, though the Vizing's conjecture is generally not true for directed graphs, the subclass of directed graphs where the conjecture is true includes directed Cayley graphs possessing an efficient dominating set. This gives us motivation to investigate the existence of efficient dominating sets in such Cayley graphs as directed tori and infinite directed grids.

In Section 4 we provide a complete characterization of efficient dominating sets for 3-dimensional tori and 3-dimensional grid. In Section 5 we prove the non-existence of efficient d-dominating sets for $d > 1$ and any dimension $n \geq 2$ thus showing, that the conjecture of Golomb and Welch ([8]) for undirected graphs (stated originally in terms of d-correcting codes), is true in the directed case.

2 Preliminaries

We will consider only directed graphs without multiple arcs. For the terminology and notation not given here, the reader is referred to [10]. Let G be a directed graph. We will denote the set of vertices and the set of edges of G as $V(G)$ and $E(G)$, respectively. A vertex y is a *successor* of a vertex x and x is a *predecessor* of y if $\overrightarrow{xy} \in E$. The vertices x and y are *adjacent* if $\overrightarrow{xy} \in E$ or $\overrightarrow{yx} \in E$. For a vertex $u \in V$ let $N_G^+(u) = \{v \in V | \overrightarrow{uv} \in E\}$, $N_G^-(u) = \{v \in V | \overrightarrow{vu} \in E\}$ and $N_G^+[u] = N_G^+(u) \cup \{u\}$, $N_G^-[u] = N_G^-(u) \cup \{u\}$. The reverse digraph of G is defined as a digraph G^{-1} with the vertex set $V(G^{-1}) = V(G)$ and the arc set $E(G^{-1}) = \{\overrightarrow{uv} | \overrightarrow{vu} \in E(G)\}$. The *distance* of a vertex y from a vertex x, denoted as $\delta(x, y)$, is the length of the shortest directed path from x to y, if such path exists, and it is ∞ otherwise. Let $d \geq 1$. A vertex y is *d-dominated* by x if $\delta(x, y) \leq d$. Thus y is 1-dominated by x iff $y \in N_G^+[x]$. A set D of vertices is *d-dominating* if each vertex in $V - D$ is d-dominated by a vertex from D. The minimum cardinality over all 1-dominating sets in G is called domination number, and denoted by $\gamma(G)$. A set D of vertices is *d-independent* if there do not exist vertices $x \neq y$ in D such that x is d-dominated by y. A d-dominating set D is *perfect* if each vertex in $V - D$ is d-dominated by exactly one vertex from D. A d-domination set is *efficient* if it is d-independent and perfect. We will say that two efficient d-dominating sets D, D' in G are equivalent, if there is an automorphism of G mapping D on D'.

The Cartesian product of two directed graphs G_1, G_2 is the directed graph $G_1 \Box G_2$ with the vertex set $V(G_1) \times V(G_2)$ and a vertex (y_1, y_2) is a successor of a vertex (x_1, x_2) if $x_1 = y_1$ and $\overrightarrow{x_2y_2} \in E(G_2)$ or $x_2 = y_2$ and $\overrightarrow{x_1y_1} \in E(G_1)$. The Cartesian product of $n \geq 3$ directed graphs is defined in a similar way, hence a vertex differs from its successor in exactly one coordinate.

Let $\mathbb{Z}$ denote the set of all integers. A *directed cycle* C_k, $k \geq 2$, is a graph with vertices that are elements of the cyclic group $(\mathbb{Z}_k, +)$, and, for all $x \in \mathbb{Z}_k$, the only successor of x is $x + 1$. The *n-dimensional directed torus* $T(k_1, k_2, ..., k_n)$, $k_i \geq 2$, for $1 \leq i \leq n$, is the Cartesian product of n directed cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_n}$. An *infinite directed path* P is a digraph with the vertex set being the set of integers $\mathbb{Z}$ and, for all $x \in \mathbb{Z}$, the only successor of x is $x + 1$. The Cartesian product of $n \geq 2$ directed paths is the *infinite n-dimensional grid* G_n.

