Cycle-Partitions with Specified Vertices and Edges

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Abstract

In this paper, we consider cycle-partition problems which deal with the case when both vertices and edges are specified and we require that they should belong to different cycles. Minimum degree and degree sum conditions are given, which are best possible.

Keywords: vertex-disjoint cycles, partition of a graph, specified vertex, specified edge

1 Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. For a vertex x of a graph G, the neighborhood of x is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. For a subgraph H of G and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subset S of $V(G)$, we write $\langle S \rangle$ for the subgraph induced by S. For a subgraph H of G and a subset S of $V(G)$, $d_H(S) = \sum_{x \in S} d_H(x)$, $N_H(S) = \cup_{x \in S} N_H(x)$ and define $G - H = \langle V(G) - V(H) \rangle$ and $G - S = \langle V(G) - S \rangle$. For a graph G, $|G| = |V(G)|$ is the order of G, $\delta(G)$ is the minimum degree of G, and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) | xy \notin E(G), x, y \in V(G), x \neq y\}$$

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is the minimum degree sum of nonadjacent vertices. (When G is complete, we define $\sigma_2(G) = \infty$.)

For a graph G, mG is the union of m copies of G. For graphs G_1 and G_2, $G_1 \cup G_2$ is the union of G_1 and G_2 and $G_1 + G_2$ is the join of G_1 and G_2. Moreover, for graphs G_1, G_2 and G_3, $G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2$. K_n is a complete graph of order n.

In this paper, 'disjoint' means 'vertex-disjoint' since we only deal with partitions of the vertex set, and n always denotes the order of a graph G. Suppose $C_1, \ldots, C_k$ are disjoint cycles of a graph G. Then $\{C_1, \ldots, C_k\}$ is called a k-cycle-packing of G. Moreover, if $V(G) = \bigcup_{i=1}^{k} V(C_i)$, $\{C_1, \ldots, C_k\}$ is called a k-cycle-partition of G.

The following result is the first step of the research on a k-cycle-partition.

Theorem 1 ([1]) *Suppose $n \geq 4k$ and $\sigma_2(G) \geq n$. Then G has a k-cycle-partition.*

Egawa et al. considered the cycle-partition with specified vertices. When k vertices $x_1, \ldots, x_k$ are specified, a cycle C is called admissible if $|V(C) \cap \{x_1, \ldots, x_k\}| = 1$, and $\{C_1, \ldots, C_k\}$ is admissible if each C_i is admissible. They proved the following theorem.

Theorem 2 ([2]) *Suppose $n \geq 6k - 2$ and $\delta(G) \geq n/2$. Then G has an admissible k-cycle-partition for any k distinct vertices.*

When k independent edges $e_1 = x_1y_1, \ldots, e_k = x_ky_k$ are specified, a cycle C is called admissible if $|E(C) \cap \{e_1, \ldots, e_k\}| = 1$ and $|V(C) \cap \{x_1, \ldots, x_k, y_1, \ldots, y_k\}| = 2$, and $\{C_1, \ldots, C_k\}$ is admissible if each C_i is admissible. In this case, the following result is obtained.

Theorem 3 ([3]) *Suppose $k \geq 2, n \geq 4k - 1$ and $\sigma_2(G) \geq n + 2k - 2$. Then G has an admissible k-cycle-partition for any k independent edges.*

In this paper, we consider the case when both vertices and edges are specified. Let $S = \{v_1, \ldots, v_p\}$ be a subset of $V(G)$, $F = \{e_1 = x_1y_1, \ldots, e_q = x_qy_q\}$ be a subset of $E(G)$, and $V(F) = \{x_1, \ldots, x_q, y_1, \ldots, y_q\}$. If $|V(F)| = 2q$ (that is, F is independent) and $S \cap V(F) = \phi$, $S \cup F$ is called *feasible*. A cycle C of G is called *admissible* if one of the following holds:

(a) $V(C) \cap (S \cup V(F)) = \phi$,

(b) $|V(C) \cap S| = 1$ and $V(C) \cap V(F) = \phi$,

(c) $|E(C) \cap F| = 1$ and $|V(C) \cap (S \cup V(F))| = 2$.

If $C_1, \ldots, C_k$ are admissible disjoint cycles and $S \cup V(F)$ is contained in $\cup_{i=1}^{k} V(C_i)$, $\{C_1, \ldots, C_k\}$ is called an admissible k-cycle-packing. An admissible k-cycle-partition is defined similarly.

The main result is the following theorem.

Theorem 4 *Suppose $n \geq 10k$, $k \geq p+q$, $p \geq 0$, $q \geq 1$ and either*

$$\delta(G) \geq \max\left\{\frac{n+q}{2}, \frac{n+p+2q-3}{2}\right\},$$

or

$$\sigma_2(G) \geq \max\{n+q, n+2p+2q-2\}.$$

Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible k-cycle-partition.

To prove Theorem 4, we first solve the packing problem.

Theorem 5 *Suppose $n \geq 9k$, $k \geq p+q$, $p \geq 0$, $q \geq 1$ and either $\delta(G) \geq (n+p+2q-3)/2$ or $\sigma_2(G) \geq n+2p+2q-2$. Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible k-cycle-packing.*

Note that the assumption $n \geq 9k$ is not sharp, but it cannot be dropped. The degree conditions in Theorem 5 are sharp in the following sense.

Example 1. Let $G = K_m + K_{p+2q-2} + K_m$ with an edge e_1 which joins the two K_ms. Take p distinct vertices $v_1, \ldots, v_p$ and $q-1$ independent edges $e_2, \ldots, e_q$ in K_{p+2q-2} such that $\{v_1, \ldots, v_p, e_1, \ldots, e_q\}$ is feasible. Then there is not an admissible k-cycle-packing, while $\delta(G) = (n+p+2q-4)/2$.

Example 2. Let $G = K_{p+q} + K_{2p+2q-1} + K_m$. Take p distinct vertices $v_1, \ldots, v_p$ in K_{p+q} and q independent edges $e_1, \ldots, e_q$ between K_{p+q} and $K_{2p+2q-1}$ such that $\{v_1, \ldots, v_p, e_1, \ldots, e_q\}$ is feasible. Then G does not contain an admissible k-cycle-packing, while $\sigma_2(G) = n+2p+2q-3$.

