Upper minus total domination of a 5-regular graph*

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ABSTRACT. A function $f : V(G) \rightarrow \{-1, 0, 1\}$ defined on the vertices of a graph G is a minus total dominating function (MTDF) if the sum of its function values over any open neighborhood is at least one. That is, for every $v \in V$, $f(N(v)) \geq 1$, where $N(v)$ consists of every vertex adjacent to v. The weight of a MTDF is the sum of its function values over all vertices. A MTDF f is minimal if there does not exist a MTDF $g : V(G) \rightarrow \{-1, 0, 1\}, f \neq g$, for which $g(v) \leq f(v)$ for every $v \in V$. The upper minus total domination number, denoted by $\Gamma_t^-(G)$, of G is the maximum weight of a minimal MTDF on G. A function $f : V(G) \rightarrow \{-1, 1\}$ defined on the vertices of a graph G is a signed total dominating function (STDF) if the sum of its function values over any open neighborhood is at least one. The signed total domination number, denoted by $\gamma_t^s(G)$, of G is the minimum weight of a STDF on G. In this paper we establish an upper bound on $\Gamma_t^-(G)$ of the 5-regular graph and characterize the extremal graphs attaining the upper bound. Also, we exhibit an infinite family of cubic graphs in which the difference $\Gamma_t^-(G) - \gamma_t^s(G)$ can be made arbitrary large.

Key words: Minus total domination; Regular graph; Signed total domination

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1 Introduction

All graphs under consideration are finite, undirected and simple. For standard graph theory terminology not given here we refer the reader to [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. The order of G is denoted by $n = |V(G)|$. For a vertex $v \in V$, the open neighborhood of v, denoted by $N(v)$, is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \in V |\ uv \in E\}$. The closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a subset $S \subseteq V$, the open neighborhood of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. $G[S]$ denotes the subgraph of G induced by S. For any vertex $v \in V$, the degree of v is denoted by $d(v)$. A graph is said to be k-regular if its every vertex is of degree k. For a subset $S \subseteq V$, the number of vertices in S that are adjacent to v is denoted by $d_S(v)$. The number of edges between X and Y is denoted by $e(X, Y)$, where the subsets X and Y of vertices are disjoint.

For a real-valued function $f : V(G) \rightarrow R$, the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex $v \in V$, we denote $f(N(v))$ by $f[v]$ for notational convenience.

Let $f : V(G) \rightarrow \{-1, 0, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 0, 1\}$. The function f is defined in [1] to be minus total dominating function (MTDF) of G if $f[v] \geq 1$ for every $v \in V$. A minus total dominating function f is said to be minimal if every minus total dominating function g satisfying $g(v) \leq f(v)$ for every $v \in V$, is equal to f. The minus total domination number, denoted by $\gamma_t^-(G)$, of G is the minimum weight of a MTDF on G. The upper minus total domination number, denoted by $\Gamma_t^-(G)$, of G is the maximum weight of a minimal MTDF on G. A minimal minus total dominating function of weight $\Gamma_t^-(G)$ is called a $\Gamma_t^-(G)$-function on G. For a vertex $v \in V$, if $f[v] = 1$, then v is said to be a critical vertex under f. The parameters $\gamma_t^-(G)$ and $\Gamma_t^-(G)$ were studied in [4] and [5] respectively.

If we only allow the weights -1 and 1, then this is well-known signed total domination which was first introduced by Zelinka in [6]. Let $f : V(G) \rightarrow \{-1, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 1\}$. The function f is called a signed total dominating function (STDF), if $f[v] \geq 1$ for every $v \in V$. The signed total domination number, denoted by $\gamma_t^s(G)$, of G is the minimum weight of a STDF on G. The upper signed total domination number, denoted by $\Gamma_t^s(G)$, of G is the maximum weight of a minimal STDF on G. The two parameters were studied by Henning in [3].