When investigating the existence of efficient d-dominating sets, we will assume that no vertex dominates itself by a positive distance, i.e., each directed cycle under consideration is of length greater than d. Throughout the paper, except in Section 5, "domination" means "1-domination".

Let us start, however, with an observation being valid for any $d \geq 1$. A d-domination set in a n-dimensional torus can be easily extended to some larger torus or to an n-dimensional grid.

Proposition 1 *Let D be a d-dominating set in a torus $T(k_1, k_2, ..., k_n)$, $d \geq 1$. Let $p_1 \geq 1$, ..., $p_n \geq 1$. If $H = T(p_1k_1, p_2k_2, ..., p_nk_n)$ or $H = G_n$ then $Ext_H(D) = \{(u_1, u_2, ..., u_n) \in H | (u_1 \bmod k_1, u_2 \bmod k_2, ..., u_n \bmod k_n) \in D\}$ is a d-dominating set in H. Moreover, $Ext_H(D)$ is perfect (d-independent) if and only if D is perfect (d-independent).*

It is not difficult to see that the number $\sigma(n, d)$ of vertices d-dominated by one vertex in a n-dimensional torus or grid satisfies the recurrence relation $\sigma(n, d) = \sigma(n, d-1) + \sigma(n-1, d)$, for $n \geq 2$, $d \geq 2$ and $\sigma(n, 1) = n+1$, $\sigma(1, d) = d + 1$, for $n \geq 1, d \geq 1$. Therefore $\sigma(n, d) = \binom{n+d}{d}$. If an efficient d-dominating set in a n-dimensional torus exists, then the vertex set of the torus can be partitioned into sets of the equal size $\binom{n+d}{d}$ and, consequently, the size of the vertex set must be divisible by $\binom{n+d}{d}$. It is known, that an efficient 1-dominating set exists in the n-dimensional torus T_n being the product of n copies of the cycle C_{n+1}, $n \geq 2$. The construction in the following Proposition 2 is given in [11]. Proposition 1 then provides the way to extend this result to the grid G_n.

Proposition 2 *Let $0 \leq k \leq n$. Then the set*

$$D_k^n = \{(a_1, a_2, \ldots, a_n) \in T_n | \sum_{i=1}^{n} ia_i = k \bmod (n+1)\} \quad (1)$$

is an efficient 1-dominating set in the torus T_n.

Corollary 3 *Let $n \geq 2$. If $k_i \geq 2$, for $1 \leq i \leq n$, and* $\gcd(k_1, k_2, \ldots, k_n)$ *is a multiple of $n+1$ then the torus $T(k_1, k_2, \ldots, k_n)$ contains an efficient 1-dominating set and $\gamma(T(k_1, k_2, \ldots, k_n)) = k_1 k_2 \cdots k_n/(n+1)$. The grid G_n contains an efficient dominating set.*

Remark 4 *The dominating sets D_k^n from Proposition 2 are mutually equivalent, since $D_k^n = \{(a_1 + k, a_2, \ldots, a_n) | (a_1, a_2, \ldots, a_n) \in D_0^n\}$.*

The characterization of efficient dominating sets in 2-dimensional case was given in [11], as well, and can be summarized as follows.

Theorem 5 *The torus $T(p, q)$, $p, q \geq 2$, contains an efficient dominating set if and only if both numbers p, q are multiples of 3. The only three efficient dominating sets in G_2 are the sets $Ext_{G_2}(D_i^2)$, $0 \leq i \leq 2$ from Proposition 1, where D_i^2 is as in Proposition 2.*

In Section 4 we give a complete characterization of efficient dominating sets in 3-dimensional tori and in the 3-dimensional infinite grid.

