On Large Harmonious Graphs

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Abstract. In this paper, we give one construction for constructing large harmonious graph from smaller ones. Subsequently, three families of graphs are introduced and some members of them are shown to be or not to be harmonious.

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph with p vertices and q edges. A vertex labeling of G, $g : V(G) \to \mathbf{Z}_q$, is called *harmonious* [9] if the induced edge labeling $g^* : E(G) \to \mathbf{Z}_q$ given by $g^*(uv) \equiv g(u) + g(v) \pmod{q}$ for every edge $uv \in E(G)$ is an injection when G is not a tree. In the case of trees, the vertices can be labeled from 0 to q with no vertex labels used twice and the resulting edge labels are distinct. A graph G with a harmonious labeling is called a *harmonious graph.*

Harmonious graphs naturally arose in the study by Graham and Sloane of modular versions of additive bases problems stemming from error-correcting codes.

Chang, Hsu, Rogers [3] and Grace [7] defined a stronger form of harmonious labeling. A vertex labeling g of G is called *r-sequential* (or *strongly r-harmonious* [3]) if the edge labels induced by $g^*(uv) = g(u) + g(v)$ for every edge $uv \in E(G)$ form a sequence of distinct consecutive integers r, $r+1$, $r+2$, ..., $r+q-1$. A graph G is *sequential* [7] if it has an r-sequential labeling for some positive integer r. Every sequential graph is harmonious by taking all edge labels modulo q. Results on sequential labelings can be found in [4].

In the second section of this paper, we discuss one method to cluster graphs together and obtain new harmonious graphs. Then, in the third section, we introduce three families of cycle-related graphs and show that some members of them are harmonious or sequential and some members of them are not.

2. Construction

Suppose that $G_1, G_2, \ldots, G_n$ and H are vertex-disjoint graphs. If $V(H) = \{w_1, w_2, \ldots, w_n\}$ and $v_i \in V(G_i)$, for $i = 1, 2, \ldots, n$. Attaching graph G_i to vertex w_i of H by identifying v_i with w_i, for $i = 1, 2, \ldots, n$, the resulting graph is denoted by $H[w_1, w_2, \ldots, w_n] \oplus [G_1, G_2, \ldots, G_n]$ at $[v_1, v_2, \ldots, v_n]$, as depicted in Fig. 1. In particular, if $G_i \cong G$ for each $i = 1, 2, \ldots, n$ and $v_1, v_2, \ldots, v_n$ are isomorphic to the same vertex v in $V(G)$, then the notation $H[w_1, w_2, \ldots, w_n] \oplus [G_1, G_2, \ldots, G_n]$ at $[v_1, v_2, \ldots, v_n]$ is shortened to $H \oplus [G]_v$.

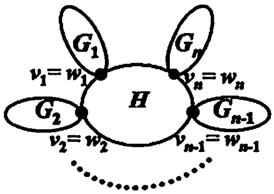


Fig. 1. $H[w_1, w_2, \ldots, w_n] \oplus [G_1, G_2, \ldots, G_n]$ at $[v_1, v_2, \ldots, v_n]$

Throughout this section, we let $\{u_0, u_1, \ldots, u_{p-1}\}$ be the vertex set of G. For convenience, the ith copy of G is called G_i and we assume that $\{u_{i,0}, u_{i,1}, \ldots, u_{i,p-1}\}$ is the vertex set of G_i where $u_{i,j}$ is the isomorphic image of u_j for $j = 0, 1, 2, \ldots, p-1$. Moreover, we also assume that h and g are the given harmonious or r-sequential labelings of graphs H and G respectively as the case may be.

Theorem 2.1. *Suppose that H is harmonious with $2n+1$ edges and $2n+1$ vertices, and G is a graph with q edges having an r-sequential labeling g. If either $g(u_0) = (r-1)/2$ or $g(u_0) = (r+q)/2$, then $H \oplus [G]_{u_0}$ is harmonious.*

Proof. Let $\{w_1, w_2, \ldots, w_{2n+1}\}$ be the vertex set of H with $h(w_i) = i-1$.