Throughout this paper, if f is a $\Gamma_t^-(G)$-function on G, then we let P, Q and M denote the sets of those vertices in G which are assigned under f the value $+1, 0$ and -1, respectively. We also define

$$P_{ij} = \{v \in P \mid d_Q(v) = i, d_M(v) = j\},$$

$$Q_{ij} = \{v \in Q \mid d_P(v) = i, d_M(v) = j\},$$

$$M_{ij} = \{v \in M \mid d_P(v) = i, d_Q(v) = j\},$$

and let $|P| = p, |Q| = q$ and $|M| = m$. Thus, $n = p + q + m, w(f) = |P| - |M| = p - m$.

In [5], the authors showed that, for every cubic graph G, $\Gamma_t^-(G) \leq 5n/7$, and for every 4-regular graph G, $\Gamma_t^-(G) \leq 7n/10$. Meanwhile, the authors characterized the regular graphs attaining these upper bounds. Furthermore, some open problems were also posed as follows:

1. Find the upper bounds on $\Gamma_t^-(G)$ for a k-regular graph G, $k \geq 5$.

2. For any positive integer k, does there exist a family of graphs satisfying $\Gamma_t^-(G) - \gamma_t^s(G) \geq k$?

In this paper, we establish an upper bound on $\Gamma_t^-(G)$ for the 5-regular graph and characterize the graphs achieving this upper bound. Also, we exhibit an infinite family of graphs in which the difference $\Gamma_t^-(G) - \gamma_t^s(G)$ can be made arbitrarily large.

2 5-regular graphs

In this section we establish an upper bound on the upper minus total domination number of a 5-regular graph in terms of its order and characterize classes of the 5-regular graphs attaining this bound.

To complete our characterization, we first construct a family $\mathcal{F} = \{G_{k,l} \mid k \geq 3, l \geq 4\}$ of 5-regular graphs. The two integers $k \geq 3$ and $l \geq 4$ satisfy $5l - 3k = 4h$, where h is positive integer. Let $G_{k,l}$ be a 5-regular graph with vertex set $\cup_{i=1}^5 A_i$ with $|A_i| = a_i, i = 1, 2, \cdots, 5$ where all a_is are integers satisfying $a_1 = k$, $a_2 = l$, $a_3 = 3k$, $a_4 = (5l - 3k)/2$, $a_5 = 6k + 5l$, and A_2 and A_3 are two independent sets. The edge set of $G_{k,l}$ is constructed as follows:

Add k edges joining vertices of A_1 so that A_1 induces a 2-regular graph. Add $(5l - 3k)/4$ edges joining vertices of A_4 so that A_4 induces a 1-regular graph. Add $12k + 10l$ edges joining vertices of A_5 so that A_5 induces a 4-regular graph. Add $3k$ edges between A_1 and A_3 so that each vertex in A_1 is adjacent to precisely three vertices of A_3 and each vertex in A_3 is adjacent to exactly one vertex of A_1. Add $6k + 5l$ edges between A_5 and $A_3 \cup A_4$ so that each vertex in A_5 is adjacent to precisely one vertex of $A_3 \cup A_4$, and each vertex in A_3 is adjacent to exactly three vertices of A_5 while each vertex in A_4 is adjacent to precisely two vertices of A_5. Add $5l$ edges between A_2 and $A_3 \cup A_4$ in such a way that each vertex in A_2 is adjacent to precisely five vertices of $A_3 \cup A_4$, and each vertex in A_3 is adjacent to exactly a vertex of A_2 while each vertex in A_4 is adjacent to precisely two vertices of A_2.

By the definition of minimal minus total dominating function, the following observation is straightforward and therefore its proof is omitted.

Observation 1 *A MTDF on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) \geq 0$, there exist a vertex $u \in N(v)$ with $f[u] = 1$.*

We next present an upper bound on the upper minus total domination number of a 5-regular graph in terms of its order.

Theorem 2 *If G is a 5-regular graph of order n , then*

$$\Gamma_t^-(G) \leq \frac{13}{17}n$$

with equality if and only if $G \in \mathcal{F}$.

