# A NOTE ON EDGE-COVER COLORING OF NEARLY BIPARTITE GRAPHS

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ABSTRACT. Let $G$ be a graph with vertex set $V(G)$. An edge coloring $C$ of $G$ is called an edge-cover coloring, if for each color, the edges assigned with it forms an edge cover of $G$. The maximum positive integer $k$ such that $G$ has a $k$-edge-cover coloring is called the edge cover chromatic index of $G$ and is denoted by $\chi'_c(G)$. It is well known that $\min\{d(v) - \mu(v) : v \in V\} \leq \chi'_c(G) \leq \delta(G)$, where $\mu(v)$ is the multiplicity of $v$ and $\delta(G)$ is the minimum degree of $G$. If $\chi'_c(G) = \delta(G)$, $G$ is called a graph of CI class, otherwise $G$ is called a graph of CII class. In this paper, we give a new sufficient condition for a nearly bipartite graph to be of CI class.

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## 1. INTRODUCTION

Throughout this paper, a graph $G(V, E)$ allows multiple edges but no loops and has a finite vertex set $V$ and a finite nonempty edge set $E$. $G$ is a simple graph if it has no multiple edges. Given two vertices $u, v \in V(G)$, the multiplicity $\mu(uv)$ is the number of edges joining $u$ and $v$ in $G$. The multiplicity of $v$ is $\mu(v) = \max\{\mu(uv) : u \in V\}$. Set $\mu = \max\{\mu(v) : v \in V\}$. When $G$ has no multiple edges (that is $\mu = 1$), $G$ is a simple graph. Let $\delta(G)$ denote the minimum degree of $G$. For the sake of simplicity, we write $\delta(G)$ by $\delta$. Let $N_G(v)$ denote the neighborhood of $v$.

A graph $G$ is called *nearly bipartite*, if there exists a vertex $u \in V(G)$ such that $G - u$ is a bipartite graph. If $G - u$ is with bipartition $(X, Y)$, then $G$ is denoted by $G(X, Y; u)$.

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An edge coloring of $G$ is an assignment of colors to the edges of $G$. Associate positive integers $1, 2, \ldots$ with colors, and we call $C$ a *k-edge coloring* of $G$ if $C$: $E \to \{1, 2, \ldots, k\}$. Let $i_C(v)$ denote the number of edges of $G$ incident with vertex $v$ receiving color $i$ in the coloring $C$. For simplification, we write $i(v) = i_C(v)$ if there is no obscurity. $C$ is called an *edge-cover coloring* of $G$, if for each vertex $v \in V$, $i_C(v) \geq 1$ for $i = 1, 2, \ldots, k$. That is, the edges assigned with the same color forms an edge cover of $G$. Let $\chi_c'(G)$ denote the maximum positive integer $k$ for which an edge-cover coloring with $k$ colors of $G$ exists, which is called the edge-cover chromatic index of $G$. The edge-cover coloring was studied in [1, 3, 5, 6, 7] and etc. In [1] and [5], the authors proved that

**Theorem 1.** ([1]) *For any graph $G$,* $\min\{d(v) - \mu(v) : v \in V\} \leq \chi_c'(G) \leq \delta$.

We say that a graph $G$ is of CI class if $\chi_c'(G) = \delta$, otherwise $G$ is of CII class.

Wang and Liu [6] gave some sufficient conditions for a nearly bipartite simple graph to be of CI class.

**Theorem 2.** ([6]) *Let $G(X, Y; u)$ be a nearly bipartite simple graph with minimum degree $\delta \geq 3$. If $d(u) \geq 2\delta - 1$, then $G(X, Y; u)$ is of CI class.*

**Theorem 3.** ([6]) *Let $G(X, Y; u)$ be a nearly bipartite simple graph with minimum degree $\delta \geq 3$. If $N_G(u)$ contains at most one minimum degree vertex, then $G(X, Y; u)$ is of CI class.*

**Theorem 4.** ([6]) *Let $G(X, Y; u)$ be a nearly bipartite simple graph with minimum degree $\delta \geq 3$ and $S = \{v \in N_G(u) | d(v) = \delta\}$. If $S \subseteq X$ (or $Y$) and $d(u) \geq \delta + |S| - 1$, then $G(X, Y; u)$ is of CI class.*

From the proof in [6], it is easy to see that Theorem 2, Theorem 3 and Theorem 4 are also true for graphs with mutiple edges.

In [6], the authors gave a graph $K_{n,n} \vee \{u\}$ to show that the condition $d(u) \geq 2\delta - 1$ in Theorem 2 can't be replaced by $d(u) \geq 2\delta - 2$. In this sense Theorem 2 is best possible. But when $d(u) \leq 2\delta - 2$, does there exist a sufficient condition for a nearly bipartite graph to be of CI class? In this note, we give such a result.