3 On the directed case of the Vizing's conjecture

One of the most studied problem in domination is the Vizing's conjecture. The conjecture can be formulated as follows. Let G and H be finite graphs. Then $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$. When we replace "graph" by "digraph" the above inequality is generally not valid any more. It is enough to consider the Cartesian product $C_3 \Box C_3$: $\gamma(C_3) = 2$ and $\gamma(C_3 \Box C_3) = 3$. The following relationship between dominating sets in directed graphs G and G^{-1} will help us to find a subclass of directed graphs where the conjecture is true.

Lemma 6 *Let D be an efficient dominating set in digraph G then $\gamma(G^{-1}) \geq |D|$.*

Proof. Let u be a vertex in D and D' any dominating set in G^{-1}. As $N_G^+[u] = N_{G^{-1}}^-[u]$ and the vertex u is dominated by at least one vertex in D', hence $|N_G^+[u] \cap D'| \geq 1$. Because $N_G^+[u] \cap N_G^+[v] = \emptyset$ for any $u, v \in D, u \neq v$ the statement follows. ■

A digraph G can have efficient dominating sets of different cardinalities, but when for both G and G^{-1} there exists an efficient dominating set, then from Lemma 6 it follows that $\gamma(G) = \gamma(G^{-1})$, and all efficient dominating sets in G and G^{-1} have the same cardinalities.

We will prove a weaker form of Vizing's conjecture using a similar procedure as in [12].

Lemma 7 *Let G and H be digraphs and let H^{-1} has an efficient dominating set F. Then $\gamma(G \Box H) \geq \gamma(G).|F|$.*

Proof. Let h be a vertex of digraph H and D a dominating set in $G \Box H$. Each vertex in $V(G) \times \{h\}$ can be dominated only by a vertex in D from the set $(V(G) \times N_H^-[h])$. It is therefore clear that $|(V(G) \times N_H^-[h]) \cap D)| \geq \gamma(G)$. As F is an efficient dominating set in H^{-1}, $\{V(G) \times N_{H^{-1}}^+[h]\}_{h \in F}$ is a partition of $V(G) \times V(H)$. So $|D| = |\bigcup_{h \in F}((V(G) \times N_{H^{-1}}^+[h]) \cap D)| = \sum_{h \in F} |(V(G) \times N_{H^{-1}}^+[h]) \cap D| = \sum_{h \in F} |(V(G)) \times N_H^-[h]) \cap D| \geq \gamma(G).|F|$.
■

Lemma 7 directly implies

Theorem 8 *Let G and H be finite digraphs such that both H and H^{-1} have efficient dominating sets then $\gamma(G \Box H) \geq \gamma(G).\gamma(H)$.*

If a digraph H is a Cayley digraph then it is isomorphic to H^{-1} and H has an efficient dominating set iff H^{-1} has. In the rest of the paper we shall investigate efficient dominating sets in directed tori and infinite grids, being special classes of directed Cayley graphs.

4 The case $d = 1, n = 3$

In this section we investigate the existence of efficient dominating sets in tori and infinite grids for $n = 3$. First we find a characterization of all efficient dominating sets in G_3.

The vertices of G_3 are ordered triples of integers (x, y, z). We will denote as the k-th *layer* of G_3 the subgraph induced by the set of all vertices with $z = k$, $k \in \mathbb{Z}$. Within one layer z, the i-th *row*, i-th *column*, i-th *diagonal*, for $i \in \mathbb{Z}$, consists of all vertices (x, y, z) with $x = i$, $y = i$, $x + y = i$, respectively. We will use the following coloring of vertices of a subgraph H of G_3 induced by an efficient dominating set D in H.