Proof. Let f be a $\Gamma_t^-(G)$-function on G. Then $\Gamma_t^-(G) = |P| - |M| = p - m$. By definition, for any vertex $v \in V, d_M(v) \leq 2, d_Q(v) \leq 4 - 2d_M(v)$ and $d_P(v) \geq d_M(v) + 1 \geq 1$ for otherwise $f[v] < 1$. Therefore P, Q and M can be partitioned into the following sets, respectively.

$$\begin{aligned}
P_{ij} &= \{v \in P |\ d_Q(v) = i, d_M(v) = j, \text{where } 0 \leq j \leq 2,\ 0 \leq i \leq 4 - 2j\}, \\
Q_{ij} &= \{v \in Q |\ d_P(v) = i, d_M(v) = j, \text{where } 0 \leq j \leq 2, j + 1 \leq i \\
&\quad \leq 5 - j\}, \\
M_{ij} &= \{v \in M |\ d_P(v) = i, d_Q(v) = j, \text{where } 0 \leq j \leq 4,\ \lfloor\frac{5-j}{2}\rfloor + 1 \leq i \\
&\quad \leq 5 - j\},
\end{aligned}$$

and let $|P_{ij}| = p_{ij}$, $|Q_{ij}| = q_{ij}$, and $|M_{ij}| = m_{ij}$. Then

$$p = p_{00} + p_{01} + p_{02} + p_{10} + p_{11} + p_{20} + p_{21} + p_{30} + p_{40},$$

$$
\begin{aligned}
q &= q_{10} + q_{20} + q_{21} + q_{30} + q_{31} + q_{32} + q_{40} + q_{41} + q_{50}, \\
m &= m_{14} + m_{22} + m_{23} + m_{30} + m_{31} + m_{32} + m_{40} + m_{41} + m_{50}.
\end{aligned}
$$

Furthermore, we write

$$P' = P_{02} \cup P_{21} \cup P_{40}, Q' = Q_{10} \cup Q_{21} \cup Q_{32}, M' = M_{14} \cup M_{22} \cup M_{30}.$$

Clearly, each vertex $v \in P' \cup Q' \cup M'$ is a critical vertex of G under f, i.e., $f[v] = 1$, while for every vertex $v \in V - (P' \cup Q' \cup M'), f[v] \geq 2$. By counting the edge number $e(P,Q), e(Q,M)$, and $e(P,M)$, we obtain the following equalities at once.

$$
\begin{aligned}
p_{10} &+ p_{11} + 2p_{20} + 2p_{21} + 3p_{30} + 4p_{40} = e(P,Q) \\
&= 5q - (4q_{10} + 3q_{20} + 3q_{21} + 2q_{30} + 2q_{31} + 2q_{32} + q_{40} + q_{41}), \quad (1) \\
q_{21} &+ q_{31} + 2q_{32} + q_{41} = e(Q,M) \\
&= 4m_{14} + 2m_{22} + 3m_{23} + m_{31} + 2m_{32} + m_{41} \quad (2)
\end{aligned}
$$

and

$$
\begin{aligned}
p_{01} &+ 2p_{02} + p_{11} + p_{21} = e(P,M) \\
&= 5m - (m_{22} + 2m_{30} + m_{31} + m_{40} + q_{21} + q_{31} + 2q_{32} + q_{41}). \quad (3)
\end{aligned}
$$

By Observation 1, for every vertex $v \in P - P' = P_{00} \cup P_{01} \cup P_{10} \cup P_{11} \cup P_{20} \cup P_{30}$, there exists a vertex $u \in N(v)$ such that $f[u] = 1$. It follows that for every vertex $v \in P - P'$, there must exist a neighbor of v that belongs to $P' \cup Q' \cup M'$. Hence we have

$$
\begin{aligned}
p_{00} &+ p_{01} + p_{10} + p_{11} + p_{20} + p_{30} \leq e(P - P', P' \cup Q' \cup M') \\
&= e(P - P', P') + e(P - P', Q' \cup M') \\
&= e(P - P', P_{02}) + e(P - P', P_{21}) + e(P - P', P_{40}) \\
&\quad + e(P - P', Q' \cup M'). \quad (4)
\end{aligned}
$$