Next, we extend a packing to a partition.

Theorem 6 *Let $S \cup F$ be a feasible set with $|S| = p$ and $|F| = q$. Suppose $n \geq 10k$, $k \geq 1$, $k \geq p+q$, $p \geq 0$, $q \geq 0$, $\delta(G) \geq p+q+1$, $\sigma_2(G) \geq n+q$, and G has an admissible k-cycle-packing. Then G has an admissible k-cycle-partition.*

The assumption $n \geq 10k$ is not sharp, but it cannot be dropped. The degree conditions in Theorem 6 are sharp in the following sense.

Example 3. Let $G = K_1 + K_{p+q} + K_m$. Take p distinct vertices in K_{p+q} and q independent edges between K_{p+q} and K_m such that these p vertices and q edges form a feasible set. Then G has an admissible k-cycle-packing but has no admissible k-cycle-partition, while $\delta(G) = p + q$.

Example 4. Let $G = K_{m+q} + (m+1)K_1$. Take p distinct vertices and q independent edges in K_{m+q} such that these p vertices and q edges form a feasible set. Then G has an admissible k-cycle-packing but does not contain an admissible k-cycle-partition, while $\sigma_2(G) = n + q - 1$.

By Theorem 5 and Theorem 6, we get Theorem 4 as a corollary.

If we put $p = 0$ and $q = k$ in Theorem 4, we get the following.

Corollary 7 *Suppose $n \geq 10k$, $k \geq 2$, and either*

$$\sigma_2(G) \geq n + 2k - 2$$

or

$$\delta(G) \geq \frac{n + 2k - 3}{2}.$$

Then G has an admissible k-cycle-partition for any k independent edges.

This corollary shows that the minimum degree condition in Theorem 3 is not sharp when n is odd.

Let $P = u_1 u_2 \cdots u_l$ be a path. Then we say that P connects u_1 and u_l, and P is a u_1–u_l path. We will use the notation $P[u_i, u_j]$, $1 \leq i < j \leq l$, for a subpath of P from u_i to u_j.

We will also use $C[u, v]$ to denote the segment of the cycle C from u to v (including u and v) under some orientation of C, and $C[u, v) = C[u, v] - \{v\}$ and $C(u, v) = C[u, v] - \{u, v\}$. Given a cycle C with an orientation, we let v^+ (resp. v^-) denote the successor (resp. the predecessor) of v along C according to this orientation.

2 Proof of Theorem 5

To prove Theorem 5, we first prove the following two theorems.

Theorem 8 *Suppose $n \geq 9p+8q-2$, $p \geq 0, q \geq 1$ and $\delta(G) \geq (n+p+2q-3)/2$. Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible $(p+q)$-cycle-packing such that all $p+q$ cycles are of length at most* 5.

Theorem 9 *Suppose $n \geq 4p+4q-1$, $p \geq 0, q \geq 1$ and $\sigma_2(G) \geq n+2p+2q-2$. Then for any feasible set $S \cup F$ with $|S| = p$ and $|F| = q$, G has an admissible $(p+q)$-cycle-packing such that all $p+q$ cycles are of length at most* 4.

The sharpness of the assumptions in Theorems 8 and 9 is already shown in Section 1.

In this section, we will use the following results to prove above theorems.

Theorem 10 ([3]) *Suppose $k \geq 1$, $n \geq 4k-1$ and $\sigma_2(G) \geq n+2k-2$. Then for any k independent edges, G has an admissible k-cycle-packing such that each cycle is length at most* 4.

Theorem 11 ([4], [5]) *Suppose $k \geq 1$, $n \geq 3k$ and $\sigma_2(G) \geq 4k-1$. Then G has a k-cycle-packing.*

Let $S \cup F$ be a feasible set with $S = \{v_1 \ldots, v_p\} \subseteq V(G)$ and $F = \{e_1, \ldots, e_q\} \subseteq E(G)$. If $C_1, \ldots, C_h$ are admissible disjoint cycles and $S \cup V(F) - \{v_i\}$ for some $v_i \in S$ or $S \cup V(F) - V(e_j)$ for some $e_j \in F$ is contained in $\bigcup_{l=1}^{h} V(C_l)$, $\{C_1, \ldots, C_h\}$ is called a *semi-admissible* h-cycle-packing.

2.1 Proof of Theorem 8

Let G be an edge-maximal counterexample to Theorem 8, $S \cup F$ be a feasible set with $S = \{v_1, \ldots, v_p\} \subseteq V(G)$ and $F = \{e_{p+1}, \ldots, e_{p+q}\} \subseteq E(G)$, and $e_i = x_i y_i$ for $p+1 \leq i \leq p+q$. In the rest of the proof, a cycle is called *short* if its length is at most 5. Since if G is a complete graph, G contains an admissible $(p+q)$-cycle-packing, G is not complete. Let x and y be nonadjacent vertices of G and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Then G' is not a counterexample by the maximality of G, and so G' contains an admissible $(p+q)$-cycle-packing $\{C_1, \ldots, C_{p+q}\}$. Since $xy \in E(C_i)$ for some i, $1 \leq i \leq p+q$, G has a semi-admissible $(p+q-1)$-cycle-packing. We take these $p+q-1$ cycles so that admissible cycles which contain specified edges are as many as possible. Subject to this, we take these cycles so that the sum of the length of cycles is as small as possible.

We consider the following two cases.

Case 1 Some specified edge is not contained in the admissible cycles.

We may assume that G contains a semi-admissible $(p+q-1)$-cycle-packing $\{C_1, \ldots, C_{p+q-1}\}$ such that $v_i \in V(C_i)$ for $1 \le i \le p$, $e_i \in E(C_i)$ for $p+1 \le i \le p+q-1$ and $|C_i| \le 5$ for $1 \le i \le p+q-1$. Let $L = \langle \bigcup_{i=1}^{p+q-1} V(C_i) \rangle$, $M = G - L$, and $D = M - \{x_{p+q}, y_{p+q}\}$.

