A sufficient condition for Set Reconstruction*

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Abstract

A graph is called set ***reconstructible*** *if it is determined uniquely (up to isomorphism) by the set of its vertex-deleted subgraphs. We prove that all graphs are set reconstructible if all 2-connected graphs G with diam(G) = 2 and all 2-connected graphs G with diam(G) = diam($\overline{G}$) = 3 are set reconstructible.*

1. Introduction

All graphs in this paper are finite, simple and undirected. We use the terminology in Harary [1]. The degree of a vertex v of a graph G is denoted by **deg v** (or **$\deg_G$ v**). A vertex v with deg v = m is referred to as an **m-vertex.** A 1-vertex is called an **endvertex.** A vertex-deleted unlabeled subgraph G–v of a graph G is called a **card** of G. A graph H is called a **set reconstruction** of G if H has the same set of (non isomorphic) cards as G. A graph is said to be **set reconstructible (set-rec)** if it is isomorphic to all its set reconstructions. Equivalently, a graph is **set-rec** if it is determined uniquely up to isomorphism from the set of its cards. A family of graphs is called **set recognizable** if for each graph G in that family, all set reconstructions of G are again in that family. If a parameter Q of a graph G is uniquely determined by the set of cards of G then Q is called a **set reconstructible parameter.**

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In this paper, we study the following conjecture.

Set Reconstruction Conjecture (SRC) [4]. All graphs with at least four vertices are set reconstructible.

Many parameters and several classes of graphs are already proved to be set-rec [1, 2, 6, 7]. It is known [8] that graphs with less than 12 vertices are set-rec. So we consider the set reconstructibility of graphs with at least 12 vertices.

It has been shown [9] that SRC is true if all 2-connected graphs are set-rec. Here we prove a stronger result that SRC is true if all 2-connected graphs G with diam(G) = 2 and all 2-connected graphs G with diam(G) = diam($\overline{G}$) = 3 are set-rec.

2. Set Reconstruction of P-graphs.

In the proof of our main theorem, the set reconstructibility of a special type of graph called P-graph is used as a tool. In this section we discuss it.

Definition [10]. A graph G with p vertices is called a **P-graph** if

(i) there exist only two blocks in G and one of them has just two vertices (denote the endvertex by x and its base by r) and

(ii) there exists a vertex $u \neq r$ with deg $u = p-2$.

Notation. For a P-graph G, the letters u, r and x are always used in the sense of the above definition, the letter T denotes the set of neighbours of the 2-vertices of G other than u and r and the letter G' denotes the subgraph $G-x-ur$.

Note that in a P-graph, u is adjacent to all vertices other than x and no two 2-vertices are adjacent (as the P-graph has only two blocks)

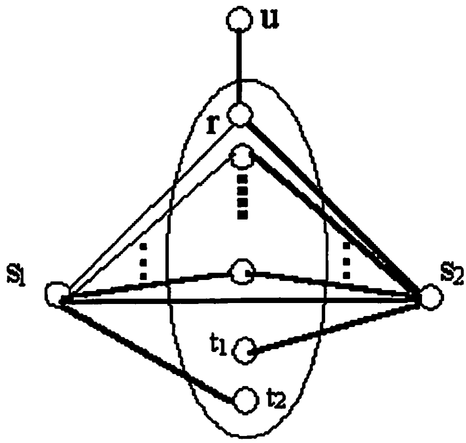
We first prove a lemma about diameters of G' and its complement. This lemma will be useful while proving the set reconstructibility of P-graphs.

Lemma 1. Let G be a P-graph with at least two 2-vertices. If r is not adjacent to 2-vertices in G and $|T| \geq 2$, then diam(G') = 2 or 3 and diam($\overline{G'}$) = 3.

Proof. Let $t_1, t_2 \in T$ and s_1 and s_2 be the 2-vertices adjacent with t_1 and t_2 respectively in G (and hence in G').