Furthermore, we note that for every vertex $v \in P_{02}$, there must exist a neighbor u of v satisfying $f[u] = 1$, that is, $u \in P' \cup M'$. If $u \in P'$, then v is adjacent to at most two vertices of $P - P'$, while if $u \in M'$, then v is adjacent to at most three vertices of $P - P'$. Hence we can write P_{02} as the disjoint union of two sets P'_{02} and P''_{02} where $P'_{02} = \{v \in P_{02} | \, d_{P-P'}(v) = 3\}$ and $P''_{02} = P_{02} - P'_{02}$. Let $|P'_{02}| = p'_{02}$, and so $|P''_{02}| = p''_{02} = p_{02} - p'_{02}$. Since each vertex $v \in P'_{02}$ is adjacent to at least one vertex of M', it follows that $p'_{02} \leq e(P'_{02}, M')$. So we get

$$
\begin{aligned}
e(P - P', P_{02}) &= e(P - P', P'_{02} \cup P''_{02}) \\
&\leq 3p'_{02} + 2(p_{02} - p'_{02}) \\
&= 2p_{02} + p'_{02} \\
&\leq 2p_{02} + e(P'_{02}, M'). \quad (5)
\end{aligned}
$$

Similarly, it follows that for every vertex $v \in P_{21}$, there must exist a neighbor u of v that belongs to $P' \cup Q' \cup M'$. If $u \in P'$, then v is adjacent to at most a vertex of $P - P'$, while if $u \in Q' \cup M'$, then v is adjacent to at most two vertices of $P - P'$. Therefore we can partition P_{02} into two subsets $P'_{21} = \{v \in P_{21} | \ d_{P-P'}(v) = 2\}$ and $P''_{21} = P_{21} - P'_{21}$. Let $|P'_{21}| = p'_{21}$, and so $|P''_{21}| = p''_{21} = p_{21} - p'_{21}$. Because each vertex $v \in P'_{21}$ is adjacent to at least one vertex of $Q' \cup M'$, we have $p'_{21} \le e(P'_{21}, Q' \cup M')$. Hence we obtain

$$\begin{aligned} e(P - P', P_{21}) &= e(P - P', P'_{21} \cup P''_{21}) \\ &\le 2p'_{21} + (p_{21} - p'_{21}) \\ &= p_{21} + p'_{21} \\ &\le p_{21} + e(P'_{21}, Q' \cup M'). \end{aligned} \tag{6}$$

By the minimality of f, for each vertex $v \in P_{40}$, there must exist a critical neighbor u of v. If $u \in P'$, then v is adjacent to no vertex of $P - P'$, while if $u \in Q'$, then v is adjacent to at most one vertex of $P - P'$. So, we can write P_{40} as the disjoint union of two sets P'_{40} and P''_{40} where $P'_{40} = \{v \in P_{40} | \ d_{P-P'}(v) = 1\}$ and $P''_{40} = P_{40} - P'_{40}$. Let $|P'_{40}| = p'_{40}$, and so $|P''_{40}| = p''_{40} = p_{40} - p'_{40}$. Since each vertex $v \in P'_{40}$ is adjacent to at least one vertex of Q', it follows that

$$\begin{aligned} e(P - P', P_{40}) &= e(P - P', P'_{40} \cup P''_{40}) \\ &= e(P - P', P'_{40}) \\ &= p'_{40} \\ &\le e(P'_{40}, Q') \end{aligned} \tag{7}$$