Claim 2.1.1 *For any $z \in V(D)$, $d_{C_i}(z) \le 3$ for $1 \le i \le p+q-1$.*

(*Proof.*) If $d_{C_i}(z) \ge 4$, $\langle V(C_i) \cup \{z\} \rangle$ contains a cycle passing through v_i or e_i which is shorter than C_i. □

Claim 2.1.2 *$d_D(x_{p+q}) \ge 2$ and $d_D(y_{p+q}) \ge 2$.*

(*Proof.*) Suppose $d_D(x_{p+q}) \le 1$. Then

$$\frac{n+p+2q-3}{2} \le d_G(x_{p+q}) \le |L| + 2 \le 5(p+q-1) + 2.$$

Hence we get

$$n \le 9p + 8q - 3.$$

This is a contradiction. □

Take any $z_1, z_2 \in N_D(x_{p+q})$ and $z_1', z_2' \in N_D(y_{p+q})$ and let $S = \{x_{p+q}, y_{p+q}, z_1, z_2, z_1', z_2'\}$. Since M has no short cycle passing through e_{p+q}, $d_S(y) \le 3$ for any $y \in V(M) - S$. Then,

$$d_M(S) \le 3(|M| - 6) + 14 = 3|M| - 4.$$

Therefore,

$$\begin{aligned} d_L(S) &\ge 6\delta(G) - (3|M| - 4) \\ &= 3n + 3p + 6q - 9 - 3|M| + 4 \\ &= 3|L| + 3p + 6q - 5 \\ &> \sum_{i=1}^{p} (3|C_i| + 3) + \sum_{i=p+1}^{p+q-1} (3|C_i| + 6). \end{aligned}$$

Hence $d_{C_i}(S) \ge 3|C_i| + 4$ for some i, $1 \le i \le p$, or $d_{C_i}(S) \ge 3|C_i| + 7$ for some i, $p+1 \le i \le p+q-1$.

Case 1.1 $d_{C_i}(S) \ge 3|C_i| + 4$ for some i, $1 \le i \le p$.

Suppose $d_{C_i}(\{a,b\}) \geq |C_i|+2$ for $a \in \{x_{p+q}, z_1, z_2\}$ and $b \in \{y_{p+q}, z'_1, z'_2\}$. Then we can find some $c \in N_{C_i}(a) \cap N_{C_i}(b) - \{v_i\}$ and this makes an admissible short cycle passing through e_{p+q}. Hence $d_{C_i}(\{a,b\}) \leq |C_i|+1$ and $d_{C_i}(S) \leq 3|C_i|+3$. This is a contradiction.

Case 1.2 $d_{C_i}(S) \geq 3|C_i|+7$ for some i, $p+1 \leq i \leq p+q-1$.

Since $d_{C_i}(\{z_1, z'_1, z_2, z'_2\}) \leq 12$, $d_{C_i}(\{x_{p+q}, y_{p+q}\}) \geq 10$ if $|C_i| = 5$ and $d_{C_i}(\{x_{p+q}, y_{p+q}\}) \geq 7$ if $|C_i| = 4$. These mean that there is an admissible triangle passing through e_{p+q}.

If $|C_i| = 3$, $d_{C_i}(S) \geq 16$. Suppose $d_{C_i}(x_{p+q}) = d_{C_i}(y_{p+q}) = 3$. Then $d_{C_i}(a) = 3$ for some $a \in \{z_1, z'_1, z_2, z'_2\}$, but this means that there are two admissible triangles passing through e_i and e_{p+q}. Otherwise, since $d_{C_i}(\{z_1, z'_1, z_2, z'_2\}) \geq 11$, we may assume that $d_{C_i}(z_1) = d_{C_i}(z'_1) = d_{C_i}(z_2) = 3$. Then there are two admissible cycles passing through e_i and e_{p+q}. This completes the proof of Case 1.

Case 2 Some specified vertex is not contained in the admissbile cycles.

We may assume that G has a semi-admissible $(p+q-1)$-cycle-packing $\{C_2, \ldots, C_{p+q}\}$ such that $v_i \in V(C_i)$ for $2 \leq i \leq p$, $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$ and $|C_i| \leq 5$ for $2 \leq i \leq p+q$. Let $L = \langle \bigcup_{i=2}^{p+q} V(C_i) \rangle$ and $M = G - L$.

Claim 2.2.1 *$d_{C_i}(x) \leq 3$ for $x \in V(M)$ and $2 \leq i \leq p$. Moreover, if $x \neq v_1$, $d_{C_i}(x) \leq 3$ for $p+1 \leq i \leq p+q$.*

(*Proof.*) If $x \neq v_1$, the proof is similar to that of Claim 2.1.1. Suppose $d_{C_i}(v_1) \geq 4$ for $2 \leq i \leq p$. Then, $\langle V(C_i) \cup \{v_1\} - \{v_i\} \rangle$ contains a cycle passing through v_i and shorter than C_i. □

Claim 2.2.2 $d_M(v_1) \geq 3$.

(*Proof.*) Suppose $d_M(v_1) \leq 2$. Then,

$$\frac{n+p+2q-3}{2} \leq d_G(v_1) \leq 3(p-1)+5q+2 = 3p+5q-1$$

by Claim 2.2.1. Hence we get

$$n \leq 5p+8q+1.$$

This is a contradiction. □

Take $z_1, z_2, z_3 \in N_M(v_1)$ and let $S = \{v_1, z_1, z_2, z_3\}$. Since M has no short cycle passing through v_1, $d_S(y) \le 1$ for any $y \in V(M) - S$. Then

$$d_M(S) \le (|M| - 4) + 6 = |M| + 2.$$

Hence

$$\begin{aligned} d_L(S) &\ge 4\delta(G) - (|M| + 2) \\ &= 2n + 2p + 4q - 6 - |M| - 2 \\ &= 2|L| + 2p - 2 + 4q + |M| - 6 \\ &> 2|L| + 2p - 2 + 4q + 4(p-1) \\ &= 2|L| + 6p - 6 + 4q \\ &= \sum_{i=2}^{p}(2|C_i| + 6) + \sum_{i=p+1}^{p+q}(2|C_i| + 4) \qquad (1) \end{aligned}$$

since

$$\begin{aligned} |M| - 6 &\ge n - 5p - 5q + 5 - 6 \ge 9p + 8q - 2 - 5p - 5q - 1 \\ &= 4p + 3q - 3 > 4(p-1). \end{aligned}$$

Claim 2.2.3 $d_{C_i}(S) \le 2|C_i| + 4$ *for* $p + 1 \le i \le p + q$.