Clearly G' is connected and the vertex u is adjacent to all the vertices of G' other than r and hence $d(v, w) \leq 3$ for all $v, w \in V(G')$. Since G' is not complete, diam(G') = 2 or 3.

Consider $\overline{G'}$. Now s_1 is adjacent to all the vertices other than t_1 and u. (Figure 1). Similarly, s_2 is adjacent to all the vertices other than t_2 and u. Hence $d(v, w) \leq 3$ for all $v, w \in V(\overline{G'})-\{u\}$. Clearly u is the only endvertex and is adjacent with r (and hence $\overline{G'}$ is connected). This r is adjacent to s_1 and s_2. Hence $d(u, v) \leq 3$ for all $v \in V(\overline{G'})$. Since r is not adjacent to at least one vertex, say r' in $\overline{G'}$, $d(u, r')=3$ and hence $diam(\overline{G'})=3$. □



$\overline{G'}$

Figure 1

The following three results for P-graphs are proved in [9].

Lemma 2. A P-graph G is set-rec if

(i) G has a 2-vertex adjacent with r

or

(ii) G has at least two 2-vertices and there is a $t \in T$ adjacent with all the 2-vertices of G.

Theorem 1. A P-graph G is set-rec if

(i) G has at most one 2-vertex

or

(ii) G has at least two 2-vertices and there is a $t \in T$ with $\deg t \geq p-3$.

Theorem 2. A P-graph G with at least two 2-vertices and $\deg t \leq p-4$ for all $t \in T$ is set-rec if all 2-connected graphs are set-rec.

By Lemma 2, it is enough to prove Theorem 2 by adding the additional conditions that 'G has no 2-vertex adjacent with r' and 'no $t \in T$ is adjacent with all the 2-vertices of G' (and hence $|T| \geq 2$). Now for such P-graphs G, diam(G′) = 2 or 3 and diam($\overline{G'}$) = 3 by Lemma 1.

However, in the proof of Theorem 2 in [9], the hypothesis "set reconstructibility of 2-connected graphs" is used only as the set reconstructibility of the graph G′. Hence in Theorem 2, instead of the assumption "all 2-connected graphs are set-rec", the weaker assumption "all 2-connected graphs H with diam(H) = 2 or 3 and diam($\overline{H}$) = 3 are set-rec" is enough. Consequently we have the following theorem.

Theorem 3. A P-graph G with at least two 2-vertices and deg $t \leq p-4$ for all $t \in T$ is set-rec if all 2-connected graphs H with diam(H) = 2 or 3 and diam($\overline{H}$) = 3 are set-rec.

Theorem 4. P-graphs are set set-rec if all 2-connected graphs F with diam(F) = 2 or 3 and diam($\overline{F}$) = 3 are set-rec.
Proof. Follows by Theorems 1 and 3. □

3. Main Result.

The following five results are proved in Manvel [7].

Theorem 5. G is set-rec iff $\overline{G}$ is set-rec.
Theorem 6. The minimum degree, the maximum degree and the number of edges of a graph are set-rec.
Theorem 7. The degree sequence of a graph G with minimum degree at most 3 is set-rec.
Theorem 8. Disconnected graphs are set-rec.
Theorem 9. Separable graphs without endvertices are set-rec.
The next result is well known.
Theorem 10. If diam(G) > 3, then diam($\overline{G}$) < 3.
Corollary 1. All graphs are set-rec if all graphs G with diam(G) ≤ 3 are set-rec. (Follows by Theorems 5, 8 and 10)

Lemma 3. If G is a graph on p vertices having a (p−1)-vertex, then G is set-rec.
Proof. The graph under consideration is set recognizable by Theorem 6.

Now G can be obtained uniquely from a (p−1)-vertex deleted card (which is identifiable in the set of cards of G by Theorem 6) by augmentations. □

Lemma 4. If G is separable with diam(G) = 2, then G is set-reconstructible.

Proof. All the blocks of G are endblocks and the only cutvertex of G is adjacent to all other vertices of G and hence G is set-rec by Lemma 3. □

We now prove our main theorem.