1. A vertex from D has red color.

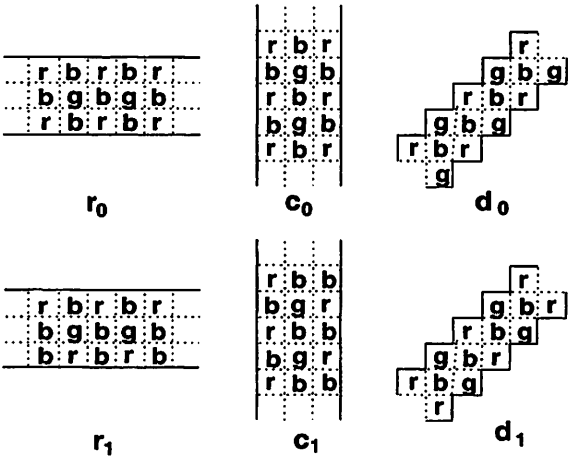

Figure 1: The basic patterns

2. A vertex dominated by a vertex from the same layer has blue color.
3. A vertex dominated by a vertex from a different layer has green color.

We will consider patterns in G_3. A *pattern* in the grid G_3 is an induced subgraph H of G_3 together with a coloring of its vertices by the red (r), blue (b) and green (g) color. We distinguish six *basic patterns*, each being a part of a single layer in G_3: the patterns $\mathbf{r}_0$, $\mathbf{r}_1$ consist of three neighbor rows, the patterns $\mathbf{c}_0$, $\mathbf{c}_1$consist of three neighbor columns and the patterns $\mathbf{d}_0$, $\mathbf{d}_1$ consist of three neighbor diagonals. The vertices in these basic patterns are colored in a regular way, as depicted in Figure 1.

An entire layer may be colored in the way, where, for each even k, the rows k, $k+1$, $k+2$ form the pattern $\mathbf{r}_0$ or $\mathbf{r}_1$ and the (bi-infinite) sequence, in which the patterns $\mathbf{r}_0, \mathbf{r}_1$ appear, can be chosen arbitrarily. We will call this colorings of a layer **r**-*colorings.* We define **c**-*colorings* and **d**-*colorings* as those containing $\mathbf{c}_0$ and $\mathbf{c}_1$ in some sequence of columns, and $\mathbf{d}_0$ and $\mathbf{d}_1$ in some sequence of diagonals, respectively. It is not difficult to see, that each such coloring of one layer may be uniquely extended to the whole grid G_3 to become a coloring induced by some efficient dominating set D. The colorings of the remaining layers are of the same type (**r**, **c**, **d**) and they are described by the same bi-infinite sequence of patterns as the one in the starting layer, they are just shifted by one row, column or diagonal. We will prove now, that these are the only possible colorings induced by efficient dominating sets in G_3.

In the following, we assume some fixed efficient dominating set D in G_3 and the coloring C induced by D. Our aim is to prove that every vertex in

G_3 is a part of some basic pattern.

Let us mention some elementary properties of the coloring C. If a vertex $v = (x, y, z)$ has red color, then there are vertices in the layers $z - 1$, z and $z + 1$ that cannot have red color. We say that the vertex v *blocks* these vertices. Thus a vertex is blocked iff it is either blue or green. In the layer z, there are 6 vertices blocked by the vertex v. Two of them are the predecessors of v, two are the blue vertices dominated by v and the last two are their predecessors in layer z, distinct from v. In the layer $z + 1$, there are three vertices blocked by v, one is the green vertex dominated by v and the next two are the predecessors of the green vertex in layer $z + 1$. In the layer $z - 1$, there are three blocked vertices that are the predecessors of the vertex v and of the two vertices in layer z dominated by v. More properties of C may be easily observed.

Proposition 9
P1. Each vertex is assigned exactly one color.
P2. No two red vertices are adjacent. No two green vertices are adjacent.
P3. The vertices (x, y, z) and $(x + 1, y - 1, z)$ cannot be both green or both red.
P4. If the vertices (x, y, z), $(x-1, y, z)$, $(x, y-1, z)$ are not red then (x, y, z) is green and $(x, y, z - 1)$ is red.

We will refer to properties *P1-P4* without explicitly mentioning Proposition 9.

Let us now consider a vertex $v = (x, y, z)$ from D (i.e., it is colored in red).