(*Proof.*) Suppose $d_{C_i}(S) \ge 2|C_i| + 5$ for some i, $p + 1 \le i \le p + q$.

If $|C_i| = 5$, $d_{C_i}(S) \ge 15$. But this contradicts Claim 2.2.1.

If $|C_i| = 4$, $d_{C_i}(S) \ge 13$. Then, $d_{C_i}(v_1) = 4$ and $d_{C_i}(z_1) = d_{C_i}(z_2) = d_{C_i}(z_3) = 3$. This means that there are two admissible short cycles passing through v_1 and e_i.

If $|C_i| = 3$, $d_{C_i}(S) \ge 11$. In this case, we may assume that $d_{C_i}(z_1) = d_{C_i}(z_2) = 3$. Then, $d_{C_i}(z_3) \le 1$. But this is a contradiction. □

By (1) and Claim 2.2.3, we may assume that $d_{C_i}(S) \ge 2|C_i| + 7$ for some i, $2 \le i \le p$. Clearly, this contradicts Claim 2.2.1. This completes the proof of Theorem 8.

2.2 Proof of Theorem 9

Let $S \cup F$ be a feasible set with $S = \{v_1, \ldots, v_p\} \subseteq V(G)$ and $F = \{e_{p+1}, \ldots, e_{p+q}\} \subseteq E(G)$. Since $\sigma_2(G) \ge n + 2p + 2q - 2$, $\delta(G) \ge 2p + 2q$. Then we can take p independent edges $e_1, \ldots, e_p$ such that $v_i \in V(e_i)$ for $1 \le i \le p$ and $\{e_1, \ldots, e_{p+q}\}$ is also a set of independent edges. Therefore, we can apply Theorem 10 and obtain a required $(p+q)$-cycle-packing. □

2.3 Proof of Theorem 5

Let $S \cup F$ be a feasible set with $|S| = p$ and $|F| = q$. By Theorem 8 and Theorem 9, G has an admissible $(p+q)$-cycle-packing $\{C_1, \ldots, C_{p+q}\}$ such that $|C_i| \leq 5$ for $1 \leq i \leq p+q$. If $k = p+q$, this is a required k-cycle-packing. Hence we may assume that $k > p+q$. Then we take these cycles so that $|\bigcup_{i=1}^{p+q} V(C_i)|$ is as small as possible. Let $L = \langle \bigcup_{i=1}^{p+q} V(C_i) \rangle$ and $H = G - L$. Note that $d_{C_i}(x) \leq 3$ for any $x \in V(H)$ and $1 \leq i \leq p+q$. Then $|H| \geq n - 5(p+q) \geq 3(k-p-q)$ and

$$\sigma_2(H) \geq n + 2p + 2q - 3 - 6(p+q) \geq 4(k-p-q) - 1.$$

Therefore, we can apply Theorem 11 and we get a $(k-p-q)$-cycle-packing of H. Hence we get an admissible k-cycle-packing of G. This completes the proof of Theorem 5.

3 Proof of Theorem 6

3.1 Preliminary Lemmas

Before proving the theorem, we prepare several definitions and lemmas.

Let D be a cycle (resp. a path) of G and $x \in V(G-D)$. We say x *can be inserted into* D if $\langle V(D) \cup \{x\} \rangle$ has a cycle (resp. a path) D' such that $V(D') = V(D) \cup \{x\}$. Moreover, if D contains a specified edge e, D' has to contain e, and if D is a u-v path, then D' also has to be a u-v path.

Lemma 1 *Let C be a cycle of G and $x \in V(G-C)$. Suppose C does not contain a specified edge and $d_C(x) \geq (|C|+1)/2$. Then x can be inserted into C.*

(*Proof.*) Since $d_C(x) \geq (|C|+1)/2$, $N_C(x)$ contains two consecutive vertices of C. Hence x can be inserted into C. □

Lemma 2 *Let $P = u_1u_2 \cdots u_l$ be a path of G and $x \in V(G-P)$. Suppose P does not contain a specified edge and $d_P(x) \geq |P|/2 + 1$. Then x can be inserted into P.*

(*Proof.*) Since $d_P(x) \geq |P|/2+1$, $N_P(x)$ contains two consecutive vertices of P. Hence x can be inserted into P. □

Lemma 3 *Let C be a cycle of G and $x \in V(G-C)$. Suppose $e \in E(C)$ is a specified edge and $d_C(x) \geq |C|/2+1$. Then x can be inserted into C.*

(*Proof.*) Let $e = aa^+$. Since $d_C(x) \geq |C|/2+1$, $N_G(x) \cap C[a^+, a^-]$ contains two consecutive vertices of C. Then x can be inserted into C. □

Lemma 4 *Let $P = u_1u_2\cdots u_l$ be a path of G and $x \in V(G-P)$. Suppose $e \in E(P)$ be a specified edge and $d_P(x) \geq (|P|+3)/2$. Then x can be inserted into P.*

(*Proof.*) Let $e = u_iu_{i+1}$, $1 \leq i \leq l-1$. Since $d_P(x) \geq (|P|+3)/2$, $N_G(x) \cap P[u_1, u_i]$ or $N_G(x) \cap P[u_{i+1}, u_l]$ contains two consecutive vertices of P. Hence x can be inserted into P. □

Let $C_1, \ldots, C_k$ be disjoint subgraphs such that C_h is a u-v path for some h, $1 \leq h \leq p+q$, the rest are all cycles, and $v_i \in V(C_i)$ for $1 \leq i \leq p$ and $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$. Let also $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle$ and $M \subseteq V(G-L)$, $M \neq \phi$. Then we say M *can be inserted into* L if $\langle V(L) \cup M \rangle$ contains disjoint subgraphs $C'_1, \ldots, C'_k$ such that C'_h is a u-v path, the rest are all cycles, $v_i \in V(C_i)$ for $1 \leq i \leq p$ and $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$, and $\bigcup_{i=1}^{k} V(C'_i) = V(L) \cup M$.