Thus, by (4),(5),(6) and (7), we get

$$\begin{aligned} p_{00} &+ p_{01} + p_{10} + p_{11} + p_{20} + p_{30} \le 2p_{02} + p_{21} + e(P'_{02}, M') \\ &+ e(P'_{21}, Q' \cup M') + e(P'_{40}, Q') + e(P - P', Q' \cup M') \\ &\le 2p_{02} + p_{21} + e(P, Q' \cup M') \\ &= 2p_{02} + p_{21} + (q_{10} + 2q_{21} + 3q_{32}) + (m_{14} + 2m_{22} + 3m_{30}). \end{aligned} \tag{8}$$

Next, we start to establish the upper bound on $\Gamma_t^-(G)$. First, we obtain

$$\begin{aligned} n &= (q + m) + p \\ &= (q + m) + (p_{00} + p_{01} + p_{10} + p_{11} + p_{20} + p_{30}) + (p_{02} + p_{21} + p_{40}) \\ &\le (q + m) + (p_{02} + p_{21} + p_{40}) + 2p_{02} + p_{21} + (q_{10} + 2q_{21} + 3q_{32}) \\ &\quad + (m_{14} + 2m_{22} + 3m_{30}) \quad \text{(by (8))} \\ &= 11(q + m) - a \quad \text{(by (1),(3))}, \end{aligned}$$

where $a = (2p_{01} + p_{02} + 2p_{10} + 4p_{11} + 4p_{20} + 4p_{21} + 6p_{30} + 7p_{40} + m_{30} + 2m_{31} + 2m_{40} + 7q_{10} + 6q_{20} + 6q_{21} + 4q_{30} + 6q_{31} + 5q_{32} + 2q_{40} + 4q_{41} - m_{14})$. Hence, it follows that

$$q + m \geq \frac{1}{11}n + \frac{1}{11}a.$$

So

$$p = n - (q + m) \leq \frac{10}{11}n - \frac{1}{11}a. \tag{9}$$

On the other hand, we have

$$\begin{aligned}
p &= (p_{00} + p_{01} + p_{10} + p_{11} + p_{20} + p_{30}) + (p_{02} + p_{21} + p_{40}) \\
&\leq (p_{02} + p_{21} + p_{40}) + 2p_{02} + p_{21} + (q_{10} + 2q_{21} + 3q_{32}) \\
&\quad +(m_{14} + 2m_{22} + 3m_{30}) \quad (\text{ by } (8)\ ) \\
&= \frac{34}{27}(p_{01} + 2p_{02} + p_{11} + p_{21}) - \frac{34}{27}(p_{01} + 2p_{02} + p_{11} + p_{21}) + (p_{02} + \\
p_{21} &\quad +p_{40}) + 2p_{02} + p_{21} + (q_{10} + 2q_{21} + 3q_{32}) + (m_{14} + 2m_{22} + 3m_{30}) \\
&= \frac{170}{27}m - \frac{34}{27}b,
\end{aligned}$$

where $b = (p_{01} + p_{11} + q_{31} + q_{41} + m_{31} + m_{40}) - \frac{1}{34}(13p_{02} + 20p_{21} + 27p_{40} + 27q_{10} + 20q_{21} + 13q_{32} + 27m_{14} + 20m_{22} + 13m_{30})$. The last equality comes from (3). So, we obtain

$$m \geq \frac{27}{170}p + \frac{34}{170}b. \tag{10}$$

Combining (2), (9) and (10), we immediately get

$$\begin{aligned}
\Gamma_t^-(G) &= p - m \\
&\leq \frac{143}{170}p - \frac{34}{170}b \\
&\leq \frac{13}{17}n - \frac{1}{170}(13a + 34b) \\
&= \frac{13}{17}n - \frac{1}{170}c \quad (\text{ by } (2)\ ) \\
&\leq \frac{13}{17}n,
\end{aligned}$$

where $c = (60p_{01} + 26p_{10} + 86p_{11} + 52p_{20} + 32p_{21} + 78p_{30} + 64p_{40} + 64q_{10} + 78q_{20} + 48q_{21} + 52q_{30} + 102q_{31} + 32q_{32} + 26q_{40} + 76q_{41} + 30m_{23} + 70m_{31} + 20m_{32} + 60m_{40} + 10m_{41})$.