**Theorem 5.** *Let $G(X, Y; u)$ be a nearly bipartite graph. If there exits a vertex $y \in N_G(u)$, such that $d(u) + d(y) \geq 3\delta - 1$, then $G(X, Y; u)$ is of CI class.*

Clearly, Theorem 5 includes Theorem 2.

## 2. Our Main Result

Before proving our main result, we need some more terminologies and results.

Let $C$ be a $k$-edge coloring of $G$. If $i(v) \geq 1$, we also say that color $i$ appears at $v$. Let $\Phi(v)$ denote the set of colors appearing at $v$ in $C$ and $\sigma(v) = |\Phi(v)|$; $\sigma(C) = \Sigma_{v \in V}\sigma(v)$. If $\sigma(v) = k$ for each $v \in V(G)$, then $C$ is a $k$-edge-cover coloring of $G$. We call a $k$-edge coloring $C'$ an improvement on $C$ if $\sigma(C') > \sigma(C)$. For a $k$-edge coloring $C_0$ of $G$, if $\sigma(C_0) = \max\{\sigma(C) : C \text{ is a } k\text{-edge coloring of } G\}$, $C_0$ is called *an optimal $k$-edge coloring.*

Let $E(i)$ be the set of edges receiving color $i$ in an edge coloring $C$ of $G$. The edge induced subgraph $E(i) \cup E(j)$ is denoted by $G(i,j)$. For $v \in V$, let $G(v;i,j)$ be the component of $G(i,j)$ containing $v$. We call a subgraph $H$ of $G$ an *obstruction* (about $C$), if $H = G(x;i,j)$ is an odd cycle and $i_C(x) = 2$, $j_C(x) = 0$, $i_C(v) = j_C(v) = 1$ for each $v \in V(H) \setminus \{x\}$.

Let $G$ be edge colored and let $\alpha$ and $\beta$ be two of the used colors. An *$(\alpha,\beta)$-exchange chain* $K$ of $G$ is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{r-1}, e_r, v_r)$ of vertices and edges of $G$ in which

(i) for $1 \leq i \leq r$, the vertices $v_{i-1}$ and $v_i$ are distinct and are both incident with the edge $e_i$,

(ii) the edges are all distinct and are colored alternately by $\alpha$ and $\beta$,

(iii) $e_1$ is colored by $\alpha$ and $\alpha_C(v_0) > \beta_C(v_0)$. Similarly, let $\gamma$ denote the color of $e_r$ and $\overline{\gamma}$ denote the other color of $\{\alpha,\beta\}$. When $v_0 \neq v_r$, $\gamma_C(v_r) > \overline{\gamma}_C(v_r)$ or $\gamma_C(v_r) > 1$; when $v_0 = v_r$, then $\gamma = \alpha$ and $\alpha_C(v_0) > \beta_C(v_0) + 1$.

An $(\alpha,\beta)$-exchange chain $K$ is called minimal if there exists no other $(\alpha,\beta)$-exchange chain $K'$ which is starting at the same vertex as $K$ and $K' \subset K$. If $\alpha_C(v_0) > \beta_C(v_0)$, the existence of an $(\alpha,\beta)$-exchange chain starting at $v_0$ is proved in [4].

**Lemma 1.** ([2]) *Let $G(V,E)$ be a connected graph. Then $G$ has a 2-edge coloring $C$ such that:*

(a) *If $G$ is Eulerian and $|E|$ is odd, then for an arbitrarily selected $x \in V$, we have $|1(x) - 2(x)| = 2$ and $1(v) - 2(v) = 0$ for all $v \in V \setminus \{x\}$.*

(b) *If $G$ is Eulerian and $|E|$ is even, then $1(v) - 2(v) = 0$ for all $v \in V$.*

(c) *If $G$ is not Eulerian, then $|1(v) - 2(v)| \leq 1$ for all $v \in V$.*

Remark 1. Let $C$ be an optimal $\delta$-edge coloring of $G$. If $G$ is not of CI class. Then there exists $v \in V(G)$, $i,j \in \{1,2,\cdots,\delta\}$, such that $i(v) \geq 2$, $j(v) = 0$. If $G(v;i,j)$ is not an obstruction, by Lemma 1, we can get an improved coloring. Which contradicts to the fact that $C$ is optimal. So $H = G(v;i,j)$ is an obstruction. Since $G(X,Y;u)$ is nearly bipartite, so $u \in V(H)$. By Lemma 1, we can recolor $G(v;i,j)$ such that $i(u) = 2$, $j(u) = 0$ and $i(v) = j(v) = 1$ for all $v \in V(H) \setminus \{u\}$. By iterating this process, we can get an optimal $\delta$-edge coloring $C'$ of $G$ such that $i(u) \leq 2$, and $i(v) \geq 1$ for any $v \in V(G) \setminus \{u\}$ and $i \in \{1,2,\cdots,\delta\}$. Such a coloring is called *a standard optimal $\delta$-edge coloring.*