Lemma 5 *Let L be a subgraph of G defined in the above definition, $M \subseteq V(G-L)$ and $M \neq \phi$. Suppose $N_G(M) \subseteq V(L) \cup M$ and*

$$d_G(x) \geq \frac{|L|+q}{2} + (|M|-1) + \frac{3}{2}$$

for any $x \in V(M)$. Then M can be inserted into L.

(*Proof.*) Take any $x \in V(M)$. Then

$$\begin{aligned} d_L(x) &\geq \frac{|L|+q}{2} + (|M|-1) + \frac{3}{2} - (|M|-1) = \frac{|L|+q}{2} + \frac{3}{2} \\ &= \sum_{i=1}^{p} \frac{|C_i|}{2} + \sum_{i=p+1}^{p+q} \frac{|C_i|+1}{2} + \sum_{i=p+q+1}^{k} \frac{|C_i|}{2} + \frac{3}{2}. \end{aligned}$$

Hence one of the following holds.

(a) $1 \leq h \leq p$ and $d_{C_h}(x) \geq \frac{|C_h|}{2} + 1$.

(b) $p+1 \leq h \leq p+q$ and $d_{C_h}(x) \geq \frac{|C_h|+3}{2}$.

(c) $d_{C_i}(x) \geq \frac{|C_i|+1}{2}$ for some $i \neq h$, $1 \leq i \leq p$ or $p+q+1 \leq i \leq k$.

(d) $d_{C_i}(x) \geq \frac{|C_i|}{2}+1$ for some $i \neq h$, $p+1 \leq i \leq p+q$.

Then, by Lemmas 1, 2, 3, and 4, x can be inserted into C_h or C_i.

Let $L' = \langle V(L) \cup \{x\} \rangle$ and $M' = M - \{x\}$, and suppose $M' \neq \phi$. Then $N_G(M') \subseteq V(L') \cup M'$ and for any $y \in V(M')$,

$$\begin{aligned} d_G(y) &\geq \frac{|L|+q}{2} + (|M|-1) + \frac{3}{2} \\ &= \frac{|L'|-1+q}{2} + (|M'|+1-1) + \frac{3}{2} \\ &= \frac{|L'|+q}{2} + (|M'|-1) + 2. \end{aligned}$$

Again, y can be inserted into L'. By repeating this operation, M can be inserted into L. □

3.2 Proof of Theorem 6

Suppose $\mathcal{C} = \{C_1, \ldots, C_k\}$ and $\mathcal{C}' = \{C'_1, \ldots, C'_k\}$ are two admissible k-cycle-packing. We say $\mathcal{C}$ *is larger than* $\mathcal{C}'$ if $|\bigcup_{i=1}^k V(C_i)| > |\bigcup_{i=1}^k V(C'_i)|$.

In the rest of this section, $N(x)$ and $N(H)$ will be used instead of $N_G(x)$ and $N_G(H)$ for $x \in V(G)$ and a subgraph H of G.

Let $S \cup F$ be a feasible set with $S = \{v_1, \ldots, v_p\} \subseteq V(G)$ and $F = \{e_{p+1}, \ldots e_{p+q}\} \subseteq E(G)$, and $e_i = x_i y_i$ for $p+1 \leq i \leq p+q$. Since G contains an admissible k-cycle-packing, we take an admissible k-cycle-packing $\{C_1, \ldots, C_k\}$ such that $|\bigcup_{i=1}^k V(C_i)|$ is as large as possible. We may assume that $v_i \in V(C_i)$ for $1 \leq i \leq p$ and $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$. Let $L = \langle \bigcup_{i=1}^k V(C_i) \rangle$ and $H = G - L$. If $H = \phi$, we have nothing to prove. Hence we may assume that $H \neq \phi$.

By Lemmas 1 and 3, the next claim holds.

Claim 3.1 *For $x \in V(H)$, $d_{C_i}(x) \leq |C_i|/2$ for $1 \leq i \leq p$ and $p+q+1 \leq i \leq k$, and $d_{C_i}(x) \leq (|C_i|+1)/2$ for $p+1 \leq i \leq p+q$.*

Claim 3.2 *H is connected.*

(*Proof.*) Let H_0 be a connected component of H, $x \in V(H_0)$ and $y \in V(H - H_0)$. Then,

$$n + q \leq d_G(x) + d_G(y)$$

$$\leq \ |H_0| - 1 + \sum_{i=1}^{k} d_{C_i}(x) + |H - H_0| - 1 + \sum_{i=1}^{k} d_{C_i}(y)$$
$$\leq \ |H| - 2 + \sum_{i=1}^{k} |C_i| + q = n + q - 2$$

by Claim 3.1. But this is a contradiction. □

Claim 3.3 *Suppose $b_1, b_2 \in N(H) \cap V(C_i)$, $b_1 \neq b_2$, and $v_i \notin V(C_i(b_1, b_2))$ if $1 \leq i \leq p$ and $e_i \notin E(C_i[b_1, b_2])$ if $p+1 \leq i \leq p+q$. Then $V(C_i(b_1, b_2)) \neq \phi$.*

(*Proof.*) Take $a_1, a_2 \in V(H)$ such that $a_1b_1, a_2b_2 \in E(G)$ (possibly $a_1 = a_2$) and suppose $b_2 = b_1^+$. Then we can get an admissible cycle $b_1a_1Pa_2b_2C_i(b_2, b_1)b_1$ which is longer than C_i, where P is a path in H connecting a_1 and a_2. This contradicts the maximality of L. □

Claim 3.4 $|N(H) \cap V(C_i)| \leq 1$ *for* $1 \leq i \leq k$.