(1) If $g(u_0) = (r-1)/2$, then we identify w_i with the isomorphic image $u_{i,0}$ of u_0, $i = 1, 2, \ldots, 2n+1$. In this case, let the vertex labeling f of $H \oplus [G]_{u_0}$ be defined by

$$f(u_{i,j}) = g(u_j) + (i-1)(q+1),\ i = 1, 2, \ldots, 2n+1, \text{ and } j = 0, 1, \ldots, p-1.$$

Note that we have labeled the vertices of H because w_i and $u_{i,0}$ are identified. We shall show that all edge labels are distinct. For any number $s \in \{r, r+1, \ldots, r+q-1\} = g^*(E(G))$, there is exactly one edge e_{j_s} of G such that $g^*(e_{j_s}) = s$. Let e_{i,j_s} be the isomorphic image of e_{j_s} in G_i, $i = 1, 2, \ldots, 2n+1$. Then

$$\{f^*(e_{i,j_s}) \mid i = 1, 2, \ldots, 2n+1\} = \{s, s+2(q+1), s+4(q+1), \ldots, s+4n(q+1)\}$$

$\equiv \{ s,\ s+(q+1),\ s+2(q+1), \ldots,\ s+2n(q+1) \} \quad (\text{mod}\ (2n+1)(q+1))$.

So the set of edge labels of all the $e_{i,js}$'s induced by f is

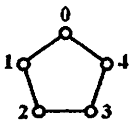
$$\{ f^*(e_{i,js}) \mid i = 1, 2, \ldots, 2n+1 \} =$$

$$\begin{cases} \{ s, s+(q+1), s+2(q+1), \ldots, s+2n(q+1)\}, & \text{if } s < q+1; \text{ and} \\ \{ s-(q+1), s, s+(q+1), \ldots, s+(2n-1)(q+1)\}, & \text{if } s \geq q+1, \end{cases}$$

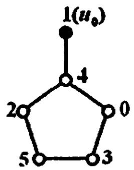
where $s \in \{ r, r+1, \ldots, r+q-1 \}$.

Next, let's consider the edge labels of H. Suppose that $\{e'_0, e'_1, \ldots, e'_{2n}\}$ is the edge set of H such that $h^*(e'_j) = j$ or $h^*(e'_j) \equiv j$ (mod $2n+1$). Then $f^*(e'_j) = (r-1) + h^*(e'_j)(q+1) \equiv (r-1) + j(q+1) \equiv (r-1) + j(q+1)$ (mod $(2n+1)(q+1)$) That is, $f^*(e'_j) = (r-1) + j(q+1)$, $j = 0, 1, 2, \ldots, 2n$. One can now easily observe that the edge labels are all distinct and f is therefore harmonious. (We present an example in Fig. 2 to demonstrate how the labeling works).

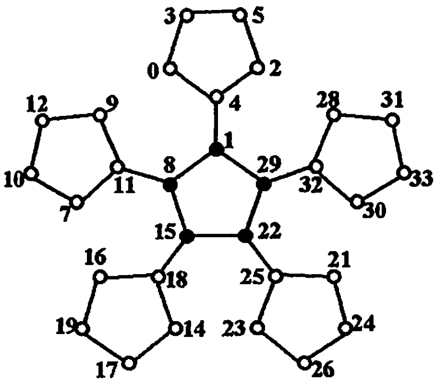
(2) When $g(u_0) = (r+q)/2$, the proof is analogous to (1) and is omitted. ▌



(1) $H = C_5$



(2) $G = C_5^{+1}$ is 3-sequential.



(3) $C_5 \oplus [C_5^{+1}]_{u_0}$ is harmonious.

Fig. 2.

The result holds when G_i's are not all isomorphic. In the following, let g_i be an r-sequential labeling of G_i, and $V(G_i) = \{u_{i,0}, u_{i,1}, ..., u_{i,p_i-1}\}$, for $i = 1, 2, ..., 2n+1$.

Corollary 2.2. *Suppose that H is harmonious with $2n+1$ edges and $2n+1$ vertices $\{w_1, w_2, ..., w_{2n+1}\}$, and G_i is a graph with q edges having an r-sequential labeling g_i, $i = 1, 2, ..., 2n+1$. If either $g_i(u_{i,0}) = (r-1)/2$ or $g_i(u_{i,0}) = (r+q)/2$ for all $i = 1, 2, ..., 2n+1$, then $H[w_1, w_2, ..., w_{2n+1}] \oplus [G_1, G_2, ..., G_{2n+1}]$ at $[u_{1,0}, u_{2,0}, ..., u_{2n+1,0}]$ is harmonious.*

Remark: Theorem 2.1 and Corollary 2.2 also hold when G_i's are trees since the vertices of trees are labeled from 0 to q with no repeated vertex labels.