For a 5-regular graph G of order n, we next show that if $\Gamma_t^-(G) = 13n/17$, then $G \in \mathcal{F}$. Suppose that $\Gamma_t^-(G) = 13n/17$, then equalities hold for the above inequalities. By $c = 0$, we immediately have

$$\begin{aligned} p_{01} &= p_{10} = p_{11} = p_{20} = p_{21} = p_{30} = p_{40} = 0 \\ q_{10} &= q_{20} = q_{21} = q_{30} = q_{31} = q_{32} = q_{40} = q_{41} = 0 \\ m_{23} &= m_{31} = m_{32} = m_{40} = m_{41} = 0 \end{aligned}$$

and by the equalities (1) and (2), it follows that $q = 0$ and $m_{22} = m_{14} = 0$. Consequently, we obtain $V(G) = P_{00} \cup P_{02} \cup M_{30} \cup M_{50}$. Applying the equality (3) and the equality from (8), we get

$$2p_{02} = 3m_{30} + 5m_{50} \tag{11}$$

and

$$p_{00} = 2p_{02} + e(P'_{02}, M_{30}) = 2p_{02} + 3m_{30} \tag{12}$$

Furthermore, according to the equality from (5), we have

$$e(P_{00}, P_{02}) = 3p'_{02} + 2p''_{02} = 2p_{02} + p'_{02} = 2p_{02} + e(P'_{02}, M_{30}) \tag{13}$$

Combining the equalities (11),(12) and (13), it follows that

$$\begin{aligned} p'_{02} &= e(P'_{02}, M_{30}) = 3m_{30}; \\ p''_{02} &= p_{02} - p'_{02} = (5m_{50} - 3m_{30})/2; \\ p_{00} &= e(P_{00}, P_{02}) = 3p'_{02} + 2p''_{02}. \end{aligned}$$

So, each vertex in P_{00} is adjacent to precisely one vertex of $P_{02} = P'_{02} \cup P''_{02}$, and each vertex in P'_{02} is adjacent to exactly three vertices of P_{00} while each vertex in P''_{02} is adjacent to exactly two vertices of P_{00}. Obviously, $G[P_{00}]$ and $G[M_{30}]$ are 4-regular and 2-regular graphs , respectively. By the definition of P''_{02}, we get $e(M_{50}, P''_{02}) = 2p''_{02}$. Hence, $G[P''_{02}]$ is a 1-regular graph. Since each vertex in P'_{02} is adjacent to no vertex of P_{02}, it follows that P'_{02} is an independent set. Moreover, it is obvious that M_{50} is also an independent set. Let $m_{30} = k \geq 3$, $m_{50} = l \geq 4$ and k and l be integers satisfying $5l - 3k = 4h$, where h is positive integer. Hence, we get $p'_{02} = 3k$, $p''_{02} = (5l - 3k)/2$, $p_{00} = 6k + 5l$. Thus, $G = G_{k,l}$ with vertex set $\cup_{i=1}^{5} A_i$, where $A_1 = M_{30}$, $A_2 = M_{50}$, $A_3 = P'_{02}$, $A_4 = P''_{02}$, $A_5 = P_{00}$. Therefore, $G \in \mathcal{F}$.

Conversely, suppose that $G \in \mathcal{F}$. Let $G = G_{k,l}$ for two integers $k \geq 3$ and $l \geq 4$ that satisfy $5l - 3k = 4h$, where h is positive integer. Let f be a function on $G_{k,l}$ which assigns to every vertex of $A_1 \cup A_2$ and $A_3 \cup A_4 \cup A_5$ the value -1 and $+1$, respectively. Then the set $A_1 \cup A_3 \cup A_4$ is critical set of $G_{k,l}$ under f, which implies that for every vertex $v \in V$, there exists a vertex $u \in N(v)$ such that $f[u] = 1$. So, f is a minimal minus total dominating function with weight $w(f) = \sum_{i=3}^{5} a_i - (a_1 + a_2) = 13(k + l)/2 = 13n/17$. Consequently, $\Gamma_t^-(G) = 13n/17$.