(*Proof.*) Suppose $|N(H) \cap V(C_i)| \geq 2$ for some i, $1 \leq i \leq k$. Choose two vertices $b_1, b_2 \in V(C_i)$ and vertices $a_1, a_2 \in V(H)$ (possibly $a_1 = a_2$) such that $a_jb_j \in E(G)$ for $j = 1, 2$, $v_i \notin V(C_i(b_1, b_2))$ if $1 \leq i \leq p$, $e_i \notin E(C_i[b_1, b_2])$ if $p + 1 \leq i \leq p + q$ and $N(H) \cap V(C_i(b_1, b_2)) = \phi$. Take $x \in V(H)$ and $y \in V(C_i(b_1, b_2))$. Then,

$$\begin{aligned} n + q &\leq d_G(x) + d_G(y) \\ &\leq |H| - 1 + \sum_{h=1}^{p} \frac{|C_h|}{2} + \sum_{h=p+1}^{p+q} \frac{|C_h| + 1}{2} \\ &\quad + \sum_{h=p+q+1}^{k} \frac{|C_h|}{2} - \frac{|C_i(b_1, b_2)|}{2} + \frac{1}{2} + d_G(y) \\ &\leq |H| - \frac{1}{2} + \frac{|L|}{2} + \frac{q}{2} - \frac{|C_i(b_1, b_2)|}{2} + d_G(y). \end{aligned}$$

Hence

$$d_G(y) = d_L(y) \geq \frac{|L| + q + |C_i(b_1, b_2)| + 1}{2}. \tag{2}$$

Let $L' = \langle V(C_i[b_2, b_1]) \cup (\bigcup_{h=1}^{k} V(C_h) - V(C_i)) \rangle$. Then by (2),

$$d_G(y) \ \geq \ \frac{|L| + q + |C_i(b_1, b_2)| + 1}{2}$$

$$= \frac{|L'| + |C_i(b_1, b_2)| + q + |C_i(b_1, b_2)| + 1}{2}$$
$$= \frac{|L'| + q}{2} + (|C_i(b_1, b_2)| - 1) + \frac{3}{2}.$$

Hence by Lemma 5, $V(C_i(b_1, b_2))$ can be inserted into L'. By adding $b_1 a_1 P a_2 b_2$ where P is a path in H connecting a_1 and a_2, we get a larger admissible k-cycle-packing. This is a contradiction. □

Claim 3.5 $|N(H) \cap V(C_i)| = \phi$ *for* $p + q + 1 \leq i \leq k$.

(*Proof.*) Suppose $|N(H) \cap V(C_i)| \neq \phi$ for some i, $p+q+1 \leq i \leq k$. Without loss of generality, we may assume that $i = k$. Take $y \in N(H) \cap V(C_k)$.

Subclaim 3.5.1 $|N(H) \cap V(C_i)| \neq \phi$ *and* $d_{C_i}(y^+) + d_{C_i}(y^-) \geq 2|C_i| - 1$ *for some* i, $1 \leq i \leq p$ *or* $p + q + 1 \leq i \leq k - 1$.

(*Proof.*) Suppose the subclaim does not hold. Let $r = |\{h | N(H) \cap V(C_h) \neq \phi, 1 \leq h \leq p, p + q + 1 \leq h \leq k\}|$, $r' = |\{h | N(H) \cap V(C_h) \neq \phi, p + 1 \leq h \leq p + q\}|$. Then

$$d_L(y^+) + d_L(y^-) \leq \sum_{h=1}^{k} 2|C_h| - 2r = 2|L| - 2r.$$

Without loss of generality, we may assume that $d_L(y^+) = d_G(y^+) \leq |L| - r$. Take any $x \in V(H)$, then

$$\begin{aligned} n + q &\leq d_G(x) + d_G(y^+) \leq |H| - 1 + r + r' + |L| - r \\ &= n + r' - 1. \end{aligned}$$

Hence we get $q \leq r' - 1$, but this is a contradiction. □

We may assume that $N(H) \cap V(C_i) \neq \phi$ and $d_{C_i}(y^+) + d_{C_i}(y^-) \geq 2|C_i| - 1$ for some i, $1 \leq i \leq p$ or $p + q + 1 \leq i \leq k - 1$. Take $z \in N(H) \cap V(C_i)$. By symmetry, we may assume that $y^+ z^-, y^+ z^+, y^- z \in E(G)$. Let $y a_1, z a_2 \in E(G)$, $a_1, a_2 \in V(H)$ (possibly $a_1 = a_2$). We replace C_i to $C_i' = y^+ z^+ C_i(z^+, z^-) z^- y^+$ and, let $P = y y^- z$, $L' = \langle (\bigcup_{h=1}^{k} V(C_h) - V(C_i \cup C_k)) \cup V(C_i' \cup P) \rangle$ and $M = V(C_k) - \{y, y^+, y^-\}$. For any $x \in M$, since $d_G(a_1) \leq |H| - 1 + k$ and $x a_1 \notin E(G)$,

$$\begin{aligned} d_G(x) &\geq n + q - (|H| - 1 + k) = |L| + q - k + 1 \\ &= |L'| + |M| + q - k + 1 \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{|L'|+q}{2} + (|M|-1) + \frac{3(k-1)}{2} + \frac{q}{2} - k + 2 \\
&= \frac{|L'|+q}{2} + (|M|-1) + \frac{k+q+3}{2} \\
&> \frac{|L'|+q}{2} + (|M|-1) + \frac{3}{2}.
\end{aligned}
$$

Then by Lemma 5, M can be inserted into L'. By adding $za_2P'a_1y$ where P' is a path in H connecting a_1 and a_2, we get a larger admissible k-cycle-packing. □

Let $N(H) \cap V(C_h) = \{u_h\}$ for $1 \leq h \leq r_1$ and $p+1 \leq h \leq r_2$ and $N(H) \cap V(C_h) = \phi$ for $r_1+1 \leq h \leq p$ and $r_2+1 \leq h \leq p+q$. Since $\sigma_2(G) \geq n+q$, G is $(q+2)$-connected. Hence $r_1 \geq 2$. Let also $|N(u_h) \cap V(H)| \geq 2$ for $1 \leq h \leq s_1$, $|N(u_h) \cap V(H)| = 1$ for $s_1+1 \leq h \leq r_1$ and $r = r_1 + r_2 - p$. Let $U_1 = \{u_1, \ldots u_{s_1}\}$ and $U = \{u_1, \ldots, u_{r_1}, u_{p+1}, \ldots, u_{r_2}\}$. If r_2 does not exist, let $r = r_1$ and $U = \{u_1, \ldots, u_{r_1}\}$.