Among known 1-sequential graphs are wheels W_m ($m \equiv 0$ or 1 (mod 3)) [3], friendship graphs $C_3^{(m)}$ ($m \equiv 0$ or 1 (mod 4)) [10,11], stars S_m, fans F_m [3], helms H_m [14], French windmills $K_4^{(t)}$ [10,11], $S_m + \overline{K_t}$ [6] and $K_{m,n} + K_1$ [3]. In addition, paths P_{2m+1} are both m and $(m+1)$-sequential. P_{4m+2} are $(2m+1)$- sequential [8]. Odd cycles C_{4m+3} are $(2m+1)$-sequential [16]. All these graphs satisfy the hypotheses for G in Theorem 2.1. Therefore the following corollary is straightforward.

Corollary 2.3. *Suppose that H is harmonious with $2n+1$ edges and $2n+1$ vertices. In the following, let u_j be the vertex labeled j of the graph in brackets.*

(1) *If $m \equiv 0$ or 1 (mod 3), then $H \oplus [W_m]_{u_0}$ is harmonious.*

(2) *If $m \equiv 0$ or 1 (mod 4), then $H \oplus [C_3^{(m)}]_{u_0}$ is harmonious.*

(3) *$H \oplus [S_m]_{u_0}$, $H \oplus [F_m]_{u_0}$, $H \oplus [H_m]_{u_0}$, $H \oplus [K_4^{(t)}]_{u_0}$, $H \oplus [S_m + \overline{K_t}]_{u_0}$, $H \oplus [K_{m,n} + K_1]_{u_0}$, $H \oplus [P_{2m+1}]_{u_{\lfloor m/2 \rfloor}}$, $H \oplus [P_{4m+2}]_{u_m}$ and $H \oplus [C_{4m+3}]_{u_m}$ are harmonious.*

If we restrict the case to harmonious cycles, that is, odd cycles. Theorem 2.1 guarantees the harmoniousness of $C_{2n+1} \oplus [C_{4m+3}]$. On the other hand, $C_{2n+1} \oplus [C_{4m+1}]$ is not harmonious because it violates the following theorem by Graham and Sloane. Hence, Theorem 2.5 is obvious.

Theorem 2.4. [9, Theorem 11] *If a harmonious graph has $2t$ edges and the degree of every vertex is divisible by 2^k, then t is divisible by 2^k.*

Theorem 2.5. *$C_{2n+1} \oplus [C_{2m+1}]$ is harmonious if and only if m is odd.*

The graph obtained by taking the union of graphs G_1, G_2, ... and G_n with disjoint vertex sets is the *disjoint union* of G_1, G_2, ..., and G_n, written $\bigcup_{i=1}^{n} G_i$.

In general, nG is the disjoint union of n pairwise disjoint copies of G.

By a method similar to Theorem 2.1, we may construct harmonious labelings of $\bigcup_{i=1}^{2n+1} G_i$ and $(2n+1)G$.

Theorem 2.6.

(1) *If G is harmonious, then* $(2n+1)G$ *is harmonious.*

(2) *If* $G_1, G_2, \ldots,$ *and* G_{2n+1} *are harmonious with* $g_1^*(E(G_1)) = g_2^*(E(G_2)) = \ldots = g_{2n+1}^*(E(G_{2n+1}))$, *then* $\bigcup_{i=1}^{2n+1} G_i$ *is harmonious.*

Proof. (1) Let us define the desired labeling f of $(2n+1)G$ by

$$f(u_{i,j}) = g(u_j) + (i-1)q,\ i = 1, 2, \ldots,\ 2n+1, \text{ and } j = 0, 1, \ldots, p-1.$$

(2) Similar to (1) and is omitted. ▌

Remark. Theorem 2.6(1) has been shown by M. Z. Youssef [18].

3. Some families of harmonious graphs

(I) The book with n m-polygonal pages:

In [13], M. Murugan and G. Arumugan called the graph nC_5 with an edge in common the *book with n pentagonal pages*. Generalizing this definition, we let $\Theta(C_m)^n$ denote the *book with n m-polygonal pages* which consists of n copies of an m-cycle that share a common edge.