**Lemma 2.** ([7]) *Let $C$ be an edge coloring of $G$ and $G(v;i,j)$ be an obstruction, where $e = vy$ is with color $i$. We can get an improved coloring if one of the following is satisfied.*

(a) $\alpha(v) > 2$ *for some color* $\alpha$ *of* $C$*;*

(b) $\alpha(y) > 2$ *for some color* $\alpha$ *of* $C$*;*

(c) $\alpha(v) = 2$ *and* $\alpha(y) = 2$ *for some color* $\alpha$ *of* $C$.

**The Proof of Theorem 5** Since all bipartite graph is of CI class. So we can assume that $G(X,Y;u)$ is connected.

For $\delta = 1$, the assertion is trivial. For $\delta = 2$, by Lemma 1, a connected graph is of CII class if and only if it is an odd cycle. While by the condition that there exits a vertex $y \in N_G(u)$ such that $d(u) + d(y) \geq 3\delta - 1$. So $G(X,Y;u)$ is not an odd cycle, which means $G$ is of CI class.

Now suppose that $\delta \geq 3$. Let $C$ be a standard optimal $\delta$-edge coloring of $G$. Suppose, for the sake of contradiction, $C$ is not an edge-cover coloring of $G$. Then there exists a color $\beta \in \{1,2,\cdots,\delta\}$ such that $\beta(u) = 0$. If there exists a color $i \neq \beta$ with $i(u) > 2$, by Lemma 2, we can get an improved coloring. Which contradicts to the chosen of $C$. So for any $i \in \{1,2,\cdots,\delta\}\backslash\{\beta\}$, $i(u) \leq 2$ and $\beta(u) = 0$. Thus if $d(u) \geq 2\delta - 1$, we get a contradiction. That is $G$ is of CI class if $d(u) \geq 2\delta - 1$.

Now set $d(u) = \delta + p$, $0 \leq p \leq \delta - 2$. Thus there are at least $p + 1$ colors, each of which appears twice at $u$. Set the set of such colors be $M_u$. Clearly, $\beta \notin M_u$. For $y \in N_G(u)$, suppose that one of the edge $e = uy$ is colored $\alpha$. Without loss of generality, we can suppose that $\alpha(u) = 2$. In fact, if $\alpha(u) = 1$, there is an other color $\gamma$ such that $\gamma(u) = 2$ for $|M_u| \geq p + 1$. Let $K$ be a minimum $(\gamma, \alpha)$-exchange chain starting at $u$. Since $\gamma(u) = 2, \alpha(u) = 1$ and $K$ is minimal, so $K$ doesn't contain the edge $e$ and doesn't end at $u$. Exchanging the colors on $K$, we get an other standard optimal $\delta$-edge coloring $C'$ such that $\alpha(u) = 2$, and $e$ is still with color $\alpha$. From the above discussion, we can assume that $\alpha(u) = 2$ in coloring $C$. Since $C$ is optimal, $H = G(u;\alpha,\beta)$ is an obstruction. From $d(u)+d(y) \geq 3\delta-1$, we get $d(y) \geq 2\delta-1-p$. Since $C$ is a standard optimal coloring and by Lemma 2, we have $1 \leq i(y) \leq 2$ for any $i \in \{1,2,\cdots,\delta\}$. That is, there are $\delta - 1 - p$ colors each of which appears twice at $y$. Set the set of such colors be $M_y$. From the above discussion, $|M_u| + |M_y| \geq \delta$ and $M_u \cup M_y \subseteq \{1,2,\cdots,\delta\}$.

Noting that $H = G(u;\alpha,\beta)$ is an obstruction, so $\alpha(y) = \beta(y) = 1$. While $\beta(u) = 0$. That is, $\beta \notin M_u \cup M_y$. So $|M_u \cup M_y| \leq \delta - 1$. While $|M_u| + |M_y| \geq \delta$, so there exists a color $\gamma \in M_u \cap M_y$. That is $\gamma(u) = \gamma(y) = 2$, by Lemma 2, we get an improved coloring. Which contradicts to the chosen of $C$.

From the above discussion, we get that $C$ is an edge-cover coloring of $G$. That is, $G$ is of CI class. □

The example $K_{n,n} \vee \{u\}$ also shows that the bound in Theorem 5 is sharp.

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