Lemma 10 *At least one of the vertices $(x + 1, y + 1, z)$, $(x, y + 2, z)$, $(x + 1, y + 2, z)$ is red.*

Proof. Assume in contrary, that none of the three vertices is red. The vertices $(x, y + 1, z)$, $(x + 1, y, z)$ are blocked by v. *P4* implies that both $(x + 1, y + 1, z)$ and $(x + 1, y + 2, z)$ are green, in contradiction to *P2*. ∎

We will distinguish three cases. The proving technique in each of the cases is similar. We therefore provide the complete proof of the assertions just in Case 1.

Case 1. $(x + 1, y + 1, z)$ is red.

We will first investigate the situation in the layer z.

Lemma 11 *The vertices $(x - 1, y + 2, z), (x + 2, y - 1, z)$ are blocked.*

Proof. Suppose that the vertex $(x+2, y-1, z)$ is colored by red color. Then the vertex $(x+2, y, z)$ is blue and the vertices $(x+i, y, z-1)$, $(x+i, y+1, z-1)$, $i = 0, 1, 2$, are blocked, because their descendants in layer z are red or blue. Neither of the adjacent vertices $(x+1, y+1, z-1)$, $(x+2, y+1, z-1)$ can be dominated by vertices from layer $z-1$, so they should be green, what is a contradiction to *P3*. The case of the vertex $(x-1, y+2, z)$ can be proved in a similar way. ∎

Now we can observe how the coloring of the neighbor layers can be influenced.

Lemma 12 *D contains the vertices* $(x+1, y-1, z+1)$, $(x+2, y, z+1)$, $(x-1, y+1, z+1)$, $(x, y+2, z+1)$.

Proof. The vertex $(x+1, y, z+1)$ is blocked, as it is the predecessor of a green vertex in the same layer. Since $(x+1, y-1, z+1)$ is the only non-blocked predecessor of $(x+1, y, z+1)$, it must be red. Now all predecessors of $(x+2, y, z+1)$ are blocked, so this vertex must be red, as well. A similar argument can be used to prove that the vertices $(x-1, y+1, z+1)$, $(x, y+2, z+1)$ are red . ∎

Lemma 13 *D contains the vertices* $(x+1, y-1, z-1)$, $(x+2, y, z-1)$, $(x-1, y+1, z-1)$, $(x, y+2, z-1)$.

Proof. From Lemma 11 it follows that both predecessors of the blocked vertex $(x, y+2, z)$ in layer z are blocked, so *P3* implies that it is green and $(x, y+2, z-1)$ is red. The vertex $(x+1, y+1, z-1)$ is blocked and so are both its predecessors in layer $z-1$, so this vertex must be green. By *P2*, $(x+1, y, z-1)$ cannot be green. The only predecessor of $(x+1, y, z-1)$, that is not blocked, is $(x+1, y-1, z-1)$. So the latter vertex must be red. By symmetry, $(x-1, y+1, z-1)$ and $(x, y+2, z-1)$ are red, as well. ∎

Repeated application of Lemma 12 and 13 yields that red vertices in the diagonals $x+y$ and $x+y+2$ are $(x+2i, y-2i, z)$, $(x+2i+1, y-2i+1, z)$ for all $i \in \mathbb{Z}$. This uniquely determines the coloring of the remaining vertices in the diagonals $x+y$, $x+y+1$ and $x+y+2$.

Corollary 14 *If in the coloring C the vertices* (x, y, z) *and* $(x+1, y+1, z)$ *are red then they are contained in a pattern* $\mathbf{d}_1$.

Case 2. $(x+1, y+2, z)$ is red and v is not contained in a pattern $\mathbf{d}_1$.

In this case v blocks two additional vertices: $(x+1, y+1, z)$ and $(x-1, y-1, z)$.

Lemma 15 *D contains the vertices* $(x+2, y, z)$, $(x-1, y+2, z)$,$(x-2, y, z)$ *and* $(x+3, y+2, z)$.