Theorem 11. All graphs are set reconstructible iff all 2-connected graphs E with diam(E) = 2 and all 2-connected graphs F with diam(F) = diam($\overline{F}$) = 3 are set reconstructible.

Proof. Let all 2-connected graphs E with diam(E) = 2 and all 2-connected graphs F with diam(F) = diam($\overline{F}$) = 3 be set-rec. --- (1)

Let G be any graph. To show that G is set-rec.

By Corollary 1, we can take that diam(G) $\leq$ 3.

If diam(G) = 1, then G is complete and hence G is set-rec. If diam(G) = 2, then G is set-rec by Lemma 4 (when G is separable) or by **(1)** (when G is 2-connected). --- **(2)**

Now let **diam(G) = 3.** --- **(3)**

If diam($\overline{G}$) > 3, then by Theorem 9 applied to $\overline{G}$, diam(G) < 3 and hence G is set-rec as in **(2)**.

If diam($\overline{G}$) < 3, then $\overline{G}$ is set-rec as in **(2)** and hence G is set-rec.

So, we have to consider the case **diam(** $\overline{G}$ **) = 3.** --- **(3.1)**

If G is 2-connected then G is set-rec by **(1)** since diam(G) = diam($\overline{G}$) = 3.
If G is separable without endvertices then G is set-rec by Theorem 9.
So we can take **G to be separable with endvertices.** --- **(3.11)**
Then the degree sequence of G is set-rec by Theorem 5.

We have two subcases.

Case 1. G has at least two (p–2)-vertices.

Let v and w be two (p–2)-vertices in G. Then v and w are adjacent, as otherwise G cannot have endvertices, contradicting (3.11). Hence taking v′ (respectively w′) as the unique vertex which is nonadjacent with v (respectively w), we have v′ ≠ w and w′≠ v. (See Figure 2)

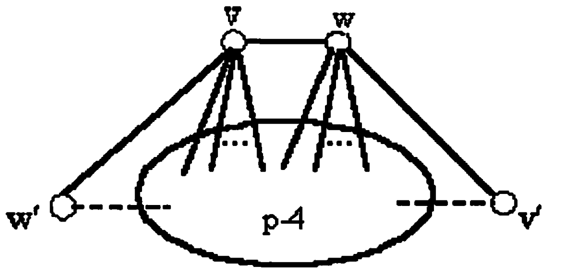


Figure 2. The graph G

Now the subgraph (G–v′)–w′ is 2-connected as $K_{2,\,p-4}$ is a spanning subgraph of (G–v′)–w′. If v′ = w′, then the two (p–2)-vertices v and w are adjacent to all other vertices except v′(= w′) and hence G-v′ is a block so that G is a P-graph (as v′ must be an endvertex by (3.11)). If at least one among v′ and w′ (say v′) is not an endvertex, then the other (namely w′) must be an endvertex by (3.11) and hence G is a P-graph (since (G–w′)–v′ is a block, w′ is an endvertex not adjacent with v′ and v′ is not an endvertex, G–w′ is a block). Thus G is a P-graph except when 'v′ ≠ w′ and both v′ and w′ are endvertices" and hence G is set-rec by Theorem 4. But in the exempted case, G is set recognizable from its degree sequence alone and it must be the graph obtained from a card of G with exactly one endvertex by adding a vertex and joining it to a (p–3)-vertex of the card.

Case 2. G has at most one (p–2)-vertex.

If G has no (p–2)-vertex, then $\overline{G}$ has no endvertex and hence it is either 2-connected or separable without endvertices. Hence $\overline{G}$ is set-rec by **(1)** or by Theorem 9, and so is G.

Now let G have exactly one (p–2)-vertex, say w. - - - **(4)**

Then $\overline{G}$ is a separable graph with an endvertex. If $\overline{G}$ has at least two (p–2)-vertices, then $\overline{G}$ is set-rec by Case 1, and so is G. Now let $\overline{G}$ have at most one (p–2)-vertex.