Claim 3.6 *$u_i \neq v_i$ for $u_i \in U_1$.*

(*Proof.*) Suppose $u_i = v_i$ for some $i \in U_1$. Without loss of generality, we may assume that $i = 1$. Let $a_1, a_2 \in N(v_1) \cap V(H)$ and $L' = \langle \bigcup_{i=2}^{k} V(C_i) \rangle$. Since $d(x) \leq |H| - 1 + k$ and $xv \notin E(G)$ for any $x \in V(H)$ and $v \in V(C_1) - \{v_1\}$,

$$
\begin{aligned}
d_G(v) &\geq n + q - (|H| - 1 + k) \\
&= |L| + q - k + 1 \\
&= |L'| + |C_1| + q - k + 1 \\
&\geq \frac{|L'|+q}{2} + \frac{3(k-1)}{2} + \frac{q}{2} + (|C_1|-1) - k + 2 \\
&= \frac{|L'|+q}{2} + (|C_1|-1) + \frac{k}{2} + \frac{q}{2} + \frac{1}{2} \\
&\geq \frac{|L'|+q}{2} + (|C_1|-1) + \frac{3}{2}
\end{aligned}
$$

Since $N(v) \subseteq V(L)$, $V(C_1) - \{v_1\}$ can be inserted into L' by Lemma 5. Let $C_1' = v_1a_1Pa_2v_1$, where P is a path in H connecting a_1 and a_2. Then we get a larger admissible k-cycle-packing. □

Claim 3.7 *For $v \in V(H)$, $|N(v) \cap L| \geq q+2$.*

(*Proof.*) Take $v \in V(H)$ and $y \in V(C_i) - \{u_i\}$ for $1 \le i \le r_1$. Then $vy \notin E(G)$, and

$$\begin{aligned} n+q &\le d_G(v)+d_G(y) \le |H|-1+|N(v)\cap L|+|L|-1 \\ &= n-2+|N(v)\cap L|. \end{aligned}$$

Therefore, $|N(v)\cap L| \ge q+2$. □

Claim 3.8 $s_1 \ge 2$.

(*Proof.*) Suppose $s_1 \le 1$. Then $|H| \le r-(q+1) \le r_1 - 1$ by Claim 3.7. Note that $|H|(p+q+1-(|H|-1)) \le |E(H,L)| \le s_1|H|+(r_1-s_1)+q|H|$. (This inequality will be used several times.) Then $|H|(p+q+2-|H|) \le s_1(|H|-1)+r_1+q|H| \le |H|-1+p+q|H|$ and $(p+q)|H|+2|H|-|H|^2 \le |H|-1+p+q|H|$. Hence $|H|^2-|H|-1 \ge p(|H|-1) \ge r_1(|H|-1) \ge (|H|+1)(|H|-1) = |H|^2-1$. This is impossible. □

Claim 3.9 $|H| > r_1 - s_1$.

(*Proof.*) Suppose $|H| \le r_1 - s_1 \le p - s_1$. Then, $|H|(p+q+2-|H|) \le s_1(|H|-1)+r_1+q|H| \le (p-|H|)(|H|-1)+p+q|H|$. This shows $2|H| \le |H|$, but this is a contradiction. □

Claim 3.10 $d_G(y) = d_L(y) \ge |L| - s_1 + 1$ *for any* $y \in V(L-U)$.

(*Proof.*) For any $x \in V(H)$, $xy \notin E(G)$. Since

$$\sum_{x\in V(H)} d_G(x) \le |H|(|H|-1)+s_1|H|+r_1-s_1+q|H|,$$

we get

$$\begin{aligned} d_G(y) &\ge n+q-(|H|-1)-s_1-q-\frac{r_1-s_1}{|H|} \\ &> |L|-s_1 \end{aligned}$$

by Claim 3.9. Hence the claim holds. □

Claim 3.11 $N(v_1)\cap(U_1-\{u_1\}) \ne \phi$.

(*Proof.*) If $N(v_1)\cap(U_1-\{u_1\}) = \phi$, $d_G(v_1) \le |L|-1-(s_1-1) = |L|-s_1$. On the other hand, $d_G(v_1) \ge |L|-s_1+1$ by Claim 3.10. This is a contradiction. □

Without loss of generality, we may assume that $u_2 \in N(v_1) \cap (U_1 - \{u_1\})$. Give orientations to C_1 and C_2 such that $C_1(v_1, u_1) \neq \phi$ and $C_2(v_2, u_2) \neq \phi$, and take $z = u_1^- \in C_1(v_1, u_1)$ and $y = v_2^+ \in C_2[u_2^+, u_2^-]$. Here and in the following, $C_j[v_j^+, u_j^-]$ will be used as the abbreviation for $V(C_j[v_j^+, u_j^-])$.

Claim 3.12 *There exist no disjoint subgraphs $C_1', C_2', \ldots, C_k'$ in L satisfying C_1' is a path connecting u_1 and u_2, $C_2', \ldots, C_k'$ are cycles, $v_i \in V(C_i)$ for $1 \leq i \leq p$, $e_i \in E(C_i)$ for $p+1 \leq i \leq p+q$ and $|\bigcup_{i=1}^{p+q} V(C_i') \cap U| \geq r-1$.*

(*Proof.*) Let $L' = \langle \bigcup_{i=1}^{k} V(C_i') \rangle$ and $M = V(L) - \bigcup_{i=1}^{k} V(C_i') - U$. For any $x \in M$, $d_G(x) = d_L(x)$ and by Claim 3.10,

$$\begin{aligned} d_L(x) &\geq |L| - s_1 + 1 \geq |L| + q - k + 1 \\ &\geq |L'| + |M| + q - k + 1 \\ &\geq \frac{|L'| + q}{2} + (|M| - 1) + \frac{3(k-1)+2}{2} + \frac{q}{2} - k + 2 \\ &= \frac{|L'| + q}{2} + (|M| - 1) + \frac{k+q+3}{2} \\ &> \frac{|L'| + q}{2} + (|M| - 1) + \frac{3}{2}. \end{aligned}$$

Then by Lemma 5, M can be inserted into L'. Choose any $y \in N_H(u_1)$. Then there exists $y' \in N_H(u_2) - \{y\}$. By adding a path connecting y and y' in H, we get a larger admissible k-cycle-packing. This contradicts the minimality of $|L|$. (We may miss one vertex in U, but they contain two vertices in H.) □

Claim 3.13 $d_{C_1}(z) + d_{C_1}(y) + d_{C_1}(v_2) \leq 2|C_1| + 1$.