Corollary 16 *If in the coloring C the vertices (x, y, z) and $(x+1, y+2, z)$ are red and (x, y, z) is not contained in a pattern $\mathbf{d}_1$, then they are contained in a pattern $\mathbf{c}_1$.*

By symmetry we obtain:

Corollary 17 *If in the coloring C the vertices (x, y, z) and $(x+2, y+1, z)$ are red and (x, y, z) is not contained in a pattern $\mathbf{d}_1$, then they are contained in a pattern $\mathbf{r}_1$.*

Case 3. $(x, y+2, z)$ is red and v is not contained in any of the patterns $\mathbf{d}_1$, $\mathbf{c}_1$ or $\mathbf{r}_1$.

In this case v blocks additional vertices, in particular $(x-1, y-1, z)$ and $(x+2, y+1, z)$.

Lemma 18 *D contains the vertices $(x+2, y, z)$, $(x+2, y-2, z)$, $(x, y-2, z)$, $(x-2, y+2, z)$, $(x-2, y, z)$ and one of the vertices $(x-2, y+4, z)$, $(x+2, y+2, z)$.*

Corollary 19 *If in the coloring C the vertices (x, y, z) and $(x, y+2, z)$ are red and (x, y, z) is not contained in any of the patterns $\mathbf{d}_1$, $\mathbf{c}_1$ or $\mathbf{r}_1$ then (x, y, z) is contained in a pattern $\mathbf{d}_0$, $\mathbf{r}_0$ or $\mathbf{c}_0$.*

The results obtained in the three cases are summarized in the following corollary.

Corollary 20 *In a coloring induced by an efficient dominating set in G_3, every red vertex belongs to some basic pattern.*

Since $\mathbf{r}_1$ contains a red vertex in each column, $\mathbf{c}_1$ in each row and $\mathbf{d}_1$ in each row and each column, the only different pattern, which can occur in the same layer with $\mathbf{r}_1$, is $\mathbf{r}_0$. In a similar way, $\mathbf{c}_1$ may occur just with $\mathbf{c}_0$ and $\mathbf{d}_1$ just with $\mathbf{d}_0$. Moreover, any two of the patterns $\mathbf{r}_0$, $\mathbf{c}_0$, $\mathbf{d}_0$ may occur together; in this case all the three patterns occur. We will call this coloring, which then occurs in all layers, *regular coloring.*

Theorem 21 *In a coloring induced by an efficient dominating set in G_3, all layers are $\mathbf{r}$-colored or all are $\mathbf{c}$-colored or all are $\mathbf{d}$-colored.*

Let us now consider a 3-dimensional torus being a product involving some cycle of an odd length. Any efficient dominating set in this torus, when expanded to the grid G_3 by Proposition 1 induces a coloring, where, there is a pair of vertices, differing by the length of the odd cycle in the corresponding coordinate, while having the remaining two coordinates identical. This contradicts to the following easy observation:

Observation 22 *In a* r*-coloring,* c*-coloring or* d*-coloring implied by an efficient dominating set of* G_3*, the distance of any two red vertices, differing in one coordinate only, is even.*

On the other hand, an efficient dominating set in G_3 inducing the regular coloring can be obtained, using Proposition 1 from an efficient dominating set in the torus $T(2,2,2)$. We arrived to the following characterization of the 3-dimensional tori containing an efficient domination set.

Theorem 23 *A torus* $T(p,q,r)$ *contains an efficient dominating set if and only if all the numbers* p,q,r *are even; in this case* $\gamma(T(p,q,r)) = pqr/4$.

The efficient dominating set inducing the regular coloring of $T(2,2,2)$, when extended to $T(4,4,4)$, is not equivalent to the ones provided by Lemma 2. In the regular coloring of $T(4,4,4)$ there are exactly three red vertices in distance 2 from any red vertex. On the other hand in the coloring described in Lemma 2 there are only two red vertices within distance 2 from any red vertex. So these efficient sets are not equivalent.