Therefore G has at most one endvertex.

So, by (3.11), G has exactly one endvertex, say y. - - - **(4.1)**

Case 2.1. wy is not an edge in G.

Now q (the neighbour of y) is a cut-vertex of G. If q is the only cutvertex then G is a P-graph and hence G is set-rec by Theorem 4. If G has one more cutvertex, then it must be w, and hence G is the union of three subgraphs

B_{wq} (the nonendblock containing w and q), F_w (the union of endblocks containing w) and the endblock B_y ($\cong K_2$) containing y.

If deg q = p−3 then $F_w \cong K_3$ (because G has only one endvertex). Consider a 2-vertex deleted card G−z with exactly two endvertices (the deleted 2-vertex cannot be from B_{wq} as every 2-vertex in B_{wq} is adjacent to w and q so that no additional endvertex is created). Such a G−z has an automorphism that interchanges the two endvertices, interchanges the two bases and fixes all other vertices. Hence all augmentations of G−z by introducing a 2-vertex so that the resulting graph has only one endvertex and only one endblock isomorphic to K_3 are isomorphic.

If deg q $\neq$ p−3 then deg q < p−3 (because $|F_w| \geq 3$). Now in the cards G−v that are connected and have at least one endvertex (cards for which deleted vertex is not one of w, y and q), the vertices w, y and q are identifiable as the only (p−3)-cutvertex, the only endvertex nonadjacent with w and the base of y respectively. From these cards G−v, if we choose one, say G_1 such that w and q are in the same block and the block containing w and q has maximum number of edges, then the nonendblock of G_1 is B_{wq}. Hence B_{wq} is known with w and q labeled. .... (5)

The only endvertex-deleted card in S is G−y and its only cutvertex is w. By (5), there is an isomorphism α from B_{wq} on to a block of G−y such that $\alpha(w)=w$. The graph G_α obtained from G−y by adding a vertex and joining it only with $\alpha(q)$ is a candidate for G. If β is another such isomorphism and G_β is the corresponding augmented graph, then $G_\alpha \cong G_\beta$ under the mapping ξ where

$\xi = \beta\alpha^{-1}$ on vertices of $\alpha(B_{wq})$
$= \alpha\beta^{-1}$ on vertices of $\beta(B_{wq})$
= identity on all other vertices
when $\alpha(B_{wq})$ and $\beta(B_{wq})$ are different blocks of G−y
and $\xi = \beta\alpha^{-1}$ on vertices of $\alpha(B_{wq})$
= identity on all other vertices
when $\alpha(B_{wq})$ and $\beta(B_{wq})$ are one and the same block of G−y.

Hence G is known up to isomorphism.

Case 2.2. wy is an edge in G.

Now wy is not an edge in $\overline{G}$. In $\overline{G}$, y and w are respectively the only endvertex and the only (p−2)-vertex and they are not adjacent. Hence by applying Case 2.1 to $\overline{G}$ instead of G, we get $\overline{G}$ is set-rec. □

4. Set Recognizability.

It is not known whether graphs G with diam(G)=2 are set recognizable or not, even though they are proved to be recognizable form the full collection of cards in [3] using a refinement of Kelly's Lemma. However, existence of an induced path of length k in a card of a graph does not imply that the diameter of the graph is at least k (Example: C_5). Graphs with diameter 1 are precisely complete graphs. Since complete graphs are set reconstructible, they are set recognizable. Hence, **if graphs with diameter 2 are set recognizable**, then graphs with diameter at least 3 are set recognizable. Also by Theorem 10, " $diam(G) \geq 3$ and $diam(\overline{G}) \geq 3$ " iff " $diam(G) = diam(\overline{G}) = 3$ ". Hence, **if graphs with diameter 2 are set recognizable**, then graphs G with $diam(G) = diam(\overline{G}) = 3$ are also set recognizable so that graphs covered in Theorem 11 are set recognizable.

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