Among this family of graphs, $\Theta(C_4)^n$ has been completely solved in [5], [6] and [7]. $\Theta(C_m)^2$ is in fact isomorphic to $C_{2(m-1)}$ with a chord "in the middle". S. D. Xu [17] has shown that every cycle with a chord is harmonious except for C_6 in the case when the distance in C_6 between the endpoints of the chord is 2. Whereas, we observed that, for some values of m, $\Theta(C_m)^2$ is sequential. In addition, a modification to the method used in Theorem 2.1 enables us to construct a harmonious labeling of $\Theta(C_{2m+1})^{2n+1}$ and some other results are also given in the following.

Theorem 3.1.

(1) $\Theta(C_{2m+1})^{2n+1}$ *is harmonious, for all* $n, m \geq 1$.

(2) $\Theta(C_{2m+1})^{2n}$ *is 2mn-sequential, for all* $n \geq 1$ and $1 \leq m \leq 4$.

(3) $\Theta(C_m)^2$ *is* $(m-2)$*-sequential, if* $m \geq 3$ *and* $m \equiv 2,3,4,7 \pmod 8$.

(4) $\Theta(C_{4m+2})^{4n+1}$ *and* $\Theta(C_{4m})^{4n+3}$ *are not harmonious, for all* $n, m \geq 1$.

Proof. In cases (1) and (2), let the vertices of the i-th copy of C_m run consecutively $u_0, u_{i,1}, \ldots, u_{i,m-2}, u_{m-1}$ with u_{m-1} joined to u_0, where u_0u_{m-1} is the common edge.

(1) The harmonious labeling of $\Theta(C_{2m+1})^{2n+1}$ is given as

$$f(u)=\begin{cases} j/2, & \text{if } u=u_j, \text{ and } j=0,\ 2m; \\ F(u_j)+2m(i-1)+1, & \text{if } u=u_{i,j}, j=2,4,\ldots,2m-2, \text{ and } i=n+2,n+3,\ldots,2n+1; \\ F(u_j)+2m(i-1), & \text{otherwise.} \end{cases}$$

where F is a sequential labeling of C_{2m+1} defined as follows

$$F(u_j)=\begin{cases} j/2, & \text{if } j=0,2,\ldots,2m; \text{ and} \\ m+(j+1)/2, & \text{if } j=1,3,\ldots,2m-1. \end{cases}$$

(An example is presented in Fig. 3(1).)

(2) Let the labeling f of $\Theta(C_{2m+1})^{2n}$ be defined by ($1 \le m \le 4$)

$$f(u)=\begin{cases} 2mn+in, & \text{if } u=u_i,\quad i=0,2m; \\ 2(i-1)+n(j-1), & \text{if } u=u_{i,j},\ j=1,3,\ldots,2m-1 \text{ and } i=1,2,\ldots,n; \\ 2i-1+n(j-3), & \text{if } u=u_{i,j},\ j=1,3,\ldots,2m-1 \text{ and } i=n+1,n+2,\ldots,2n; \\ 7n-i+1, & \text{if } u=u_{i,j},\ m=2, j=2 \text{ and } i=1,2,\ldots,n; \\ 7n-i, & \text{if } u=u_{i,j}, m=2, j=2 \text{ and } i=n+1,n+2,\ldots,2n; \\ (9-j/2)n-i+1, & \text{if } u=u_{i,j},\ m=3, j=2,4 \text{ and } i=1,2,\ldots,n; \\ f(u_{i-n},j)+4n-1, & \text{if } u=u_{i,j}, m=3, j=2,4 \text{ and } i=n+1,n+2,\ldots,2n; \\ n(j+3|j-4|)/4+8n-i+1, & \text{if } u=u_{i,j}, m=4, j=2,4,6 \text{ and } i=1,2,\ldots,n; \text{ and} \\ n(j-3|j-4|)/4+16n-i, & \text{if } u=u_{i,j}, m=4, j=2,4,6 \text{ and } i=n+1,n+2,\ldots,2n. \end{cases}$$

(See Fig. 3(2) for the labeling of $\Theta(C_7)^6$.)