3 The difference $\Gamma_t^-(G) - \gamma_t^s(G)$

In this section, we exhibit an infinite family of graphs in which the difference $\Gamma_t^-(G) - \gamma_t^s(G)$ can be made arbitrarily large. To do it, we will use the following result due to [5].

Theorem 3 ([5]) *Let G be a cubic graph of order n. Then the following statements are equivalent.*

(1) $\Gamma_t^s(G) = \frac{5}{7}n$;

(2) $\Gamma_t^-(G) = \frac{5}{7}n$;

(3) $G \in \mathcal{T}$.

The construction of the family $\mathcal{T}$ of cubic graphs in Theorem 3 is not given here, we refer the reader to [5].

Next, for any positive integer k, we construct an infinite family $\mathcal{H} = \{G_k \mid k \geq 1\}$ of cubic graphs with $\Gamma_t^-(G_k) - \gamma_t^s(G_k) \geq 2k$ as follows.

Let G_k be a cubic graph with vertex set $\cup_{i=1}^5 A_i$ with $|A_i| = a_i$, for $1 \leq i \leq 5$ where all a_is are integers satisfying $a_1 = 2k,\ a_2 = 2k,\ a_3 = 6k,\ a_4 = 4k,\ a_5 = 14k$, and A_1 and A_4 are two independent sets. Furthermore, we write $A_1 = \{u_i \mid i = 1, 2, \cdots, 2k\}$, $A_3 = \{v_i \mid i = 1, 2, \cdots, 6k\}$. The edge set of G_k is constructed as follows.

The set of edges between A_1 and A_3 is defined as $E_{13} = \{u_i v_{3i-2}, u_i v_{3i-1}, u_i v_{3i} | \ i = 1, 2, \cdots, 2k\}$. Add $3k$ edges joining vertices of A_3 in such a way that $v_i v_{i+1} \in E(G_k)$, $i = 1, 3, \cdots, 6k-1$. So $G_k[A_3]$ is a 1-regular graph. Add k edges joining vertices of A_2 so that A_2 also induces a 1-regular graph. Add $14k$ edges between $A_3 \cup A_4$ and A_5 in such a way that each vertex of A_5 is adjacent to precisely one vertex of $A_3 \cup A_4$, and each vertex in A_3 is adjacent to exactly one vertex of A_5 while each vertex of A_4 is adjacent to exactly two vertices of A_5. Add $4k$ edges between A_2 and A_4 so that each vertex in A_2 is adjacent to precisely two vertices of A_4 while each vertex of A_4 is adjacent to exactly one vertex of A_2. At last, add $14k$ edges joining vertices of A_5 so that A_5 induces a 2-regular graph. According to our construction, G_k is a cubic graph of order $n = 28k$. Obviously, $\mathcal{H} \subseteq \mathcal{T}$.

Let $G = G_k$, we define two functions f and g on G_k as follows

$$f(v) = \begin{cases} -1 & \text{if } v \in A_1 \cup A_2 \\ +1 & \text{otherwise} \end{cases}$$

and

$$g(v) = \begin{cases} -1 & \text{if } v \in A_2 \cup \{u_{2i-1} \mid i = 1, 2, \cdots, k\} \cup \{v_{3i} \mid i = 1, 2, \cdots, 2k\} \\ +1 & \text{otherwise} \end{cases}$$

Then it is easy to check that the defined f is a minimal minus total dominating function on G with weight $f(V(G)) = 20k = 5n/7$, and g is a signed total dominating on G with weight $g(V(G)) = 18k$. By Theorem 3 , we obtain $\Gamma_t^-(G) = f(V(G)) = 20k$. Thus, we have $\Gamma_t^-(G) - \gamma_t^s(G) \geq 20k - 18k = 2k$.

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