5 The case $d > 1$

Let $n \geq 2$ and $d \geq 2$. Our aim is to prove that neither a n-dimensional torus nor the n-dimensional infinite grid can contain an efficient d-dominating set. For this section only, all terms referring to "domination" will mean "d-domination". Let us look at the infinite grid first. In the following three lemmas we assume (in contrary), that G_n contains an efficient dominating set D containing the vertex $v = (0,0,\dots,0)$.

Vertices $x = (x_1,\dots,x_n)$ and $x' = (x'_1,\dots,x'_n)$ will be called (r,s)-*brothers* (or, simply, *brothers*) if $x'_i = x_i$ for $1 \leq i \leq n, i \notin \{r,s\}$ and $x'_r = x_r - 1$, $x'_s = x_s + 1$. In the following we consider (r,s)-brothers x and x' dominated by vertices w, w' from D, respectively and satisfying the condition $\delta(v,x) = \delta(v,x') = d+1$ (which implies $x_i \geq 0$ for $1 \leq i \leq n$ and $x_r > 0$).

Lemma 24 *If, for some* $1 \leq t \leq n$, $x_t \geq 1$, *then* $w_t = x_t$.

Proof. Clearly, $w_i \leq x_i$ for $1 \leq i \leq n$. If $w_t < x_t$ than the vertex $(x_1,\dots,x_{t-1},x_t - 1,x_{t+1},\dots,x_n)$ is dominated by both v and w, in contradiction to the efficiency of D. ■

Lemma 25 *The vertices* w *and* w' *are distinct.*

Proof. Since w' dominates x', $w'_r \leq x'_r = x_r - 1$, while Lemma 24 implies that $w_r = x_r$. ■

Lemma 26 *If, for some $t \neq r, s$, $x_t = 0$ then $w_t = 0$ or $w'_t = 0$.*

Proof. Clearly, $w_t \leq 0$ and $w'_t \leq 0$. If both $w_t < 0$ and $w'_t < 0$ then the vertex $(y_1, \ldots, y_n)$, with $y_i = x_i$ for $i \neq s, t$, and $y_s = x_s + 1$, $y_t = -1$, is dominated by both w and w'. ∎

We are now ready to prove our main result of this section.

Theorem 27 *For $n \geq 2$, neither a n-dimensional torus $T(k_1, k_2, \ldots, k_n)$, where $k_i > d$ for $1 \leq i \leq n$, nor the infinite gride G_n contains an efficient d-dominating set, $d \geq 2$.*

Proof. Proposition 2 implies, that it is enough to prove the non-existence of an efficient d-dominating set in the grid G_n. Assume, in contrary, the existence of an efficient d-dominating set D in G_n. WLOG we may assume that D contains the vertex $v = (0, 0, \ldots, 0)$. Let us first assume that $d \geq n \geq 2$. Consider the $(1,2)$-brothers $x = (d - n + 2, 1, 1, \ldots, 1)$ and $x' = (d-n+1, 2, 1, \ldots, 1)$ dominated by the vertices $w, w' \in D$, respectively. Lemma 24 implies $w = x$ and $w' = x'$ and the vertex $(d - n + 2, 2, 1, \ldots, 1)$ is dominated by both w and w', in contradiction to the perfectness of D. Let now $2 \leq d \leq n$. Consider the vertices $x^{(p)} = (x_1^{(p)}, \ldots, x_n^{(p)})$, $1 \leq p \leq n$, where, for $1 \leq i \leq n$, if $i \neq p$ then $x_i^{(p)} = 1$ if $1 \leq i \leq d$ and $x_i^{(p)} = 0$ if $d + 1 \leq i \leq n$; if $p \leq d$ then $x_p^{(p)} = 2$, otherwise $x_p^{(p)} = 1$. Let $w^{(p)} = (w_1^{(p)}, \ldots, w_n^{(p)}) \in D$ dominate $x^{(p)}$. Then, by Lemma 24, $w_i^{(p)} = x_i^{(p)}$, for $i \leq d$. The vertices $x^{(p)}$ are pairwise brothers and Lemma 25 implies that the vertices $w^{(p)}$ are pairwise distinct. Assume that , for some $p_1 \neq p_2$, $w^{(p_1)} = x^{(p_1)}$ and $w^{(p_2)} = x^{(p_2)}$. Then the vertex $(y_1 \ldots, y_n)$, where $y_i = x_i^{(p_1)}$ for $i \neq p_2$ and $y_{p_2} = x_{p_2}^{(p_2)}$, is dominated by both $w^{(p_1)}$ and $w^{(p_2)}$. Therefore at most one of the vertices $w^{(p)}$ may be identical with $x^{(p)}$. Each of the remaining $n-1$ vertices $w^{(p)}$ is distinct from the corresponding vertex $x^{(p)}$. It follows from Lemma 24, that the first d coordinates of $x^{(p)}$ and $w^{(p)}$ are always identical. Thus if $w^{(p)} \neq x^{(p)}$ then $w^{(p)}$ differs from $x^{(p)}$ in at least one of the coordinates, starting from $d + 1$. Lemma 24 implies, that such coordinate is not the coordinate p. Hence in $n - 1$ of the vertices $w^{(p)}$ have at least one of the last $n - d$ coordinates negative. By Lemma 26, no two of the vertices $w^{(p)}$ may have the same of these coordinates negative. As implied by the pigeonhole principle, this is not possible, since $n - 1 > n - d$. ∎