(3) Let the vertices of C_m run consecutively $u_0, u_1, \ldots, u_{m-2}, u_{m-1}$ with u_{m-1} joined to u_0. For the sake of convenient notation, let $C_m(a,b)$ denote the graph obtained by adding the chord $u_a u_b$ to the cycle C_m. We split the proof into 3 cases:

(a) First, we give a labeling of C_{8t+4} ($t \ge 0$):

$$f(u_j)=\begin{cases} j/2, & \text{if } j=0,2,\ldots,4t; \\ 4t+2+(j-1)/2, & \text{if } j=1,3,\ldots,4t+1; \\ (j-1)/2, & \text{if } j=4t+3,4t+5,\ldots,8t+3; \text{ and} \\ 4t+2+j/2, & \text{if } j=4t+2,4t+4,\ldots,8t+2. \end{cases}$$

Then, $\Theta(C_{4t+3})^2 \cong C_{8t+4}(2t, 6t+2)$, $t \ge 0$, is ($4t+1$)-sequential. ($\Theta(C_7)^2 \cong C_{12}(2,8)$ is depicted in Fig. 3(3). Note that one may obtain another sequential labelings of $\Theta(C_7)^2$ and $\Theta(C_3)^2$ from (2).)

(b) The labeling of C_{16t+6} ($t \ge 0$) is defined as follows:

$$f(u_j)=\begin{cases} j/2, & \text{if } j=0,2,\dots,8t+2;\\ 8t+2+(j-1)/2, & \text{if } j=1,3,\dots,8t+1;\\ 12t+3, & \text{if } j=8t+4;\\ 8t+3+(j-1)/2-(-1)^{(j-1)/2}, & \text{if } j=8t+3,8t+5,\dots,16t+5; \text{ and}\\ j/2-1-(-1)^{j/2}, & \text{if } t\ge 1 \text{ and } j=8t+6,8t+8,\dots,16t+4. \end{cases}$$

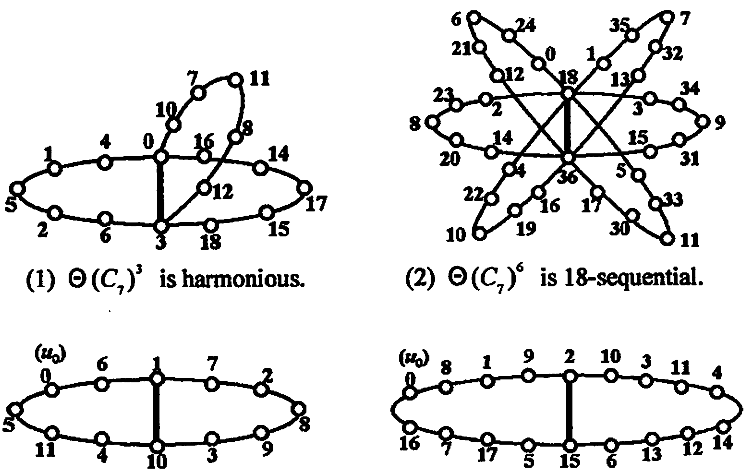
Then, $\Theta(C_{8t+4})^2 \cong C_{16t+6}(4t,12t+3)$, $t\ge 0$, is $(8t+2)$-sequential.

(c) The 8-sequential labeling of $\Theta(C_{10})^2$ is given in Fig. 3(4). When $t\ge 2$, the vertex labeling of C_{16t+2} is defined by

$$f(u_j)=\begin{cases} j/2, & \text{if } j=0,2,\dots,8t;\\ 8t+(j-1)/2, & \text{if } j=1,3,\dots,8t-1;\\ 28t-2j+4, & \text{if } j=8t+1,8t+2;\\ 12t+2+(-1)^{(j-1)/2}, & \text{if } j=8t+3,8t+5;\\ 4t+2, & \text{if } j=8t+4,\\ 4t+2+(-1)^{j/2}, & \text{if } j=8t+6,8t+8;\\ 8t+1+(j-1)/2-(-1)^{(j-1)/2}, & \text{if } j=8t+7,8t+9,\dots,16t+1; \text{ and}\\ j/2-1-(-1)^{j/2}, & \text{if } j=8t+10,8t+12,\dots,16t. \end{cases}$$

Then, $\Theta(C_{8t+2})^2 \cong C_{16t+2}(4t+1,12t+2)$, $t\ge 1$, is $8t$-sequential.

(4) A direct consequence of Theorem 2.4. ▌



(1) $\Theta(C_7)^3$ is harmonious.

(2) $\Theta(C_7)^6$ is 18-sequential.

(3) $\Theta(C_7)^2$ is 5-sequential.

(4) $\Theta(C_{10})^2$ is 8-sequential.