(*Proof.*) $N(y) \cap N(v_2) \cap (V(C_1) - \{u_1, v_1\}) = \phi$ (otherwise, we get a disjoint path P connecting u_1 and u_2 through v_1 and a cycle C_2' through v_2 in $\langle V(C_1) \cup V(C_2) \rangle$, contradicting Claim 3.12). Then $d_{C_1}(z) + d_{C_1}(y) + d_{C_1}(v_2) \leq |C_1| - 1 + |C_1| + 2 \leq 2|C_1| + 1$. □

Claim 3.14 $d_{C_2}(z) + d_{C_2}(y) + d_{C_2}(v_2) \leq 2|C_2| + 1$.

(*Proof.*) We may assume that $N(y) \cap C_2(u_2, v_2) = \phi$ and $N(v_2) \cap (C_2(y, v_2^-) - \{u_2\}) = \phi$, since otherwise we get a disjoint u_1–u_2 path C_1' passing through v_1 and a cycle C_2' passing through v_2 in $\langle V(C_1) \cup V(C_2) \rangle$, contradicting Claim 3.12. Therefore, $N_{C_2}(y) \subseteq C_2[v_2, u_2] - \{y\}$ and $N_{C_2}(v_2) \subseteq \{u_2, y, v_2^-\}$. If $N_{C_2}(z) \cap C_2(u_2, v_2] \neq \phi$ and $N_{C_2}(z) \cap C_2(v_2, u_2) \neq \phi$, we get a disjoint u_1–u_2 path C_1' passing through v_1 and a cycle C_2' passing through v_2. Then $N_{C_2}(z) \subseteq \{u_2, v_2\}$ or $C_2[u_2, v_2)$ or $C_2(v_2, u_2]$. Hence

$$\begin{aligned} d_{C_2}(z) + d_{C_2}(y) + d_{C_2}(v_2) &\leq |C_2| - 1 + |C_2| - 1 + 3 \\ &= 2|C_2| + 1. \end{aligned}$$

□

Claim 3.15 $d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq 2|C_i| + 2$ *for* $3 \leq i \leq p + q$.

(*Proof.*) Suppose $d_{C_i}(z)+d_{C_i}(y)+d_{C_i}(v_2) > 2|C_i|+2$ for some i, $3 \leq i \leq p+q$. Then $d_{C_i}(z) \geq 3$. Take $w_1, w_2 \in N_{C_i}(z)$ such that $C_i(w_1, w_2) \cap N(z) = \phi$ and $v_i \in C_i[w_1, w_2)$ if $3 \leq i \leq p$ and $e_i \in E(C_i[w_1, w_2])$ if $p+1 \leq i \leq p+q$. Then $N(v_2) \cap N(y) \cap C_i(w_2, w_1) = \phi$ and

$$\begin{aligned} d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) &\leq |C_i[w_2, w_1]| + |C_i(w_2, w_1)| + 2|C_i[w_1, w_2]| \\ &= 2|C_i| + 2. \end{aligned}$$

This is a contradiction. □

Claim 3.16 $d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq 2|C_i| + 1$ *for* $p + q + 1 \leq i \leq k$.

(*Proof.*) If $d_{C_i}(z) \leq 1$, the claim holds. Suppose $d_{C_i}(z) = t \geq 2$ and let $w_1, w_2, \ldots, w_t \in N_{C_i}(z) = W$. If $t \geq 3$, only v_2 or y can have neighbors on $C_i(w_j, w_l)$ for $1 \leq j \neq l \leq t$ by Claim 3.12. Furthermore, $N_W(v_2) \cap N_W(y) = \phi$. Then,

$$d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq 2|C_i|.$$

If $t = 2$, at least one of $N(y) \cap C_i(w_1, w_2)$ and $N(v_2) \cap C_i(w_1, w_2)$ is empty, and also at least one of $N(y) \cap C_i(w_2, w_1)$ and $N(v_2) \cap C_i(w_2, w_1)$ is empty. Hence

$$d_{C_i}(z) + d_{C_i}(y) + d_{C_i}(v_2) \leq |C_i| + 4 \leq 2|C_i| + 1.$$

□

Claim 3.17 $L - U$ *is not complete.*

(*Proof.*) $z \notin N(y) \cap N(v_2)$. □

Claim 3.18 $|L| \geq (n + q + 4)/2$.

(*Proof.*) By Claim 3.17, $2(|L|-2) \geq \sigma_2(G) = n+q$. Hence $|L| \geq (n+q+4)/2$. □

By Claim 3.10,

$$d_G(z) + d_G(y) + d_G(v_2) \geq 3|L| - 3s_1 + 3. \tag{3}$$

On the other hand, by Claims 3.13, 14, 15 and 16,

$$\begin{aligned} & d_G(z) + d_G(y) + d_G(v_2) \\ \leq\ & \sum_{i=1}^{2}(2|C_i|+1) + \sum_{i=3}^{p+q}(2|C_i|+2) + \sum_{i=p+q+1}^{k}(2|C_i|+1) \\ =\ & 2|L| + 2 + 2(p+q-2) + (k-p-q) \\ =\ & 2|L| + k + p + q - 2. \end{aligned} \tag{4}$$

By (3) and (4),

$$|L| \leq k + p + q + 3s_1 - 5.$$

By Claim 3.18,

$$(n+q+4)/2 \leq k + p + q + 3s_1 - 5.$$

Then,

$$\begin{aligned} n &\leq 2k + 2p + q + 6s_1 - 14 \\ &\leq 2k + 8p + q - 14 \\ &\leq 10k - 14. \end{aligned}$$

But this is a contradiction. This completes the proof of Theorem 6.

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