References

[1] B. AlBdaiwi, B. Bose, On resource placements in 3d tori, J. Parallel Distrib. Comput. 63 (9) (2003) 838–845.

[2] B. AlBdaiwi, B. Bose, Quasi-perfect resource placements for two-dimensional toroidal networks, J. Parallel Distrib. Comput. 65 (7) (2005) 815–831.

[3] B. AlBdaiwi, P. Horak, Perfect placements in three-dimensional tori, J. Combin. Math. Combin. Comput. 43 (2002) 159–174.

[4] D. W. Bange, A. E. Barkauskas, L. H. Host, L. H. Clark, Efficient domination of the orientations of a graph, Discrete Mathematics 178 (1-3) (1998) 1–14.

[5] A. Barkauskas, L. Host, Finding efficient dominating sets in oriented graphs, Congressus Numerantium 98 (1993) 27–32.

[6] N. Biggs, Perfect codes in graphs, Journal of Combinatorial Theory, Series B, 15 (3,) (1973) 289–296.

[7] I. J. Dejter, O. Serra, Efficient dominating sets in cayley graphs, Discrete Applied Mathematics 129 (2-3) (2003) 319–328.

[8] S. Golomb, L. Welch, "Algebric Coding and the Lee Metric", in: H. Mann (ed.), Error Correcting Codes: Proceedings of a Symposium Conducted by Mathematics Research Center, John Wiley & Sons, Inc., 1968.

[9] T. Haynes, S. T. Hedetniemi, P. J. Slater, Domination in graphs: Advanced topics., New York: Marcel Dekker, 1998.

[10] T. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of domination in graphs., New York: Marcel Dekker, 1998.

[11] J. Huang, J.-M. Xu, The bondage numbers and efficient dominations of vertex-transitive graphs, Discrete Mathematics 308 (2008) 571–582.

[12] M. Jacobson, L.F.Kinch, On domination of products of graphs ii: trees, J. Graph Theory 10 (1986) 97–106.

[13] J. Kratochvil, Perfect codes over graphs, Journal of Combinatorial Theory, Series B, 40 (1986) 224–228.

[14] M. Livingston, Q. F. Stout, Perfect dominating sets, Congressus Numerantium 79 (1990) 187–203.

[15] J. D. Masters, Q. F. Stout, D. M. V. Wieren, Unique domination in cross-product graphs, Congressus Numerantium 118 (1996) 49–71.

[16] V. G. Vizing, Some unsolved problems in graph theory. (in Russian), Uspechi Mat. Nauk 23 (6(144)) (1968) 117–134.