Fig. 3.

Koh et al. [12] introduced the notation $B(m,r,n)$ for the graph consisting of n copies of K_m with a K_r in common. As an easy consequence of Theorem

3.1, $B(3,2,n)$ which is isomorphic to $\Theta(C_3)^n$ is harmonious. We also observe some cases for which $B(m,r,n)$ is not harmonious. Before stating the results, let us introduce two theorems that are needed in the proof.

Theorem 3.2. [2, §18, Ex3] *A positive integer n is not a sum of three squares if and only if* $n = 4^e(8k+7)$ *where e and k are nonnegative integers.*

Theorem 3.3. [2, §18, Theorem 1] *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, *where* $p_1, p_2, \ldots, p_m$ *are distinct primes. Then n is not a sum of two squares if and only if there is a prime*

(*) $$p_i \equiv 3 \pmod 4 \text{ and } \alpha_i \text{ is odd.}$$

Now we are ready to give the results on $B(m,r,n)$:

Theorem 3.4.

(1) $B(m,2,3)$ *and* $B(m,3,3)$ *are not harmonious, if* $m \equiv 1 \pmod 8$.

(2) $B(m,4,2)$ *is not harmonious, if* $m+6, m-2$ *and m all satisfy condition* (*).

(3) $B(m,5,2)$ *is not harmonious, if* $m \pm 2$ *and* $m+10$ *all satisfy condition* (*).

(4) $B(m,1,n)$ *is not harmonious, if* $m \equiv 5 \pmod 8$ *and* $n \equiv 1,2,3 \pmod 4$.

(5) $B(2m+1,2m,2n+1) \cong K_{2m} + K_{2n+1}$ *is not harmonious, if* $m \equiv 2 \pmod 4$.

Proof. (1) Suppose that $B(m,2,3)$ is harmonious when $m \equiv 1 \pmod 8$. Let E_i (O_i) be the cardinality of the set of vertices of the ith copy of K_m labeled even (odd), then $E_i + O_i = m$, $i = 1, 2, 3$. Since $|B(m,2,3)|$ is even, the numbers of edges labeled odd and even are equal. That is, $E_1(E_1-1)/2 + O_1(O_1-1)/2 + E_2(E_2-1)/2 + O_2(O_2-1)/2 + E_3(E_3-1)/2 + O_3(O_3-1)/2 = E_1O_1 + E_2O_2 + E_3O_3 \pm 2$, or equivalently, $(E_1-O_1)^2 + (E_2-O_2)^2 + (E_3-O_3)^2 = 3m \pm 4$. On the other hand, $m \equiv 1 \pmod 8$ implies that $3m \pm 4$ is of the form $8k+7$ for some k. This gives a contradiction by Theorem 3.2.

Analogous reasoning leads to the fact that $B(m,3,3)$ is not harmonious when $m \equiv 1 \pmod 8$ and the proofs of cases (2) and (3).

Case (4) and (5) are direct consequences of Theorem 2.4. ∎

(II) Belts:

Next, we define a belt-like graph $b(n,m)$ and then show that it is sequential for even n and all m.

Let $b(n,0)$ be the path P_n whose vertices run consecutively $w_1, w_2, \ldots$, and w_n. For $k \geq 0$, the "*belt*" $b(n,k+1)$ is obtained by adjoining to the graph $b(n,k)$ with

$n-1$ new vertices $v_{k+1,1}$, $v_{k+1,2}$, ..., $v_{k+1,n-1}$ accompanied by edges $v_{k+1,i}\,w_i$ and $v_{k+1,i}\,w_{i+1}$, $i = 1, 2, \ldots, n-1$.

Theorem 3.5.
(1) $b(2n,m)$ *is n-sequential for all* $n \geq 1$ and $m \geq 0$.
(2) $b(4n+3,2m+1)$ *is not harmonious.*

Proof. (1) Let us construct a labeling f of $b(2n,m)$ as follows:

$$f(u) = \begin{cases} (j-1)/2, & \text{if } u = w_j \text{ and } j = 1,3,\ldots,2n-1, \\ n + j/2 - 1, & \text{if } u = w_j \text{ and } j = 2,4,\ldots,2n, \\ 2(2n-1)i - j + 1, & \text{if } u = v_{i,j} \text{ and } j = 1,3,\ldots,2n-1,\ 1 \leq i \leq m, \\ 2(2n-1)i - j + 2n, & \text{if } u = v_{i,j} \text{ and } j = 2,4,\ldots,2n-2,\ 1 \leq i \leq m. \end{cases}$$

A routine computation shows that f is an n-sequential labeling. (See Fig. 4)
(2) $b(4n+3,2m+1)$ is not harmonious because it violates Theorem 2.4. ▌

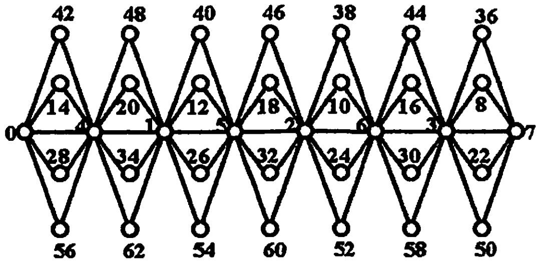


Fig. 4. $b(8,4)$ is 4- sequential.

By removing the even number parts of the belt $b(2n,m)$, we may obtain another belt-like graph which is also n-sequential. Since the pattern is pretty clear, we present an example in Fig. 5 instead of actually writing down the formula for the labeling.

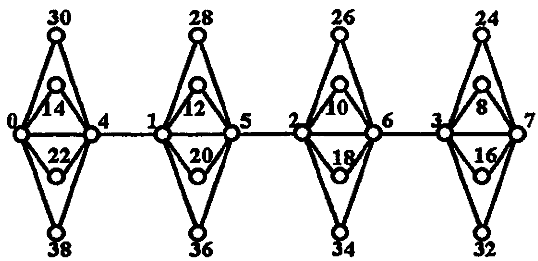


Fig. 5.

(III) Finally, we introduce the third family of graphs as the one obtained by identifying one vertex of an n-cycle C_n with the center of an (m,t) *-star tree* [1] (a star tree in which each of its m branches is a path of length t). Let the resulting graph be denoted by $C_n^{+(m,t)}$. In the following theorem, we show that some members of the family are not harmonious and we also give nice symmetrical harmonious labelings for some cases.

Theorem 3.6.

(1) $C_a^{+(1,t)}$ *is not harmonious when a+t is odd.*

(2) $C_a^{+(2m,t)}$ *is harmonious if one of the following conditions is satisfied.*

(i) $a = 3$,

(ii) $a = 2n+1$ *and* $t = n-1,\ n,\ n+1,$ *or* $2n-1$.

Proof. Suppose that the vertices of the cycle run consecutively $v_0, v_1,\ldots, v_{a-1}$ with v_{a-1} joined to v_0 and those of the ith path run consecutively $v_0, u_{i,1}, u_{i,2},\ldots, u_{i,t}$.

(1) Suppose, on the contrary, that $C_a^{+(1,t)}$ with its labeling f is harmonious where $a+t$ is odd. Since the vertices are labeled $0,1,\ldots,$ and $a+t-1$, we have

$$f(v_0)+f(v_1)+\cdots+f(v_{a-1})+f(u_{1,1})+f(u_{1,2})+\cdots+f(u_{1,t})$$
$$=0+1+2+\cdots+(a+t-1)\equiv 0 \pmod{a+t}.$$

Next, let's consider the edge labels of $C_a^{+(1,t)}$. We have

$$3f(v_0)+2[f(v_1)+f(v_2)+\cdots+f(v_{a-1})+f(u_{1,1})+f(u_{1,2})+\cdots+f(u_{1,t-1})]+f(u_{1,t})$$
$$\equiv 0+1+2+\cdots+(a+t-1) \pmod{a+t}$$
$$\equiv 0 \pmod{a+t}.$$

This implies that $f(v_0)-f(u_{1,t})\equiv 0 \pmod{a+t}$ which is a contradiction.

(2) (i) We construct a labeling f of $C_3^{+(2m,t)}$ as follows:

$$f(u)=\begin{cases} 0, & \text{if } u=v_0;\\ mt+i, & \text{if } u=v_i,\ i=1,2;\\ (t(i-1)+j)/2, & \text{if } u=u_{i,j},\ i=1,3,\ldots,2m-1, \text{ and } j=2,4,\ldots,2[t/2];\\ (ti-j+1)/2, & \text{if } u=u_{i,j},\ i=2,4,\ldots,2m, \text{ and } j=1,3,\ldots,2[(t-1)/2]+1;\\ (t(2m+i-1)+j+5)/2, & \text{if } u=u_{i,j},\ i=1,3,\ldots,2m-1, \text{ and } j=1,3,\ldots,2[(t-1)/2]+1;\\ (t(2m+i)-j)/2+3, & \text{if } u=u_{i,j},\ i=2,4,\ldots,2m, \text{ and } j=2,4,\ldots,2[t/2]. \end{cases}$$

(See Fig. 6(1) for an example.)

(ii) When $a=2n+1$, we give the desired labeling for each value of t individually.

(a) The labeling f of $C_{2n+1}^{+(2m,n-1)}$ is defined by

$$f(u)=\begin{cases}0, & \text{if } u=v_0;\\ 1+(i-1)(m+1), & \text{if } u=v_i,\quad i=1,2,\dots,n;\\ 2m(n-1)+2n+1-f(v_{2n+1-i}), & \text{if } u=v_i,\quad i=n+1,n+2,\dots,2n;\\ i+1+(j-1)(m+1), & \text{if } u=u_{i,j},\ i=1,2,\dots,m, \text{ and } j=1,2,\dots,n-1; \text{ and}\\ 2m(n-1)+2n+1-f(u_{2m+1-i,j}), & \text{if } u=u_{i,j},\ i=m+1,m+2,\dots,2m, \text{ and } j=1,2,\dots,n-1.\end{cases}$$

(Fig. 6(2) illustrates for the case.)

(b) The labeling f of $C_{2n+1}^{+(2m,t)}$, where $t=n$ or $n+1$, is given by

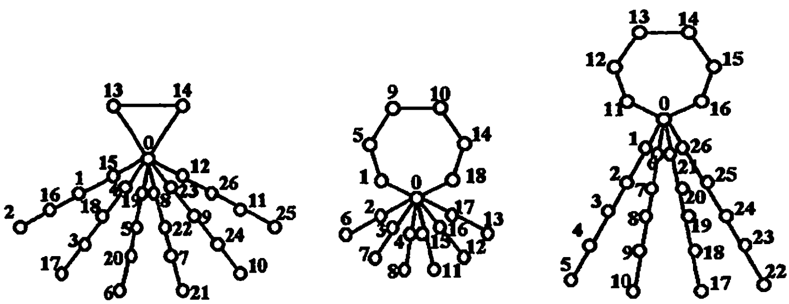
$$f(u)=\begin{cases}0, & \text{if } u=v_0;\\ i(m+1), & \text{if } u=v_i,\quad i=1,2,\dots,n;\\ 2mt+2n+1-f(v_{2n+1-i}), & \text{if } u=v_i,\quad i=n+1,n+2,\dots,2n;\\ i+(j-1)(m+1), & \text{if } u=u_{i,j},\ i=1,2,\dots,m, \text{ and } j=1,2,\dots,t; \text{ and}\\ 2mt+2n+1-f(u_{2m+1-i,j}), & \text{if } u=u_{i,j},\ i=m+1,m+2,\dots,2m, \text{ and } j=1,2,\dots,t.\end{cases}$$

(c) Let the labeling f of $C_{2n+1}^{+(2m,2n-1)}$ be defined by:

$$f(u)=\begin{cases}0, & \text{if } u=v_0;\\ m(2n-1)+i, & \text{if } u=v_i,\quad i=1,2,\dots,2n;\\ j+(i-1)(2n-1), & \text{if } u=u_{i,j},\ i=1,2,\dots,m, \text{ and } j=1,2,\dots,2n-1; \text{ and}\\ 2m(2n-1)+2n+1-f(u_{2m+1-i,j}), & \text{if } u=u_{i,j},\ i=m+1,m+2,\dots,2m, \text{ and } j=1,2,\dots,2n-1.\end{cases}$$

(Fig. 6(3) shows an example for the case.)

The remainder of the verification is just a routine computation. ∎



(1) $C_3^{+(6,4)}$ (2) $C_7^{+(6,2)}$ (3) $C_7^{+(4,5)}$

Fig. 6.

Remark. $C_a^{+(1,t)}$ is also called a "dragon"[15] or a "tadpole"[12].

Conclusion

Among the families of graphs introduced in the third section, only very few cases of them have been solved before. We intended to solve as many cases as possible. Unfortunately, none of them are completely solved. There are still some cases left in each family of graphs.

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